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DETERMINISTIC AND RANDOM VOLTERRA INTEGRAL INCLUSIONS

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Abstract. We establish the existence of solutions for a nonlinear Volterra integral inclusion, involving a nonconvex valued orientor field and defined in a separable Banach space. Next we consider a random version of it and prove the existence of random solutions. Finally we examine a perturbed version of the original inclusion, with the pertubation being multivalued. Our results extend earlier ones by Chuong, Ragimkhanov, Lyapin, Milton-Tsokos, Papageorgiou amd Tsokos.

1. Introduction

In the recent years, the study of multivalued equations has received considerable attention, in particular in conection with problems in applied mathematics (like control theory, mathematical economics, mechanics etc.) and many mathematicians have contributed interesting results, mostly in the direction of differential inclusions.

This paper is devoted in the study of integral inclusions in Banach spaces. Integral inclusions, as well as differential inclusions, arise naturally in control theory, when we deparametrize the problem and in feedback systems (see Aubin-Cellina [**1**]). Another interesting application of integral inclusions can be found in the works of Glashoff-Sprekels [**7**], [**8**], who considered problems related to thermostatic regulation, in which the heating devices controlling the temperature of the system are governed by a relay switch.

In this work we establish the existence of solutions for a large class of integral inclusions of Volterra type, defined in a separable Banach space and involving a nonconvex valued orientor field. The we examine a random version of that inclusion, establishing the existence of random solutions. Finally we consider a perturbed version of the original inclusion, allowing the pertubation to be in general multivalued.

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Our work extends the single valued results of Szufla [**19**], Vaughn [**22**] (deterministic case), Bharucha Reid [**2**], Tsokos-Padgett [**21**], Milton-Tsokos [**11**] and Tsokos [**20**] (random case), as well as the multivalued results of Ragimkhanov [**18**], Lyapin [**10**] (finite dimensional results) and Chuong [**3**] Papageorgiou [**17**] (infinite dimensional results).

2. Preliminaries

Let (Ω, Σ) be a measurable space and X a separable Banach space. We will be using the following notations:

$$P_{f(c)}(X) = \{A \subseteq X\colon \text{nonempty, closed, (convex)}\}, \quad \text{and}$$
$$P_{(w)k(c)}(X) = \{A \subseteq X\colon \text{nonempty, } (w\text{-}) \text{ compact, (convex)}\}.$$

A multifunction $F\colon \Omega \to P_f(X)$ be said to be *measurable* if it satisfies any of the following two equivalent statements:

(i) for all $x \in X$, $\omega \to d(x, F(\omega)) = \inf\{\|x - z\|\colon z \in F(\omega)\}$ is measurable,

(ii) there exist $\{f_n\}_{n\geq 1}$ measurable selectors of $F(\cdot)$ s.t. for all $\omega \in \Omega$ $F(\omega) = \mathrm{cl}\{f_n(\omega)\}_{n\geq 1}$.

If Σ admits a σ-finite measure $\mu(\cdot)$, with respect to which is complete, then statements (i) and (ii) above are equivalent to

(iii) $\mathrm{Gr}\, F = \{(\omega, x) \in \Omega \times X\colon x \in F(\omega)\} \in \Sigma \times B(X)$, $B(X)$ being the Borel σ-field of X (graph measurability).

By S_F^1 we will denote the set of integrable selectors of $F(\cdot)$ i.e. $S_F^1 = \{f \in L^1(X)\colon f(\omega) \in F(\omega)\ \mu\text{-a.e.}\}$. This set is nonempty if and only if $\omega \to \inf\{\|z\|\colon z \in F(\omega)\} \in L_+^1$. Using this set we can define a set valued integral for $F(\cdot)$ by setting

$$\int_\Omega F(\omega)\, d\mu(\omega) = \left\{ \int_\Omega f(\omega)\, d\mu(\omega)\colon f \in S_F^1 \right\}.$$

The vector valued integrals involved in this definition are defined in the sense of Bochner. We say that $F(\cdot)$ is integrably bounded if and only if $F(\cdot)$ is measurable and $\omega \to |F(\omega)| = \sup\{\|z\|\colon z \in F(\omega)\} \in L_+^1$. It is clear that in this case $S_F^1 \neq \varnothing$.

