PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 46 (60), 1989, pp. 79–85

# ON SOME RESULTS FOR STARLIKE FUNCTIONS OF COMPLEX ORDER

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**Abstract.** We give some results of various kinds concerning starlike functions of complex order in the unit disk. They represent extensions and generalisations of many previous results. We mainly used the subordination method.

**1. Introduction**. Let $A$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the unit disk $U = \{z : |z| < 1\}$.

A function $f \in A$ is said to be a starlike function of order $b$ ($b \neq 0$, complex), if and only if $f(z)/z \neq 0$, $z \in U$, and

$$\operatorname{Re}\left\{1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} > 0,\ z \in U.$$

We denote by $S(b)$ the class of all such functions. The class $S(b)$ was introduced by Nasr and Aouf in [**6**]. In the same paper they investigated certain properties of the class $S(b)$.

For a function $f \in A$ we say that it is a convex function of order $b$ ($b \neq 0$, complex), that is, $f \in C(b)$, if and only if $f'(z) \neq 0$ in $U$ and

$$\operatorname{Re}\left\{1 + \frac{1}{b}\frac{zf''(z)}{f'(z)}\right\} > 0, \quad z \in U. \tag{2}$$

This class $C(b)$ was introduced by Wiatrowski [**11**] and considered in [**5**]. We note that $f(z) \in C(b) \Leftrightarrow zf'(z) \in S(b)$.

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*AMS Subject Classification* (1980): Primary 30 C 45

Also, we note that for various $b$ in (1) we get the well-known subclasses of univalent functions. For example, for $b$ equal to 1, $1-\alpha$ $(0 \leq \alpha < 1)$, $\cos \lambda e^{-i\lambda}$ ($\lambda$ is real and $|\lambda| < \pi/2$) and $(1-\alpha)\cos \lambda e^{-i\lambda}\big(0 \leq \alpha < 1, |\lambda| < \pi/2\big)$ we have the classes of starlike functions $(S^*)$, starlike functions of order $\alpha$ $\big(S^*(\alpha)\big)$, spirallike functions $(S^\lambda)$ and spirallike functions of order $\alpha$ $(S^\lambda_\alpha)$. More about these classes one can see in the Goodman's book [**1**].

The object of this paper is to obtain some results for the class $S(b)$ using mainly the method of subordination. In that sense, we give some definitions, notations and lemmas we need in the next part.

Let $f$ and $F$ be analytic in the unit disk $U$. The function $f$ is subordinate to $F$, written $f \prec F$ or $f(z) \prec F(z)$, if $F$ is univalent, $f(0) = F(0)$ and $f(U) \subset F(U)$.

The general theory of differential subordinations was introduced in [**2**] by Miller and Mocanu. Some classes of the first-order differential subordinations were considered by the same authors in [**3**]. Namely let $\psi\colon \mathbf{C}^2 \to \mathbf{C}$ be analytic in a domain $D$, let $h$ be univalent in $U$, and let $p$ be analytic in $U$ with $\big(p(z), zp'(z)\big) \in D$ when $z \in U$, then $p$ is said to satisfy the first-order differential subordination if

$$\psi\big(p(z), zp'(z)\big) \prec h(z). \tag{3}$$

The univalent function $q$ is said to be a dominant of the differential subordination (3) if $p \prec q$ for all $p$ satisfying (3). If $\widetilde{q}$ is a dominant of (3) and $\widetilde{q} \prec q$ for all dominants $q$ of (3), then $\widetilde{q}$ is said to be the best dominant of (3).

Now we cite lemma on differential subordinations due to Miller and Mocanu [**3**].

