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ON p-VALENT ANALYTIC FUNCTIONS WITH REFERENCE TO BERNARDI AND RUSCHEWEYH INTEGRAL OPERATORS

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Abstract. Let $T_n(h)$ be the class of analytic functions in the unit disk E of the form $f(z) = a_p z^p + \sum_{n=p+1}^{\infty} a_n z^n$, $p \geq 1$, which satisfy the condition, $\dfrac{(n+1)}{(n+p)} \dfrac{D^{n+1} f(z)}{D^n f(z)} \prec h(z)$, $z \in E$, where h is a convex univalent function in E with $h(0) = 1$. Then it is proved that f is preserved under the Bernardi integral operator under certain conditions. It is also shown that if $f \in T_0(h)$, it is preserved under the Ruscheweyh integral operator under certain conditions.

The Hadamard product $(f * g)(z)$ of two functions $f(z) = \sum_{m=0}^{\infty} a_m z^m$ and $g(z) = \sum_{m=0}^{\infty} b_m z^m$ is given by $(f * g)(z) = \sum_{m=0}^{\infty} a_m b_m z^m$. Let

$$D^{\alpha} f(z) = \frac{z}{(1-z)^{\alpha+1}} * f(z), \ (\alpha \geq 1).$$

Ruscheweyh [**7**] observed that $D^n f(z) = z\big(z^{n-1} f(z)\big)^{(n)}/n!$ when $n \in N \cup \{0\}$, where $N = \{1, 2, 3, \dots\}$. This symbol $D^n f(z)$, $n \in N \cup \{0\}$ was called the n-th Ruscheweyh derivative of $f(z)$ by Al-Amiri [**1**].

Let $S_p(A, B)$, denote the class of all functions of the form $f(z) = a_p z^p + \sum_{n=p+1}^{\infty} a_n z^n$, $p \geq 1$ which are regular in $E = \{z : |z| < 1\}$ and satisfy the condition,

$$z\frac{f'(z)}{f(z)} = p\frac{1 + Aw(z)}{1 + Bw(z)}, \quad z \in E,$$

where A, $B \in \mathbf{C}$, $|A| \leq 1$, $|B| \leq 1$, $A \neq B$ and $w(z)$ is regular in E and satisfies the Schwarz lemma conditions, that is $w(0) = 0$, $|w(z)| < 1$ in E.

The class $S_p(1, -1)$ is the class of p-valent starlike functions and $S_p(A, B)$ for A, $B \in \mathbf{C}$ will yield subclasses of spirallike p-valent functions. For A, B real and $0 \leq A < B \leq 1$, $S_p(A, B)$ will give a subclass of p-valent starlike functions.

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For $p = 1$ such classes were investigated by many authors and were introduced by Janowski [**4**], when A, B are real and by Stankiewicz, Waniurski [**9**] when A, B are complex. Bernardi [**2**] showed that the function g defined by,

$$g(z) = (e+1)z^{-c}\int_0^z t^{c-1} f(t)\,dt,$$

where c is a positive integer, belongs to the class $S_1(1,-1)$ if f belongs to $S_1(1,-1)$.

Lakshma Reddy and Padmanabhan [**6**] showed that the function g defined by the Bernardi operator,

$$g(z) = (c+p)z^{-c}\int_0^z t^{c-1} f(t)\,dt, \quad c = 1, 2, \dots,$$

belongs to the class $S_p(A,B)$, A, B-real, $-1 \le A < B \le -1$, if $f \in S_p(A,B)$.

Kumar and Shukla [**5**] obtained a generalization of the above result by considering the Ruscheweyh integral operator which is given by

$$g(z) = \left\{(c+p\alpha)z^{-c}\int_0^z t^{c-1} f^{\alpha}(t)\,dt\right\}^{1/\alpha};$$
$$c \text{ and } \alpha \text{ real}, \ \alpha > 0, \quad c \ge -p\alpha\frac{1+A}{1+B}.$$

Ryszard Kowal and Jan Stankiewicz [**8**] obtained the solution of this problem in the case when A, B are complex numbers. We propose to study the more general problem when f satisfies the condition that $\dfrac{n+1}{n+p}\,\dfrac{D^{n+1}f}{D^n f}$ is subordinate to a convex univalent function h. We allow c to be complex and make use of the method of differential subordination introduced in [**3**].

Let $T_n(h)$ be the class of analytic functions on E of the form $f(z) = a_p z^p + \sum_{n=p+1}^{\infty} a_n z^n$, $p \ge 1$ which satisfy the condition,

$$\left(\frac{n+1}{n+p}\right)\frac{D^{n+1}f(z)}{D^n f(z)} \prec h(z), \quad z \in E,$$

where h is a convex univalent function in E with $h(0) = 1$ and the symbol $\prec$ denotes subordination.

Let $f \in T_n(h)$ and let g be given by,

$$g(z) = (c+p)z^{-c}\int_0^z t^{c-1} f(t)\,dt, \quad c \in \mathbf{C}. \tag{1}$$

Then we show that $g \in T_n(h)$ under certain conditions to be satisfied by c and h. In particular for $n = 0$, the class $T_0(h)$ consists of functions f of the form $f(z) = a_p z^p + \sum_{n=p+1}^{\infty} a_n z^n$, satisfying $(1/p)zf'(z)/f(z) \prec h(z)$.

