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# ON CONJUGATE Π-VARIATION AND THE COEFFICIENTS OF POWER SERIES

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**Abstract.** We prove a result connecting the asymptotic behaviour of the maximum modulus of an entire function with its coefficients. Application of this result gives an asymptotic relation between the moments and the tail of the distribution function of a random variable.

## Introduction and results

In this note we prove the equivalence of an asymptotic relation for the coefficients of a power series and the behaviour of the maximum modulus for a class of entire functions. For several orders of growth of the coefficients similar results are known. See e.g. Boas [**2**], Geluk [**5**], [**6**]. Suppose $f(x) = \sum_{n=0}^{\infty} c_n x^n$ is entire. The order of growth of the coefficients we consider includes cases like

$$|c_n|^{-1/n} = n \log^{\alpha} n + o(n \log^{\alpha-1} n) \ (n \to \infty), \ \alpha > 0$$

In order to formulate our results we need two definitions

*Definition* 1. Suppose $\pi : \mathbf{R}^+ \to \mathbf{R}^+$ has the property that there exists a function $a : \mathbf{R}^+ \to \mathbf{R}^+$ such that $\lim_{t\to\infty}\big(\pi(tx) - \pi(t)\big)/a(t) = \log x$ for all $x > 0$. Then we say $\pi \in \Pi^+$ or $\pi \in \Pi^+(a)$. The class $\Pi^-$ is defined similary, exept for $\pi \in \Pi^-$ we require $\lim_{t\to\infty}\big(\pi(tx) - \pi(t)\big)/a(t) = -\log x$. Finally $\Pi = \Pi^+ \cup \Pi^-$.

For properties concerning the class $\Pi$ the reader is referred to Geluk, de Haan [**8**] or Bingham, Goldie, Teugels [**1**]. Pairs of conjugate slowly varying functions are introduced by de Bruijn [**3**]. In de Haan, Resnick [**9**], this notion is extended to the class $\Pi$ as follows.

*Definition* 2. Suppose $\pi \in \Pi(a)$. Any function $\pi^* : \mathbf{R}^+ \to \mathbf{R}^+$ satisfying

$$\lim_{t\to\infty} \pi(t)\{a(t)\}^{-1}\left[\pi(t)\pi^*\big(t\pi(t)\big) - 1\right] = 0$$

is a conjugate function for $\pi$.

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Given $\pi \in \Pi(a)$ one can construct a conjugate function as follows. Choose $\pi_0$ such that $\pi(t) - \pi_0(t) = o\big(a(t)\big)$ $(t \to \infty)$ and such that $\pi_0$ is continuous and increasing. This is possible by [**8**, prop. 1.19.6].

Define $v(t) = t\pi_0(t)$ with inverse function $v^{-1}$. Then $\pi^*(t) = v^{-1}(t)/t$ is a conjugate function. Also, if $\pi \in \Pi^{\pm}(a)$, then $\pi^* \in \Pi^{\mp}(a^*)$ where the function $a^*$ satisfies $a^*\big(t\pi(t)\big) \sim a(t)/\pi^2(t)$ $(t \to \infty)$. See [**9**, theorem 1].

From the above construction it follows that if $\pi_1(t) - \pi_2(t) = o\big(a(t)\big)$, where $\pi_i \in \Pi(a)$ for $i = 1, 2$ then $\pi_1^*(t) - \pi_2^*(t) = o\big(a^*(t)\big)$ $(t \to \infty)$. Moreover it follows that $(\pi^*)^*(t) = \pi(t) + o\big(a(t)\big)$. For this reason the function $\pi$ and $\pi^*$ are said to form a conjugate pair.

Our main result is the following

THEOREM 1. *Suppose $f$ is entire, $f(x) = \sum_{n=0}^{\infty} c_n x^n$ with maximum modulus $M(s, f) = \max_{|z|=s} |f(z)|$. Then there exists $\pi \in \Pi^{\pm}(a)$ such that*

$$\varliminf_{n\to\infty}{}^{*} \frac{|c_n|^{-1/n} - n\pi(n)}{na(n)} = 0 \tag{1}$$

*if and only if there exists $\pi^* \in \Pi^{\mp}(a^*)$ such that*

$$\varlimsup_{s\to\infty} \frac{\log M(s, f) - e^{-1} s\pi^*(e^{-1}s)}{sa^*(s)} = 0. \tag{2}$$

In (1) $\varliminf^*$ denotes the limit inferior over all $n$ for which $c_n \neq 0$. If one of the above conditions holds, then we can choose the functions $\pi$ and $\pi^*$ in such a way that they from a conjugate pair.