Lemma 1. *Let $q$ be univalent in $U$ and let $\theta$ and $\varphi$ be analytic in a domain $D$ containing $q(U)$, with $\varphi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) = zq'(z)\varphi(q(z))$ $h(z) = \theta\big(q(z)\big) + Q(z)$ and suppose that*

(i) *$Q$ is starlike (univalent) in $U$ with $Q(0) = 0$ and $Q'(0) \neq 0$,*

(ii) $\operatorname{Re}\left\{z\dfrac{h'(z)}{Q(z)}\right\} = \operatorname{Re}\left\{\dfrac{\theta'\big(q(z)\big)}{\varphi\big(q(z)\big)} + z\dfrac{Q'(z)}{Q(z)}\right\} > 0, \quad z \in U.$

*If $p$ is analytic in $U$, with $p(0) = q(0)$, $p(U) \subset D$ and*

$$\theta\big(p(z)\big) + zp'(z)\varphi\big(p(z)\big) \prec \theta\big(q(z)\big) + zq'(z)\varphi\big(q(z)\big) = h(z), \tag{4}$$

*then $p \prec q$, and $q$ is the best dominant of* (4).

For the proof of Theorem 3 we need the following result given by Robertson [**9**].

Lemma 2. *Let $f \in A$ be univalent in $U$. For $0 \leq t \leq 1$ let $F(z,t)$ be analytic in $U$, let $F(z,0) \equiv f(z)$ and $F(0,t) \equiv 0$. Let $r$ be a positive real number for which*

$$F(z) = \lim_{t \to +0} \frac{F(z,t) - F(z,0)}{zt^r}$$

*exists. Let* $F(z,t)$ *be subordinate to* $f(z)$ *in* $U$ *for* $0 \le t \le 1$. *Then* $\operatorname{Re}\{F(z)/f'(z)\} \le 0$, $z \in U$. *If, in addition,* $F(z)$ *is also analytic in* $U$ *and* $\{F(0)\} \neq 0$, *then*

$$\operatorname{Re}\{f'(z)/F(z)\} < 0, \quad z \in U. \tag{5}$$

**2. Results and consequences**. First we use the differential subordinations to obtain

THEOREM 1. *Let* $f \in S(b)$ $(b \neq 0$, *complex*$)$. *Then*

$$\big(f(z)/z\big)^a \prec 1/(1-z)^{2ab}, \tag{6}$$

*where* $a \neq 0$ *is a complex number and either* $|2ab+1| \le 1$ *or* $|2ab-1| \le 1$, *and this is the best dominant.*

*Proof*. If we put $q(z) = (1-z)^{-2ab}$, $\varphi(w) = (ab)^{-1}w^{-1}$ and $\theta(w) = 1$ in Lemma 1, then it is easy to check that the conditions (i) and (ii) in that lemma are satisfied. Namely, $q(z)$ is univalent in $U$ (see, [**1**]), while

$$h(z) = \theta\big(q(z)\big) + zq'(z)\varphi\big(q(z)\big) = (1+z)/(1-z).$$

Consequently, for $p(z) = 1 + p_1 z + \dots$ ... analytic in $U$ with $p(z) \neq 0$ for $0 < |z| < 1$, from (4) we get

$$1 + \frac{1}{ab} \cdot \frac{zp'(z)}{p(z)} \prec \frac{1+z}{1-z} \Rightarrow p(z) \prec q(z). \tag{7}$$

Now, if in (7) we choose $p(z) = \big(f(z)/z\big)^a$, then we have

$$1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right) \prec \frac{1+z}{1-z} \Rightarrow \left(\frac{f(z)}{z}\right)^a \prec \frac{1}{(1-z)^{2ab}},$$

which was to be proved.

For the class $S_\alpha^\lambda$ we have the following corresponding result.

COROLLARY 1. *Let* $f \in S_\alpha^\lambda$ *where* $\lambda$ *is real and* $|\lambda| < \pi/2$, $0 \le \alpha < 1$ *and let* $a \neq 0$ *be a complex number such that*

$$|2a(1-\alpha)\cos\lambda e^{-i\lambda} - 1| \le 1 \quad \textit{or} \quad |2a(1-\alpha)\cos\lambda e^{-i\lambda} + 1| \le 1.$$

*Then*

$$(f(z)/z)^a \prec (1-z)^{-2a(1-\alpha)\cos\lambda e^{-i\lambda}} \tag{8}$$

*and this is the best dominant.*

This is an earlier result given in [**8**].

