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ABSOLUTE AND ORDINARY KÖTHE-TOEPLITZ DUALS OF SOME SETS OF SEQUENCES AND MATRIX TRANSFORMATIONS

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Abstract. We determine the ordinary Köthe-Toeplitz dual of the set $\Delta l_\infty(p)$ and the absolute Köthe-Toeplitz duals of the sets $\Delta l_\infty(p)$, $\Delta c_0(p)$ and $\Delta c(p)$ defined by Ahmad and Mursaleen. Further we investigate matrix transformations in these spaces and give a characterization of the class $\big(\Delta l_\infty(p), l_\infty\big)$.

1. Introduction

By ω we denote the set of all complex sequences $x = (x_k)_{k=1}^\infty$. Throughout the paper $p = (p_k)_{k=1}^\infty$ shall always be an arbitrary sequence of positive reals. The following sets were introduced and investigated by various authors:

$$\begin{aligned} l_\infty(p) &:= \big\{x \in \omega : \sup_k |x_k|^{p_k} < \infty\big\}, \\ c(p) &:= \big\{x \in \omega : |x_k - l|^{p_k} \to 0 \text{ for some complex } l\big\}, \\ c_0(p) &:= \big\{x \in \omega : |x_k|^{p_k} \to 0\big\}, \\ l(p) &:= \left\{x \in \omega : \sum_{k=1}^\infty |x_k|^{p_k} < \infty\right\} \quad \text{(cf. [2], [3], [5], and [7]).} \end{aligned}$$

Given any sequence $x \in \omega$ we shall write $\Delta x := (x_k - x_{k+1})$. In a recent paper (cf. [1]), Ahmad and Mursaleen defined the following sets:

$$\begin{aligned} \Delta l_\infty(p) &:= \{x \in \omega : \Delta x \in l_\infty(p)\}, \\ \Delta c(p) &:= \{x \in \omega : \Delta x \in c(p)\}, \\ \Delta c_0(p) &:= \{x \in \omega : \Delta x \in c_0(p)\}. \end{aligned}$$

In the determination of the absolute Köthe-Toeplitz duals of $\Delta l_\infty(p)$ and $\Delta c_0(p)$, they applied some arguments which do not seem to hold:

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(i) $x \in \Delta l_\infty(p)$ does not imply in general the existence of a finite number $N > \sup_k k^{-1}|x_k|$, as the following counterexample will show: If we put $p_k := k^{-1}$ and $x_k := k^2$ $(k = 1, 2, \dots)$ then $|\Delta x_k|^{p_k} \to 1$ $(k \to \infty)$, hence $x \in \Delta l_\infty(p)$, and $\sup_k k^{-1}|x_k| = \infty$.

(ii) If a is a sequence such that

$$\sum_{k=1}^{\infty} k|a_k|N^{1/p_k} = \infty \quad \text{for some } N > 1, \tag{1.1}$$

then the sequence x defined by $x_k := kN^{1/p_k} \operatorname{sgn} a_k$ is not in $\Delta l_\infty(p)$, in general. In order to see this, we put $p_k := k$ and $a_k := (-1)^k$ $(k = 1, 2, \dots)$. Then a satisfies (1.1) for all $N > 1$ and $|\Delta x_k|^{p_k} \to \infty$, hence $x \notin \Delta l_\infty(p)$.

In this paper, we shall determine the absolute Köthe-Toeplitz duals of the sets $\Delta l_\infty(p)$ and $\Delta c_0(p)$, and give new proofs for the characterizations of the matrix transformations considered in [**1**]. Further we shall state some new results.

2. Köthe-Toeplitz duals

For arbitrary set X of sequences, we define the ordinary and absolute Köthe-Toeplitz duals by

$$X^\dagger := \left\{ a \in \omega : \sum_{k=1}^{\infty} a_k x_k \text{ converges for all } x \in X \right\} \quad \text{and}$$

$$X^{|\dagger|} := \left\{ a \in \omega : \sum_{k=1}^{\infty} |a_k x_k| < \infty \text{ for all } x \in X \right\}$$

respectively; we shall write $X^{\dagger\dagger} := (X^\dagger)^\dagger$ and $X^{|\dagger\dagger|} := (X^{|\dagger|})^{|\dagger|}$.