Let Y, Z be Hausdorff topological spaces and $F\colon Y \to 2^Z \setminus \{\varnothing\}$. We say that $F(\cdot)$ is *upper semicontinuous* (*u.s.c.*) (resp. *lower semicontinuous* (*l.s.c.*)) if and only if for all $V \subseteq Z$ open, $F^+(V) = \{y \in Y\colon F(y) \subseteq V\}$ (resp. $F^-(V) = \{y \in Y\colon F(y) \cap V \neq \varnothing\}$) is open in Y. If Z is a metric space, on $P_f(Z)$ we can define a generalized metric by setting $h(A, B) = \max[\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)]$. This metric is the well known *Hausdorff metric*. If Z is complete, then so is $(P_f(Z), h)$. A multifunction $F\colon Y \to P_f(Z)$ is *Hausdorff continuous* (*h-continuous*), if it is continuous from the topological space Y into the metric space $(P_f(Z), h)$.

Also by $\beta(\cdot)$ we will denote the *Hausdorff* (*ball*) *measure of noncompactness* defined by

$$\beta(A) = \inf\{r > 0\colon A \text{ can be covered by finitely many balls of radius } r\}$$

($A \subseteq X$ bounded). By $\alpha(\cdot)$ we will denote the *Kuratowski* (*diameter*) *measure of noncompactness* defined by

$$\alpha(A) = \inf\{d > 0 \colon A \text{ admits a finite cover by sets of diameter } \leq d\}.$$

It is clear from these definitions that for all $A \subseteq X$ bounded

$$\beta(A) \leq \alpha(A) \leq 2\beta(A).$$

Finally recall that by a *Kamke function*, we will mean a function $w \colon T \times \mathbf{R}_+ \to \mathbf{R}_+$ satisfying the *Caratheodory conditions* (i.e. $t \to w(t,r)$ is measurable and $r \to w(t,r)$ is continuous increasing, $w(t,r) \leq \phi(t)$ a.e. with $\phi(\cdot) \in L^1_+$, $w(t,0) = 0$ and $w \equiv 0$ is the only solution of $r(t) \leq \int_0^t w(s, r(s))\, ds$, $t \in T$, $r(0) = 0$.)

3. Volterra integral inclusions

Let $T = [0,b]$, $\Delta = \{(t,s) \in T \times T \colon 0 \leq s \leq t \leq b\}$, X a separable Banach space and $\mathcal{L}(X)$ the space of bounded linear operators from X into itself.

In this section we examine the existence of solutions for the Volterra integral inclusion of the form:

$$x(t) \in p(t) + \int_0^t K(t,s) F(s, x(s))\, ds. \tag{$*$}$$

By a solution of $(*)$ we understand a continuous function $x \colon T \to X$ s.t. $x(t) = p(t) + \int_0^t K(t,s) f(s)\, ds$, $t \in T$, $f(\cdot) \in S^1_{F(\cdot, x(\cdot))}$.

We have the following existence result.

Theorem 3.1. *If* (1) $F \colon T \times X \to P_f(X)$ *is a multifunction s.t.*

- (1a) $(t,x) \to F(t,x)$ *is graph measurable and* $|F(t,x)| \leq \psi_1(t) + \psi_2(t)\|x\|$ *a.e. with* $\psi_1(\cdot), \psi_2(\cdot) \in L^1_+$,
- (1b) $x \to F(t,x)$ *is l.s.c.,*
- (1c) $\beta(F(t,B)) \leq w(t,B)$ *a.e. for all* $B \subseteq X$ *bounded and with* $2Lw(\cdot,\cdot)$ *a Kamke function for some* $L > 0$,

(2) $K \colon \Delta \to \mathcal{L}(X)$ *is a map s.t.*

- (2a) $\|K(t,s)\| \leq L$ *for all* $(t,s) \in \Delta$,
- (2b) $\lim_{t'-t \to 0} \big[\int_t^{t'} \|K(t',s)\| \psi(s)\, ds + \int_0^t \|K(t',s) - K(t,s)\| \psi(s)\, ds\big] = 0$ *with* $\psi(t) = \psi_1(t) + \psi_2(t) M$ *for some* $M > 0$.