We note that from (8) we may get the corresponding results for the classes $S^\lambda$, $S^*(\alpha)$ and $S^*$, especially, the well-known results for $S^*(\alpha)$ and $S^*$ when $a = 1$.

From Theorem 1 we directly get

COROLLARY 2. *Let $f \in C(b)$ ($b \neq 0$, complex), and let $a \neq 0$ be a complex number and either $|2ab+1| \le 1$ or $|2ab-1| \le 1$. Then*

$$\big(f'(z)\big)^a \prec (1-z)^{-2ab}$$

*and this is the best dominant.*

If we put $a = -1/2b$ in Theorem 1, then we obtain

COROLLARY 3. *Let $f \in S(b)$ ($b \neq 0$, complex), then*

$$\big(z/f(z)\big)^{1/2b} \prec 1 - z, \tag{9}$$

*and this is the best dominant.*

From (9) we have the following inequality for $f \in S(b)$:

$$|(z/f(z))^{1/2b} - 1| \le |z|, \quad z \in U.$$

Especially, for $b = 1 - \alpha$ $(0 \le \alpha < 1)$ we have

$$f \in S^*(\alpha) \Rightarrow |\big(z/f(z)\big)^{1/2(1-\alpha)} - 1| \le |z|, \quad z \in U,$$

which is a known result given in [**10**].

THEOREM 2. *Let $f \in S(b)$ ($b \neq 0$, complex) and let $a \neq 0$ be a complex number with $0 < 2ab \le 1$. Then*

$$\text{(i)}\ \ \mathrm{Re}\,\{(f(z)/z)^a\} > 2^{-2ab}; \quad \text{(ii)}\ \ |(f(z)/z)^{-a} - 2^{2ab-1}| < 2^{2ab-1}, \quad z \in U.$$

*The estimates are the best possible.*

*Proof*. (i) From Theorem 1 we have that the function $\big(f(z)\big)^a$ is subordinate to the function $q(z) = 1/(1-z)^{2ab}$. So let us show that

$$\mathrm{Re}\,\{q(z)\} > 2^{-2ab}, \quad z \in U. \tag{10}$$

It is easy to prove that $q(z)$ is a convex function in $U$. Since $q(U)$ is convex and symmetrical with respect to the real axis we obtain

$$\inf_{|z|<1} \mathrm{Re}\,\{q(z)\} = \inf_{0\le r\le 1} \{q(-r), q(r)\} = q(-1) = 2^{-2ab}. \tag{11}$$

Finally, from (11) we get (10).

(ii) This part follows from the part (i) and Lemma 1 in [**12**].

The function $f(z) = z(1-z)^{-2ab}$ shows that the results in this theorem are sharp.

Corollary 4. *Let $f \in S_\alpha^\lambda$, where $\lambda$ and $\alpha$ are real and $|\lambda| < \pi/2$, $0 \le \alpha < 1$. If $\beta$ is real and $0 < 2\beta(1-\beta)\cos\lambda \le 1$, then*

$$\text{(i)}\quad \mathrm{Re}\Big\{ \big(f(z)/z\big)^{\beta e^{i\lambda}} \Big\} > 2^{-2\beta(1-\alpha)\cos\,\lambda},$$

$$\text{(ii)}\quad \Big| \big(f(z)/z\big)^{-\beta e^{i\lambda}} - 2^{2\beta(1-\alpha)\cos\,\lambda - 1} \Big| < 2^{2\beta(1-\alpha)\cos\,\lambda - 1}, \quad z \in U.$$

*These results are sharp.*

By using Lemma 2 we give a criterion for a function $f \in A$ to be in the class $S(b)$.