Using the above results it is possible to prove a relation between the asymptotic behaviour of the moments of a random variable and the tail of its distribution function. Related results are given in Kasahara [**10**], Dewess [**4**], Geluk [**5**], [**6**].

Before we formulate the result we need one more definition.

*Definition* 3. Suppose $\varphi : \mathbf{R}^+ \to \mathbf{R}^+$ is non-decreasing. We say $\varphi \in \Gamma$ if there exists a positive function $b$ such that

$$\lim_{t\to\infty} \varphi\big(t + xb(t)\big)/\varphi(t) = e^x \quad \text{for} \quad x \in \mathbf{R}.$$

The function $b$ is called an *auxiliary function* for $\varphi$.

THEOREM 2. *Suppose the random variable $Y$ has an entire characteristic function and $\mathbf{P}(|Y| > y) > 0$ for $y \in \mathbf{R}$. Then the following statements are equivalent.*

(i) *There exists a function $\varphi \in \Gamma$ such that*

$$\lim_{y\to\infty} \frac{-\log \mathbf{P}(|Y| > y)}{\varphi(y)} = 1.$$

(ii) *There exists a function* $\pi \in \Pi^{-}(a)$ *such that* $\pi(n) \to 0$ *as* $n \to \infty$ *and*

$$\lim_{n\to\infty} \frac{e^{-1}(\mathbf{E}|Y|^n)^{-1/n} - \pi(n)}{a(n)} = 0.$$

If one of the above conditions holds, then we can choose the functions $\varphi$ and $\pi$ in such a way that $\varphi^c(s) = e^{-1}s\pi^*(e^{-1}s)$, where $\pi^*$ is a conjugate function for $\pi$ and $\varphi^c$ is the function complementary to $\varphi$, defined by $\varphi^c(s) = \sup_{y>0}\{sy - \varphi(y)\}$ for $s > 0$.

## 2. Proofs

*Proof of theorem* 1. Suppose (1) holds true with $\pi \in \Pi^{-}(a)$ and $\varepsilon > 0$ is arbitrary. We omit the case $\pi \in \Pi^{+}(a)$ which can be proved similarly. Since $\pi \in \Pi^{-}(a)$, there exists $n(\varepsilon)$ such that $|c_n|^{-1/n} \geq n\pi(n) - \varepsilon n a(n)/2 \geq n\pi(ne^{\varepsilon})$ for all $n \geq n(\varepsilon)$ satisfying $c_n \neq 0$. Hence for $|z| = s$ we have

$$\left|\sum_{n=1}^{\infty} c_n z^n\right| \leq \sum_{n=1}^{\infty} |c_n| s^n \leq \sum_{n<n_\varepsilon} |c_n| s^n + \sum_{n\geq n_\varepsilon} \left\{s/\bigl(n\pi(ne^{\varepsilon})\bigr)\right\}^n. \tag{3}$$

We denote the last sum by $\sum$. Without loss of generality we may assume $\pi$ to be differentiable with derivative $\pi'(t) \sim -a(t)/t$ as $t \to \infty$. See [**8**, prop. 1.19.6]. We estimate the index $n_0$ of the maximal term in $\sum$ for large values of $s$. Considering $n$ as a continuous variable, by differentiation $n_0$ satisfies

$$\xi = y\pi(y)\exp(y\pi'(y)/\pi(y)), \tag{4}$$

where $\xi = se^{-1+\varepsilon}$ and $y = n_0 e^{\varepsilon}$.

Since $\pi \in \Pi^{-}(a)$ and $-y\pi'(y) \sim a(y) = o\bigl(\pi(y)\bigr)$ this implies $\xi = y\pi(ey) + o\bigl(ya(y)\bigr)$ as $y \to \infty$. Inversion then gives $y = \xi\pi^*(e\xi) + o\bigl(\xi a^*(\xi)\bigr)$ as $\xi \to \infty$ and substitution of the expression for $\xi$ and $y$ leads to

$$n_0 = se^{-1}\pi^*(e^{\varepsilon}s) + o\bigl(sa^*(s)\bigr),\ s \to \infty. \tag{5}$$

Now we write $\sum = \sum_1 + \sum_2$, where the summations in $\sum_1$ and $\sum_2$ are taken over $n_\varepsilon \leq n < 2s\pi^*(s)$ and $n \geq 2s\pi^*(s)$ respectively. Observe that if we choose $\pi^*(t) = v^{-1}(t)/t$ as in the introduction the summation in $\sum_2$ is over the values of $n$ such that $n\pi(n/2) \geq 2s$. Hence

$$\sum\nolimits_2 \leq \sum_{n=1}^{\infty} \left\{\frac{\pi(n/2)}{2\pi(ne^{\varepsilon})}\right\}^n < \infty \tag{6}$$

by Cauchy's root test, since the function $\pi$ is slowly varying. The first sum can be estimated by

$$\sum\nolimits_1 \leq 2s\pi^*(s)\left\{\frac{s}{n_0\pi(n_0e^{\varepsilon})}\right\}^{n_0}. \tag{7}$$