Theorem 2.1. *For every strictly positive sequence $p = (p_k)$, we have*

(a) $$\big(\Delta l_\infty(p)\big)^{|\dagger|} = D_\infty^{(1)}(p) := \bigcap_{N=2}^{\infty} \left\{ a \in \omega := \sum_{k=1}^{\infty} |a_k| \sum_{j=1}^{k-1} N^{1/p_j} < \infty \right\},$$

(b) $$\big(\Delta l_\infty(p)\big)^{|\dagger\dagger|} = D_\infty^{(2)}(p) := \bigcup_{N=2}^{\infty} \left\{ a \in \omega : \sup_{k \geq 2} |a_k| \left[\sum_{j=1}^{k-1} N^{1/p_j} \right]^{-1} < \infty \right\},$$

(c) $$\big(\Delta c_0(p)\big)^{|\dagger|} = D_0^{(1)}(p) := \bigcup_{N=2}^{\infty} \left\{ a \in \omega : \sum_{k=1}^{\infty} |a_k| \sum_{j=1}^{k-1} N^{-1/p_j} < \infty \right\},$$

(d) $$\big(\Delta c_0(p)\big)^{|\dagger\dagger|} = D_0^{(2)}(p) := \bigcap_{N=2}^{\infty} \left\{ a \in \omega : \sup_{k \geq 2} |a_k| \left[\sum_{j=1}^{k-1} N^{-1/p_j} \right]^{-1} < \infty \right\}.$$

(*We adopt the usual convention that $\sum_{j=1}^{m} y_j = 0$ $(m < 1)$ for arbitrary y_i.*)

Proof: (a) Let $a \in D_\infty^{(1)}(p)$ and $x \in \Delta l_\infty(p)$. We choose $N > \max\{1, \sup|\Delta x_k|^{p_k}\}$. Then

$$\sum_{k=1}^{\infty}|a_k x_k| \le \sum_{k=1}^{\infty}|a_k|\left|\sum_{j=1}^{k-1}\Delta x_j\right| + |x_1|\sum_{k=1}^{\infty}|a_k| \tag{2.1}$$
$$\le \sum_{k=1}^{\infty}|a_k|\sum_{j=1}^{k-1}N^{1/p_j} + |x_1|\sum_{k=1}^{\infty}|a_k| < \infty.$$

(Note: Since $\sum_{j=1}^{k-1}N^{1/p_j} \ge 1$ for arbitrary $N > 1$ $(k = 2, 3, \dots)$, $a \in D_\infty^{(1)}(p)$ implies $\sum_{k=1}^{\infty}|a_k| < \infty$.)

Conversely let $a \notin D_\infty^{(1)}(p)$. Then we have $\sum_{k=1}^{\infty}|a_k|\sum_{j=1}^{k-1}N^{1/p_j} = \infty$ for some integer $N > 1$.

We define the sequence x by $x_k := \sum_{j=1}^{k-1}N^{1/p_j}$ $(k = 1, 2, \dots)$. Then it is easy to see that $x \in \Delta l_\infty(p)$ and $\sum_{k=1}^{\infty}|a_k x_k| = \infty$, hence $a \notin \big(\Delta l_\infty(p)\big)^{|\dagger|}$.

(b) Let $a \in D_\infty^{(2)}(p)$ and $x \in \big(\Delta l_\infty(p)\big)^{|\dagger|} = D_\infty^{(1)}(p)$, by part (a). Then for some $N > 1$, we have

$$\sum_{k=2}^{\infty}|a_k x_k| = \sum_{k=2}^{\infty}|a_k|\left[\sum_{j=1}^{k-1}N^{1/p_j}\right]^{-1}|x_k|\sum_{j=1}^{k-1}N^{1/p_j}$$
$$\le \sup_{k\ge 2}\left[|a_k|\left[\sum_{j=1}^{k-1}N^{1/p_j}\right]^{-1}\right]\sum_{k=2}^{\infty}|x_k|\sum_{j=1}^{k-1}N^{1/p_j} < \infty.$$

Conversely let $a \notin D_\infty^{(2)}(p)$. Then for all integers $N > 1$, we have

$$\sup_{k\ge 2}|a_k|\left[\sum_{j=1}^{k-1}N^{1/p_j}\right]^{-1} = \infty.$$

Hence there is a strictly increasing sequence $\big(k(m)\big)$ of integers $k(m) \ge 2$ such that

$$|a_{k(m)}|\left[\sum_{j=1}^{k(m)-1}m^{1/p_j}\right]^{-1} > m^2 \quad (m = 2, 3, \dots).$$

We define the sequence x by

$$x_k := \begin{cases} |a_{k(m)}|^{-1} & \big(k = k(m)\big) \\ 0 & \big(k \ne k(m)\big)\ (m = 2, 3, \dots). \end{cases}$$