Theorem 3. *Let $f \in A$ with $f(z)/z \neq 0$ in $U$, and let the function*

$$g(z) = \frac{1}{b}\left[ f(z) - (1-b)\int_0^z \frac{f(s)}{s}\,ds \right] = z + \cdots, \tag{12}$$

*be univalent in $U$, where $b \neq 0$ is a complex number. If the function*

$$G(z,t) = \frac{1}{b}\left[ (1-bt)f(z) - (1-b)(1-t^2)\int_0^z \frac{f(s)}{s}\,ds \right] \tag{13}$$

*is subordinate to $g(z)$ for a fixed $b$ and for each $0 \le t \le 1$, then $f \in S(b)$.*

*Proof.* It is evident that $G(z,0) \equiv g(z)$ and $G(0,t) \equiv 0$. In Lemma 2 we choose $r = 1$ and $F(z,t)$ to be the function $G(z,t)$ defined by (13). Then we have that

$$G(z) = \lim_{t\to+0} \frac{G(z,t) - G(z,0)}{zt} = \frac{1}{z}\lim_{t\to+0} \frac{\partial G(z,t)}{\partial t} = -\frac{f(z)}{z},$$

and $G(z)$ is analytic in $U$ with $\mathrm{Re}\{G(0)\} = -1 \neq 0$. Since from (12)

$$g'(z) = b^{-1}[f'(z) - (1-b)f(z)/z],$$

then from (5) we have $\mathrm{Re}\{g'(z)/G(z)\} < 0$, $z \in U$ which is equivalent to the inequality

$$\mathrm{Re}\left\{ 1 + \frac{1}{b}\left( \frac{zf'(z)}{f(z)} - 1 \right) \right\} > 0, \quad z \in U,$$

i.e., $f \in S(b)$.

For example, for $b = 1 - \alpha$ we have the following result for starlike functions of order $\alpha$.

Corollary 5. *Let $f \in A$ with $f(z)/z \neq 0$, in $U$, and let the function*

$$g(z) = \frac{1}{1-\alpha}\left[f(z) - \alpha \int_0^z \frac{f(s)}{s}\,ds\right], \ 0 \le \alpha < 1,$$

*be univalent in $U$. If*

$$G(z,t) = \frac{1}{1-\alpha}\left[\big(1-(1-\alpha)t\big)f(z) - (1-t^2)\int_0^z \frac{f(s)}{s}\,ds\right] \prec g(z),$$

*then $f \in S^*(\alpha)$.*

A result similar to the previous one was given in [**7**].

Finally, we give a transformation which maps the class $S(b_1)$ to the class $S(b_2)$,

Theorem 4. *Let $b_1$, $b_2$ and $\gamma$ be complex numbers, different from zero, such that*

$$\gamma b_1/b_2 \ge 1. \tag{14}$$

*If $f \in S(b_1)$, then*

$$g(z) = z\big(f(z)/z\big)^{\gamma}(1-z)^{-2(\gamma b_1 - b_2)} \in S(b_2). \tag{15}$$

*Proof*. From (15), after logarithmic differentiation and some simple transformations, we get

$$1 + \frac{1}{b_2}\left(\frac{zg'(z)}{g(z)} - 1\right) = \frac{\gamma b_1}{b_2}\left(1 + \frac{1}{b_1}\left(\frac{zf'(z)}{f(z)} - 1\right)\right) + \left(\frac{\gamma b_1}{b_2} - 1\right)\frac{1+z}{1-z},$$

and, from there, the statement of Theorem 4.

For special selection of $b_1$, $b_2$ and $\gamma$, we get the following

Corollary 6. (i) *If $b_1 = (1-\alpha)\cos\lambda e^{-i\lambda}$, $b_2 = (1-\beta)\cos\lambda e^{-i\lambda}$ $(0 \le \alpha, \beta < 1)$ and $\gamma(1-\alpha)/(1-\beta) \ge 1$, then*

$$f \in S_\alpha^\lambda \Rightarrow g(z) = z\big(f(z)/z\big)^{\gamma}(1-z)^{-2\nu\cos\lambda e^{-i\lambda}} \in S_\beta^\lambda,$$

*where $\nu = \gamma(1-\alpha) - (1-\beta)$.*

(ii) *Let $\gamma = 1$ and $b_1/b_2 \ge 1$; then we obtain*

$$f \in S(b_1) \Rightarrow g(z) = f(z)(1-z)^{-2(b_1-b_2)} \in S(b_2).$$

The result (i) is given in [**8**] but there were some mistakes which are not difficult to remove.

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(Received 06 06 1988)


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