(3) $p(\cdot) \in C(T,X)$,

then $(*)$ *admits a solution.*

Proof. First we obtain an a priori bound for the solutions of $(*)$. So let $x(\cdot)$ be such a solution. We have:

$$\|x(t)\| \leq \|p\|_\infty + \int_0^t L|F(s,x(s))|\,ds \leq \|p\|_\infty + \int_0^t L(\psi_1(s)+\psi_2(s)\|x(s)\|)\,ds$$
$$\leq \|p\|_\infty + L\|\psi_1\|_1 + \int_0^t \psi_2(s)\|x(s)\|\,ds.$$

Applying Gronwall's inequality, we get that

$$\|x(t)\| \leq (\|p\|_\infty + L\|\psi_1\|_1)\exp\|\psi_2\|_1 = M.$$

Next define

$$\widehat{F}(t,x) = \begin{cases} F(t,x) & \text{if } \|x\| \leq M \\ F(t, Mx/\|M\|) & \text{if } \|x\| > M. \end{cases}$$

Note that $\widehat{F}(t,x) = F(t,p_M(x))$, where $p_M(\cdot)$ is the M-radial retraction. Let $\eta: T\times X\times X \to T\times X\times X$ be defined by $\eta(t,x,y) = (t,p_M(x),y)$. Clearly $\eta(\cdot,\cdot,\cdot)$ is measurable. So $\eta^{-1}(\operatorname{Gr} F) = \{(t,x,y)\in T\times X\times X: \eta(t,x,y)\in \operatorname{Gr} F\} \in \Sigma\times B(X)\times B(X)$. But $\eta^{-1}(\operatorname{Gr} F) = \{(t,x,y): (t,p_M(x),y)\in \operatorname{Gr} F\} = \{(t,x,y): y\in F(t,p_M(x)) = \widehat{F}(t,x)\} = \operatorname{Gr}\widehat{F}$. So $\widehat{F}(\cdot,\cdot)$ is graph measurable. Also recalling that $p_M(\cdot)$ is Lipschitz, we immediately have that $\widehat{F}(t,\cdot)$ is l.s.c. Furthermore for every $B\subseteq X$ bounded we have

$$\beta(\widehat{F}(t,B)) = \beta(F(t,p_M(B))) \leq w(t,\beta(p_M(B))).$$

But $\beta(p_M(B)) \leq \beta(\overline{\operatorname{conv}}(B\cup\{0\})) = \beta(B)$. Thus we get

$$\beta(\widehat{F}(t,B)) \leq w(t,\beta(B)).$$

Finally observe that

$$|\widehat{F}(t,x)| \leq \psi_1(t)+\psi_2(t)M = \psi(t) \text{ for a.e. } \psi(\cdot)\in L^1_+.$$

So $\widehat{F}(\cdot,\cdot)$ has same properties as $F(\cdot,\cdot)$ and in addition is integrably bounded in t, uniformly for $x\in X$. Next we will solve $(*)$ using the orientor field $\widehat{F}(t,x)$ instead.

Let

$$W_0 = \left\{ y\in C(T,X): y(t) = p(t) + \int_0^t K(t,s)u(s)\,ds,\ t\in T,\ \|u(t)\|\leq\psi(t) \text{ a.e.}\right\}.$$

Clearly this is a nonempty, bounded and because of hypothesis (2b), equicontinuous subset of $C(T,X)$. Then define

$$W_1 = \left\{ y\in C(T,X): y(t) = p(t) + \int_0^t K(t,s)u(s)\,ds,\ t\in T,\ u(\cdot)\in S^1_{\overline{\operatorname{conv}}\widehat{F}(\cdot,W_0(\cdot))}\right\}.$$

We claim that $W_1 \neq \varnothing$. To see this let $y(\cdot) \in W_0$. Then since $\widehat{F}(\cdot,\cdot)$ is graph measurable, $t \to \widehat{F}(t, y(t))$ is measurable and moreover integrably bounded. Hence $\varnothing \neq S^1_{\widehat{F}(\cdot, y(\cdot))} \subseteq S^1_{\overline{\text{conv}}\widehat{F}(\cdot, W_0(\cdot))}$ and this clearly implies that $W_1 \neq \varnothing$. Also note that if $y \in W_1$, then

$$y(t) = p(t) + \int_0^t K(t,s)u(s)\,ds, \quad t \in T, \quad u \in S^1_{\overline{\text{conv}}\widehat{F}(\cdot, W_0(\cdot))}.$$