Then for all integers $N \geq 2$, we have

$$\sum_{k=1}^{\infty}|x_k|\sum_{j=1}^{k-1}N^{1/p_j} \leq \sum_{m=2}^{\infty}|a_{k(m)}|^{-1}\sum_{j=1}^{k(m)-1}N^{1/p_j} \leq$$
$$\leq \sum_{m=2}^{N-1}|a_{k(m)}|^{-1}\sum_{j=1}^{k(m)-1}N^{1/p_j} + \sum_{m=N}^{\infty}|a_{k(m)}|^{-1}\sum_{j=1}^{k(m)-1}m^{1/p_j} \leq$$
$$\leq \sum_{m=2}^{N-1}|a_{k(m)}|^{-1}\sum_{j=1}^{k(m)-1}N^{1/p_j} + \sum_{m=N}^{\infty}m^{-2} < \infty,$$

hence $x \in \big(\Delta l_\infty(p)\big)^{|\dagger|}$, and

$$\sum_{k=1}^{\infty}|a_k x_k| = \sum_{N=2}^{\infty}1 = \infty,$$

hence $a \notin \big(\Delta l_\infty(p)\big)^{|\dagger\dagger|}$.

(c) Let $a \in D_0^{(1)}(p)$. Since $|a_k| \leq |a_k|N^{1/p_1}\sum_{j=1}^{k-1}N^{-1/p_j}$ $(k = 2,3,\dots)$, we have $\sum_{k=1}^{\infty}|a_k| < \infty$. Let $x \in \Delta c_0(p)$. Then there is an integer k_0 such that $\sup_{k>k_0}|\Delta x_k|^{p_k} \leq N^{-1}$, where N is the number in $D_0^{(1)}(p)$. We put $M := \max_{1\leq k\leq k_0}|\Delta x_k|^{p_k}$, $m := \min_{1\leq k\leq k_0}p_k$, $L := (M+1)N$ and define the sequence y by $y_k := x_k L^{-1/m}$ $(k = 1,2,\dots)$. Then it is easy to see that $\sup_k|\Delta y|^{p_k} \leq N^{-1}$, and as in (2.1) with N replaced by N^{-1}, we have

$$\sum_{k=1}^{\infty}|a_k x_k| = L^{1/m}\sum_{k=1}^{\infty}|a_k y_k| < \infty.$$

Conversely, let $a \notin D_0^{(1)}(p)$. Then we can determine a strictly increasing sequence $\big(k(s)\big)$ of integers such that $k(1) := 1$ and

$$M_s := \sum_{k=k(s)}^{k(s+1)-1}|a_k|\sum_{j=1}^{k-1}(s+1)^{-1/p_j} > 1 \quad (s = 1,2,\dots).$$

We define the sequence x by

$$x_k := \sum_{l=1}^{s-1}\sum_{j=k(l)}^{k(l+1)-1}(l+1)^{-1/p_j} + \sum_{j=k(s)}^{k-1}(s+1)^{-1/p_j}$$
$$\big(k(s) \leq k \leq k(s+1)-1;\ s = 1,2,\dots\big).$$

Then it is easy to see that $|\Delta x_k|^{p_k} = 1/(s+1)$ $\big(k(s) \leq k \leq k(s+1)-1;\ s = 1,2,\dots\big)$ hence $x \in \Delta c_0(p)$, and $\sum_{k=1}^{\infty}|a_k x_k| \geq \sum_{s=1}^{\infty}1 = \infty$, i.e. $a \notin \big(\Delta c_0(p)\big)^{|\dagger|}$.

(d) For $N = 2, 3, \dots$, we put

$$E_N := \Big\{ a \in \omega : \sum_{k=1}^{\infty} |a_k| \sum_{j=1}^{k-1} N^{-1/p_j} < \infty \Big\}$$

$$F_N := \Big\{ a \in \omega : \sup_{k \geq 2} |a_k| \Big[\sum_{j=1}^{k-1} N^{-1/p_j} \Big]^{-1} < \infty \Big\}.$$

By a well known result (cf. [**3**, Lemma 4 (iv)]), we have to show $F_N = E_N^{|\dagger|}$ $(N = 2, 3, \dots)$. The proof of this is standard and therefore omitted.