Let $f \in T_0(h)$ and define F by

$$(2) \qquad F(z) = \left[(c+p\alpha)z^{-c}\int_0^z t^{c-1}f^{\alpha}(t)\,dt\right]^{1/\alpha}, \quad c \in \mathbf{C},\ \alpha > 0.$$

In this paper it is shown that F also belongs to $T_0(h)$ under certain conditions to be satisfied by c and h. We need the following theorem due to Eenigenburg, Miller, Mocanu and Reade.

THEOREM A [**3**]. *Let $\beta,\ \gamma \in \mathbf{C}$, $h \in H(E)$ be convex univalent in E with $h(0) = 1$ and $\operatorname{Re}\big(\beta h(z) + \gamma\big) > 0$, $z \in E$ and let $p \in H(E)$, $p(z) = 1 + p_1 z + \cdots$. Then*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z),$$

implies that $p(z) \prec h(z)$.

THEOREM 1. *Suppose $f \in T_n(h)$ and g is given by* (1). *Then $g \in T_n(h)$ provided* $\operatorname{Re}\{(n+p)h(z) + (c-n)\} > 0$.

Proof: Differentiating (1) we get

$$g'(z) = -\frac{c(c+p)}{z^{c+1}}\int_0^z t^{c-1}f(t)\,dt + \frac{c+p}{z^c}z^{c-1}f(z),$$

which gives

$$zg'(z) + cg(z) = (c+p)f(z).$$

Therefore we have

$$(3) \qquad D^n\big(zg'(z)\big) + D^n\big(cg(z)\big) = D^n\big((c+p)f(z)\big).$$

Using the fact that $D^n\big(zg'(z)\big) = z\big(D^n g(z)\big)'$, (3) reduces to

$$z\big(D^n g(z)\big)' + c\big(D^n g(z)\big) = (c+p)D^n f(z).$$

Again using the result

$$(4) \qquad z\big(D^n g(z)\big)' = (n+1)\big(D^{n+1}g(z)\big) - n\big(D^n g(z)\big)$$

we get

$$(5) \qquad (n+1)\frac{D^{n+1}g(z)}{D^n g(z)} + (c-n) = (c+p)\frac{D^n f(z)}{D^n g(z)}.$$

Set

$$(6) \qquad P(z) = \frac{n+1}{n+p}\frac{D^{n+1}g(z)}{D^n g(z)}.$$

Then (4) takes the form

$$P(z)+\frac{c-n}{n+p}=\frac{c+p}{n+p}\frac{D^n f(z)}{D^n g(z)}. \tag{7}$$

Taking logarithmic derivatives and multiplying by z, we get,

$$\frac{zP'(z)}{P(z)+(c-n)/(n+p)}=z\frac{\big(D^n f(z)\big)'}{D^n f(z)}-z\frac{\big(D^n g(z)\big)'}{D^n g(z)}.$$

Using (4) and (6) this takes the form,

$$P(z)+\frac{zP'(z)}{(n+p)P(z)+(c-n)}=\frac{n+1}{n+p}\frac{D^{n+1}f(z)}{D^n f(z)}\prec h(z),$$

Since $f\in T_n(h)$.

Now, if $\mathrm{Re}\{(n+p)h(z)+(c-n)\}>0$, we can conclude by Theorem A that $P(z)\prec h(z)$, that is $P\in T_n(h)$.

Remark 1. For $n=0$ the above theorem reduces to the following result, namely, $g\in T_0(h)$, that is, $(1/p)zg'(z)/g(z)\prec h(z)$ whenever $f\in T_0(h)$ if $\mathrm{Re}\{ph(z)+c\}>0$. For the choice of $h(z)=(1+Az)/(1+Bz)$, this clearly includes the result in [**6**].

THEOREM 2. *Let $f\in T_0(h)$ and let F be given by* (2). *Then $F\in T_0(h)$ provided* $\mathrm{Re}\big(\alpha ph(z)+c\big)>0$.

Proof:

$$F^\alpha(z)=\frac{c+p\alpha}{z^c}\int_0^z t^{c-1}f^\alpha(t)\,dt.$$

Differentiation gives

$$\frac{z}{p}\frac{F'(z)}{F(z)}+\frac{c}{\alpha p}=\frac{f^\alpha(z)}{F^\alpha(z)}\frac{(c+p\alpha)}{p\alpha}.$$

Setting, $P(z)=zF'(z)/(pF(z))$, this reduces to

$$P(z)+\frac{c}{\alpha p}=\frac{z^c f^\alpha(z)}{p^\alpha\int_0^z t^{c-1}f^\alpha(t)\,dt}.$$

Taking logarithmic derivatives and multiplying by z, we obtain, after some simplification,

$$\frac{zP'(z)}{P(z)+c/(\alpha p)}=\frac{\alpha zf'(z)}{f(z)}-\alpha pP(z),$$

$$P(z)+\frac{zP'(z)}{\alpha pP(z)+c}=\frac{1}{p}z\frac{f'(z)}{f(z)}\prec h(z).$$

Provided $\operatorname{Re}\bigl(\alpha p h(z) + c\bigr) > 0$, this implies that $P(z) \prec h(z)$ by Theorem A.

Remark: If we choose $h(z) = (1 + Az)/(1 + Bz)$, A, B real with $-1 \leq A < B \leq 1$ and condition on c and h reduces to

$$\operatorname{Re} c > -\alpha p \operatorname{Re} h(z) > -\alpha p \frac{1 - A}{1 - B}.$$

This condition is clearly an improvement on the condition

$$c \geq -p\alpha \frac{1 + A}{1 + B}$$

in [**5**].

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