Hence $\|u(t)\| \leq |\overline{\text{conv}}\widehat{F}(t, W_0(t))| = |\widehat{F}(t, W_0(t)| \leq \psi(t)$ a.e. $\Rightarrow$ $u(\cdot) \in W_0$ $\Rightarrow$ $W_1 \subseteq W_0$.

Next define

$$W_2 = \Big\{ y \in C(T,X)\colon y(t) = p(t) + \int_0^t K(t,s)u(s)\,ds,\ t \in T,\ u \in S^1_{\overline{\text{conv}}\widehat{F}(\cdot, W_1(\cdot))} \Big\}.$$

Note that $\widehat{F}(t, W_1(t)) \subseteq \widehat{F}(t, W_2(t))$ a.e. and so as before $\varnothing \neq W_2 \subseteq W_1 \subseteq W_0$. Continuing this way we produce a decreasing sequence $\{W_n\}_{n \geq 1}$ of nonempty, closed, bounded and equicontinuous subsets of $C(T,X)$.

Now we will determine $\beta(W_n)$. Let $\{y^n_m\}_{m\geq 1}$ be dense in W_n. We have:

$$\beta(W_n) = \beta[y^n_m\colon m \geq 1] \leq \alpha[y^n_m : m \geq 1].$$

But from Ambrosetti's theorem (see Deimling [**5**]), we know that:

$$\alpha\,[y^n_m\colon m \geq 1] = \sup_{t\in T} \alpha\,[y^n_m(t)\colon m \geq 1]$$

Also recall that

$$\begin{aligned}\alpha\,[y^n_m(t)\colon m \geq 1] &\leq 2\beta\,[y^n_m(t)\colon m \geq 1] \\ &= 2B\left[\int_0^y K(t,s)u^n_m(s)\,ds\colon m \geq 1\right].\end{aligned}$$

Using Monch's theorem [**12**] (proposition 1.6) (for an extension see also Orlicz--Szufla [**13**]), we get that

$$\begin{aligned}2\beta\left[\int_0^t K(t,s)u^n_m(s)\,ds\colon\, m \geq 1\right] &\leq \int_0^t 2\beta\left(K(t,s)u^n_m(s)\colon\, m \geq 1\right)\,ds \\ &\leq \int_0^t 2\beta\big(K(t,s)\widehat{F}(s, W_{n-1}(s))\big)\,ds \\ &\leq \int_0^t 2Lw\left(s, \beta\left(W_{n-1}(s)\right)\right)\,ds.\end{aligned}$$

Set $h_n(t) = \beta\left(W_n(t)\right),\, n \geq 1,\, t \in T$. We have

$$h(t) \leq \int_0^t 2Lw\big(s, h_{n-1}(s)\big)\, ds.$$

Note that $h_n \leq \beta(W_0(t)) \leq \|p\|_\infty + \int_0^t \psi(s)\, ds \Rightarrow \{h_n\}_{n\geq 1}$ is equicontinuous and bounded. Thus applying the Arzela-Ascoli theorem, we may assume that $h_n \to h$ in $C(T)$. Therefore in the limit as $n \to \infty$, we have:

$$h(t) \leq \int_0^t 2Lw\big(s, h(s)\big)\, ds, \quad t \in T.$$

Because $h(0) = 0$ and recalling that $2Lw(\cdot,\cdot)$ is a Kamke function, we conclude that $h(t) = 0,\ t \in T$. But recall that

$$\beta(W_n) \leq \sup_{t\in T} \alpha\big(W_n(t)\big) \leq \sup_{t\in T} 2\beta\big(W_n(t)\big) = 2 \sup_{t\in T} h_n(t) \Rightarrow \beta(W_n) \to 0.$$

Set $W = \bigcap_{n\geq 1} W_n$. Then by Kuratowski's theorem (see Deimling [**5**], p. 42) W is nonempty and compact. We will now show that the functions in W have an integral representation.