Now we shall give some new results:

Theorem 2.2. *For every strictly positive sequence $p = (p_k)$, we have*

(a) $\big(\Delta c(p)\big)^{|\dagger|} = D^{(1)}(p) := D_0^{(1)} \cap \{a \in \omega : \sum_{k=1}^{\infty} |a_k| k < \infty\}$ *and*

(b) $\big(\Delta l_\infty(p)\big)^{\dagger} = D_\infty(p)$

$$:= \bigcap_{N=2}^{\infty} \Big\{ a \in \omega : \sum_{k=1}^{\infty} a_k \sum_{j=1}^{k-1} N^{1/p_j} \text{ converges and } \sum_{k=1}^{\infty} N^{1/p_k} |R_k| < \infty \Big\},$$

where $R_k := \sum_{\nu=k+1}^{\infty} a_\nu$ $(k = 1, 2, \dots)$.

Proof: (a) Let $a \in D^{(1)}(p)$ and $x \in \Delta c(p)$. Then there is a complex number l such that $|\Delta x_k - l|^{p_k} \to 0$ $(k \to \infty)$. We define y by $y_k := x_k + lk$ $(k = 1, 2, \dots)$. Then $y \in \Delta c_0(p)$ and

$$\sum_{k=1}^{\infty} |a_k x_k| \leq \sum_{k=1}^{\infty} |a_k| \Big| \sum_{j=1}^{k-1} \Delta y_j \Big| + |l| \sum_{k=1}^{\infty} |a_k| k < \infty$$

by Theorem 2.1.(c) and since $a \in D^{(1)}(p)$. Now let $a \in \big(\Delta c(p)\big)^{|\dagger|} \subset \big(\Delta c_0(p)\big)^{|\dagger|} = D_0^{(1)}(p)$ by Theorem 2.1.(c). Since the sequence x defined by $x_k := k$ $(k = 1, 2, \dots)$ is in $\Delta c(p)$ we have $\sum_{k=1}^{\infty} |a_k| k < \infty$.

(b) Let $a \in D_\infty(p)$ and $x \in \Delta l_\infty(p)$. Then there is an integer $N > \max\{1, \sup_k |\Delta x_k|^{p_k}\}$. We have

$$\sum_{k=1}^{n} a_k x_k = -\sum_{j=1}^{n-1} \Delta x_j R_j + R_n \sum_{j=1}^{n-1} \Delta x_j + x_1 \sum_{k=1}^{n} a_k \quad (n = 1, 2, \dots). \tag{2.2}$$

Obviously the last term on the right in (2.2) is convergent. Since $\sum_{j=1}^{\infty} |\Delta x_j| \times |R_j| \leq \sum_{j=1}^{\infty} N^{1/p_j} |R_j| < \infty$, the first term on the right in (2.2) is absolutely convergent. Finally by Corollary 2 in [**4**], the convergence of $\sum_{k=1}^{\infty} a_k \sum_{j=1}^{k-1} N^{1/p_j}$ implies $\lim_{n\to\infty} R_n \sum_{j=1}^{k-1} N^{1/p_j} = 0$. Conversely let $a \in \big(\Delta l_\infty(p)\big)^{\dagger}$. Since $e := (1, 1, \dots) \in \Delta l_\infty(p)$ and $x = \Big[\sum_{j=1}^{k-1} N^{1/p_j}\Big] \in \Delta l_\infty(p)$, we conclude the convergence of $\sum_{k=1}^\infty a_k$ and $\sum_{k=1}^\infty a_k \sum_{j=1}^{k-1} N^{1/p_j}$ respectively. Applying Corollary 2 in [**4**] again we have

$$\lim_{n\to\infty} R_n \sum_{j=1}^{k-1} N^{1/p_j} = 0.$$

From (2.2), we obtain the convergence of $\sum_{k=1}^\infty \Delta x_k R_k$ for all $x \in \Delta l_\infty(p)$. Since $x \in \Delta l_\infty(p)$ if and only if $y := \Delta x \in l_\infty(p)$, this implies $R \in l_\infty^\dagger(p)$, hence $\sum_{k=1}^\infty N^{1/p_k}|R_k| < \infty$ for all $N > 1$ by a well known theorem (cf. [**2**, Th. 2]).

3. Some matrix transformations

For any complex matrix $A = (a_{nk})$, we shall write $A_n := (a_{nk})_k$ for the sequence in the n-th row of A. Given A we define the matrix B by

$$b_{nk} := a_{nk} - a_{n+1,k} \quad (n, k = 1, 2, \dots).$$

Let X, Y be two subsets of ω. By (X, Y) we denote the class of all matrices A such that the series $A_n x := \sum_{k=1}^\infty a_{nk} x_k$ converge for all $x \in X$ $(n = 1, 2, \dots)$ and the sequence $Ax := (A_n x)$ is in Y for all $x \in X$.