With $n_0$ and $s$ connected by (5) we have by (4)

$$\log \frac{s}{n_0\pi(n_0 e^{\varepsilon})} = 1 + \frac{y\pi'(y)}{\pi(y)} = 1 - \frac{a(y)}{\pi(y)}\big(1+o(1)\big) = 1 - a^*(s)\pi\big(s\pi^*(s)\big)\big(1+o(1)\big)$$

as $s \to \infty$, the last equality being justified by $y = n_0 e^{\varepsilon} \sim e^{\varepsilon-1} s\pi^*(s)$ hence

$$\frac{a(y)}{\pi(y)} \sim \frac{a\big(s\pi^*(s)\big)}{\pi\big(s\pi^*(s)\big)} \sim a^*(s)\pi\big(s\pi^*(s)\big).$$

Substituting this with (5) in the right-hand side of (7) gives

$$\sum\nolimits_1 \le 2s\pi^*(s) \times \exp\Big[\big\{se^{-1}\pi^*(e^{\varepsilon}s) + o\big(sa^*(s)\big)\big\}\big\{1 - a^*(s)\pi\big(s\pi^*(s)\big)\big(1+o(1)\big)\big\}\Big]. \qquad (8)$$

Since $\pi^*(e^{\varepsilon}s) \sim \pi^*(s)$ and $\pi^*(s)\pi(s\pi^*(s)) \to 1$, $s \to \infty$, combination of (3), (6) and (8) gives

$$\varlimsup_{s\to\infty} \frac{\log M(s,f) - se^{-1}\pi^*(e^{\varepsilon}s)}{sa^*(s)} \le -e^{-1}.$$

Since $\pi^* \in \Pi^+(a^*)$ this implies

$$\varlimsup_{s\to\infty} \frac{\log M(s,f) - se^{-1}\pi^*(s)}{sa^*(s)} \le (\varepsilon - 1)e^{-1}.$$

Since $\varepsilon > 0$ is arbitrary, the left hand side is at most $-e^{-1}$. By (1) for any $\varepsilon > 0$ there exists a sequence $n_k \to \infty$ $(k \to \infty)$ such that for $n = n_k$

$$|c_n|^{-1/n} \le n\pi(n) + \frac{\varepsilon}{2} na(n) \le n\pi(ne^{-\varepsilon}).$$

Application of Cauchy's inequality gives

$$M(s,f) \ge |c_n| s^n \ge \left\{\frac{s}{n\pi(ne^{-\varepsilon})}\right\}^n \quad \text{for } n = n_k \text{ and } s > 0. \qquad (9)$$

Substituing $s = s_n = en\pi(n)$ we find for $n = n_k > n(\varepsilon)$

$$\begin{aligned}\log M(s_n,f) &\ge n\left\{1 + \log\left(1 + \frac{\pi(n)-\pi(e^{-\varepsilon}n)}{a(e^{-\varepsilon}n)}\,\frac{a(e^{-\varepsilon}n)}{\pi(e^{-\varepsilon}n)}\right)\right\} \\ &\ge n\left(1 - 2\varepsilon\frac{a(n)}{\pi(n)}\right) \quad \text{since } \pi \in \Pi^-(a).\end{aligned}$$

As a consequence,

$$\begin{aligned}\lim_{n\to\infty} \frac{\log M(s_n,f) - e^{-1}s_n\pi^*(s_n)}{s_n a^*(s_n)} &\ge \lim_{n\to\infty} \frac{1 - 2\varepsilon a(n)/\pi(n) - \pi(n)\pi^*(en\pi(n))}{e\pi(n)a^*\big(n\pi(n)\big)} \\ &= -2\varepsilon e^{-1} + \lim_{n\to\infty} \frac{1 - \pi(n)\pi^*\big(en\pi(n)\big)}{e\pi(n)a^*\big(n\pi(n)\big)} = (-1-2\varepsilon)e^{-1}.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, by combination of the above results it follows that

$$\overline{\lim_{s\to\infty}}\,(\log M(s,f) - e^{-1}s\pi^*(s))/(sa^*(s)) = -e^{-1},$$

which is equivalent to (2) since $\pi^* \in \Pi^+(a^*)$. The proof of the converse part of theorem 1 is similar to the proof of the corresponding part in [**6**] and is omitted.