Let $y \in W$. Then $y \in W_n$ for all $n \geq 1$ and so

$$y(t) = p(t) + \int_0^t K(t,s) u_n(s)\, ds, \quad t \in T, \quad u_n \in S^1_{\overline{\text{conv}}\widehat{F}(\cdot, W_{n-1}(\cdot))}, \quad n \geq 1.$$

So $u_n(t) \in \overline{\text{conv}}\widehat{F}\big(t, W_{n-1}(t)\big)$ a.e. $n \geq 1$ and $u_m(t) \in \overline{\text{conv}}\widehat{F}\big(t, W_{n-1}(t)\big)$ a.e. $m \geq n$. Hence we can write that

$$\beta\Bigg(\bigcup_{m\geq n} u_n(t)\Bigg) \leq \beta(\widehat{F}(t, W_{n-1}(t))) \leq w(t, h_{n-1}(t)) \quad \text{a.e.}$$

Since $w(t, h_{n-1}(t)) \to 0$ as $n \to \infty$ and since the measure of noncompactness of a finite number of points is clearly zero, we get that

$$\beta\Bigg(\bigcup_{m\geq 1} u_m(t)\Bigg) \leq \varepsilon \quad \text{for all } \varepsilon > 0$$

$$\Rightarrow \overline{\bigcup_{m\geq 1} u_m(t)} \in P_k(X) \ \Rightarrow\ G(t) = \overline{\text{conv}} \bigcup_{m\geq 1} u_m(t) \in P_{kc}(X).$$

and clearly $G(\cdot)$ is integrably bounded. So from [**14**], we deduce that S^1_G is w-compact in $L^1(X)$ and from the Eberlein-Smulian theorem is sequentially w-compact. So by passing to a subsequence if necessary, we may assume that $u_n \xrightarrow{w} u$ in $L^1(X)$. Hence for all $t \in T$

$$y(t) = p(t) + \int_0^t K(t,s) u_n(s)\, ds \xrightarrow{w} p(t) + \int_0^t K(t,s) u(s)\, ds.$$

So $y(t) = p(t) + \int_0^t K(t,s)u(s)\,ds$, which proves that

$$W = \Big[y \in C(T,X) : y(t) = p(t) + \int_0^t K(t,s)u(s)\,ds,\ t \in T,\ u(\cdot) \in \bigcap_{n \geq 1} S^1_{\overline{\text{conv}}\widehat{F}(\cdot, W_n(\cdot))} \Big].$$

Next define $R: W \to P_f(L^1(X))$ by $R(x) = S^1_{\widehat{F}(\cdot, x(\cdot))}$. Since $\widehat{F}(t,\cdot)$ is l.s.c., from theorem 4.1 of [**16**], we get that $R(\cdot)$ is l.s.c. too. So applying Fryszkowski's selection theorem [**6**], we can find $r: W \to L^1(X)$ continuous map s.t. $r(x) \in R(x)$ for all $x \in W$. Set $v(x)(t) = p(t) + \int_0^t K(t,s)r(x)(s)\,ds$. Clearly from what we proved for the elements of W, $v(x) \in W$. So $v: W \to W$ and is continuous. Apply Schauder's fixed point theorem to get $\widehat{x} \in W$ s.t. $v(\widehat{x}) = \widehat{x}$. It is easy to see that $\widehat{x}(\cdot)$ is a solution of $(*)$ with orientor field $\widehat{F}(\cdot,\cdot)$. But note that $\|\widehat{x}(t)\| \leq |\widehat{F}(t,\widehat{x}(t))| \leq \psi_1(t) + \psi_2(t)\|\widehat{x}(t)\|$ a.e. and as in the beginning of the proof, through Gronwall's inequality, we get $\|\widehat{x}(t)\| \leq M \Rightarrow \widehat{F}(t,\widehat{x}(t)) = F(t,\widehat{x}(t)) \Rightarrow \widehat{x}(\cdot)$ is the desired solution of $(*)$. QED

4. Random integral inclusions

In this section let (Ω, Σ, μ) be a complete probability space, $T = [0,b]$ and X a finite dimensional Banach space. We will examine the following random Volterra integral inclusion:

$$x(\omega,t) \in p(\omega,t) + \int_0^t K(\omega,t,s)F(\omega,s,x(\omega,s))\,ds. \qquad (**)$$

By a *random solution* of $(**)$ we understand a stochastic process $x: \Omega \times T \to X$ with continuous paths s.t. for every $\omega \in \Omega$, $x(\omega,\cdot)$ is a solution of the corresponding deterministic problem.