The following is obvious and therefore stated without proof:

Lemma 3.1. *Let X, Y be linear sequence spaces. We put $\Delta Y := \{y \in \omega : \Delta y \in Y\}$. Then $A \in (X, \Delta Y)$ if and only if $B \in (X, Y)$ and $A_1 \in X^\dagger$.*

Lemma 3.1 and well known results together yield for instance the characterization of the following classes for strictly positive sequences $q \in l_\infty$: $\big(l(p), \Delta l_\infty(q)\big)$, $\big(l(p), \Delta c_0(q)\big)$, $\big(l(p), \Delta c(q)\big)$, (cf. [**5**, Th. 5 (i), (ii) and (iii)] if $0 < p_k \le 1$ $(k = 1, 2, \dots)$, [**5**, Th. 8 and Th. 9] if $1 < p_k \le H < \infty$ $(k = 1, 2, \dots)$). Now we give a characterization for the class $\big(\Delta l_\infty(p), l_\infty\big)$:

Theorem 3.1. *For every strictly positive sequence p, we have $A \in \big(\Delta l_\infty(p), l_\infty\big)$ if and only if the following three conditions hold:*

$$\text{(i)} \quad M_1(N) := \sup_n \left| \sum_{k=1}^\infty a_{nk} \sum_{j=1}^{k-1} N^{1/p_j} \right| < \infty \quad \textit{for all } N > 1,$$

$$\text{(ii)} \quad M_2(N) := \sup_n \left[\sum_{\nu=1}^\infty N^{1/p_\nu} \left| \sum_{k=\nu+1}^\infty a_{nk} \right| \right] < \infty \quad \textit{for all } N > 1,$$

$$\text{(iii)} \quad M_3 := \sup_n \left| \sum_{k=1}^\infty a_{nk} \right| < \infty.$$

Proof: Let conditions (i), (ii) and (iii) be satisfied. Then $A_n \in \big(\Delta l_\infty(p)\big)^\dagger$ $(n = 1, 2, \dots)$ by Theorem 2.2.(b). Hence the series $A_n x$ converge for all $x \in$

$\Delta l_\infty(p)$ $(n = 1, 2, \dots)$. Further as in the proof of Theorem 2.2.(b), we have for $x \in \Delta l_\infty(p)$ such that $\sup_k |\Delta x_k|^{p_k} < N$:

$$\left|\sum_{k=1}^{\infty} a_{nk}x_k\right| \le \sum_{\nu=1}^{\infty} N^{1/p_\nu}\left|\sum_{k=\nu+1}^{\infty} a_{nk}\right| + |x_1|\left|\sum_{k=1}^{\infty} a_{nk}\right| \le M_2(N) + |x_1|M_3 \quad (n = 1, 2, \dots),$$

hence $Ax \in l_\infty$.

Conversely let $A \in \big(\Delta l_\infty(p), l_\infty\big)$. The necessity of conditions (i) and (iii) follows from the fact that $(x_k) := \left[\sum_{j=1}^{k-1} N^{1/p_j}\right]$ and e are in $\Delta l_\infty(p)$. In order to show the necessity of condition (ii), we assume that $M_2(N) = \infty$ for some $N > 1$.

Then for the matrix C defined by

$$c_{n\nu} := \sum_{k=\nu+1}^{\infty} a_{nk} \quad (n, \nu = 1, 2, \dots),$$

we have $C \notin \big(l_\infty(p), l_\infty\big)$. (cf. [**2**, Th. 3]) Hence there is a sequence $x \in l_\infty(p)$ such that $\sup_\nu |x_\nu|^{p_\nu} = 1$ and $\sum_{\nu=1}^{\infty} c_{n\nu}x_\nu \ne O(1)$. We define the sequence y by $y_\nu := -\sum_{j=1}^{\nu-1} x_j + x_1$ $(\nu = 1, 2, \dots)$. Then $y \in \Delta l_\infty(p)$ and $\sum_{\nu=1}^{\infty} a_{n\nu}y_\nu = \sum_{\nu=1}^{\infty} c_{n\nu}x_\nu + x_1 \sum_{\nu=1}^{\infty} a_{n\nu} \ne 0(1)$, a contradiction to the assumption $A \in \big(\Delta l_\infty(p), l_\infty\big)$. Therefore we must have $M_2(N) < \infty$ for all $N > 1$.

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