For the proof of theorem 2 we need three lemmas.

*Definition* 4. Suppose $\varphi : \mathbf{R}^+ \to \mathbf{R}^+$ is measurable and $\varphi(t)/t \to \infty$ as $t \to \infty$. Define the function $\widetilde{\varphi}$ by $\widetilde{\varphi}(s) = \log s \int_0^\infty e^{us-\varphi(u)}du,\ s > 0$. Define the function $\varphi^c$ by

$$\varphi^c(x) = \sup_{y>0}\{xy - \varphi(y)\} \tag{10}$$

Lemma 1. *If $\varphi : \mathbf{R}^+ \to \mathbf{R}$ is locally bounded and $\varphi(t) \sim \int_0^t s(u)\,du\ (t \to \infty)$ with $s \in \Gamma$, then*

$$\varphi^c(x) = \int_0^x s^{\leftarrow}(u)\,du + o\big(xa(x)\big) = \widetilde{\varphi}(x) \ \text{ as } x \to \infty$$

*where the inverse function $s^{\leftarrow}$ of $s$ satisfies $s^{\leftarrow} \in \Pi^+(a)$.*

Hence $\varphi^c(x)/x$ and $\widetilde{\varphi}(x)/x$ are in $\Pi^+(a)$.

Lemma 2. *Suppose $F$ is the distribution function of a non-negative random variable with entire characteristic function and $\widehat{F}$ is defined by $\widehat{F}(s) = \int_0^\infty e^{su}dF(u)$ for $s \in \mathbf{R}$. Let $\varphi : \mathbf{R}^+ \to \mathbf{R}^+$ be such that $\varphi(t)/t \to \infty\ (t \to \infty)$. Then $\widehat{F}(s) \le e^{\varphi(s)}$ for $s > 0$ implies $1 - F(x) \le e^{-\varphi^c(x)}$ for $x > 0$, where $\varphi^c$ is defined by* (10)*. Conversely, if there exists an $x_0$ such that $1 - F(x) \le e^{-\varphi(x)}$ for $x > x_0$ and with $\varphi \in \Gamma$, then $\widehat{F}(s) \le \exp\{\varphi^c(s) + o(sa(s))\}$ as $s \to \infty$, where $\varphi^c(x)/x \in \Pi^+(a)$* (see Lemma 1).

Lemma 3. *Suppose the assumptions of lemma 2 are satisfied and suppose moreover that $\varphi \in \Gamma$. Then $\varphi^c(s)/s \in \Pi^+$ and $\varphi^c(s)/s \to \infty$ as $s \to \infty$. Moreover*

$$\underline{\lim_{x\to\infty}}\,\frac{-\log\big(1 - F(x)\big)}{\varphi(x)} = 1 \quad \text{if and only if} \quad \overline{\lim_{s\to\infty}}\,\frac{\log \widehat{F}(s) - \varphi^c(s)}{sa(s)} = 0.$$

Lemma 1 above is a combination of theorems 1.5 and 1.9 in [**7**]. The proof of the lemmas 2 and 3 is omitted since it is similar to the lemmas in [**6**].

*Proof of theorem* 2. With $F$ the distribution function of $|Y|$ we have $\widehat{F}(s) = M(s,f)$. Application of lemma 3 and theorem 1 shows that (i) is equivalent to

$$\underline{\lim_{n\to\infty}}\,\frac{(\mathbf{E}|Y|^n/n!)^{-1/n} - n\pi(n)}{na(n)} = 0, \quad \text{with } \pi \in \Pi^-(a),$$

where the functions $\varphi$ and $\pi$ are related by $\varphi^c(s) = e^{-1}s\pi^*(e^{-1}s)$. The last expression is equivalent to (ii) since $(n!)^{1/n} = ne^{-1} + O(\log n)\ (n \to \infty)$.

*Example.* $\pi^*(t) = \log t + o(1)$ $(t \to \infty)$ if and only if

$$\pi(t) = \frac{1}{\log t} + \frac{\log \log t}{\log^2 t} + o\left(\frac{1}{\log^2 t}\right), \quad t \to \infty.$$

Application of theorem 2 then shows that

$$\lim_{y\to\infty} \frac{-\log \mathbf{P}(|Y| > y)}{\exp(ey - 1)} = 1 \quad \text{if and only if}$$

$$\lim_{n\to\infty} e^{-1}(\log^2 n)(\mathbf{E}|Y|^n)^{1/n} - \log n - \log\log n = 0.$$

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