Our existence theorem extend the works of Milton-Tsokos [**11**] (theorem 3.1), Tsokos [**20**] (theorem 3.1) and Bharucha Reid [**2**] (chapter 6.4.B).

THEOREM 4.1. *If* (1) $F: \Omega \times T \times X \to P_{fc}(X)$ *is a multifunction s.t.*

(1a) $(\omega,t) \to F(\omega,t,x)$ *is measurable,*

(1b) $x \to F(\omega,t,x)$ *is h-continuous,*

(1c) $|F(\omega,t,x)| \leq \psi_1(\omega,t) + \psi_2(\omega,t)\|x\|$ *a.e. for all* $\omega \in \Omega$, *with* $\psi_1(\cdot,\cdot), \psi_2(\cdot,\cdot)$ *measurable and* $\psi_1(\omega,\cdot), \psi_2(\omega,\cdot) \in L^1_+$*;*

(2) $K : \Omega \times \Delta \to \mathcal{L}(X)$ *is a map s.t.*

(2a) $(\omega,t,s) \to K(\omega,t,s)$ *is measurable,*

(2b) $\|K(\omega,t,s)\| \leq L(\omega)$ *for all* $\omega \in \Omega$ *with* $L(\cdot)$ *measurable,* (2c) $\lim_{t'-t\to 0}\big[\int_t^{t'}\|K(\omega,t,s)\|\psi(\omega,s)\,ds+\int_0^t\|K(\omega,t',s)-K(\omega,t,s)\|\psi(\omega,s)\,ds\big]$ $=0$, *where* $\psi(\omega,t)=\psi_1(\omega,t)=\psi_2(\omega,t)M(\omega)$ *for some* $M:\Omega\to\mathbf{R}_+\setminus\{0\}$ *measurable, and*

(3) $p(\cdot,\cdot)$ *is a Caratheodory* X*-valued function,*

then $(**)$ *admits a random solution.*

Proof: First let $x(\cdot,\cdot)$ be a solution of $(**)$. Then through Gronwall's inequality, we get that $\|x(\omega,t)\|\le[\,\|p(\omega,\cdot)\|_\infty+L(\omega)\|\psi(\omega,\cdot)\|_1\,]\exp\big(L(\omega)\|\psi_2(\omega,\cdot)\|_1\big)=M(\omega)$. Define $\widehat{F}(\omega,t,x)$ as in the proof of theorem 3.1 and consider $(**)$ with $\widehat{F}(\cdot,\cdot,\cdot)$ being the orientor field. Consider $G\colon\Omega\times C(T,X)\to P_f\big(C(T,X)\big)$ defined by

$$G(\omega,x)=\Big\{y\in C(T,X)\colon y(t)=p(\omega,t)+\int_0^t K(\omega,t,s)f(s)\,ds,\ t\in T,\ f\in S^1_{\widehat{F}(\omega,\cdot,x(\cdot))}\Big\}.$$

We claim that $\omega\to G(\omega,x)$ is measurable. So let $f\in L^1(X)$ and $z\in C(T,X)$. Define $r\colon\Omega\times L^1(X)\to C(T,X)$ by

$$r(\omega,f)(t)=p(\omega,t)+\int_0^t K(\omega,t,s)f(s)\,ds,\quad t\in T.$$

Using the lemma in [**15**], we have that $r(\cdot,\cdot)$ is a Caratheodory function, hence jointly measurable. Also, again from [**15**], we know $\omega\to S^1_{\widehat{F}(\omega,\cdot,x(\cdot))}$ is measurable. So we can find $\widehat{f}\colon\Omega\to L^1(X)$, $n\ge 1$ measurable s.t. $\overline{\{\widehat{f}_n\}}_{n\ge1}=S^1_{\widehat{F}(\omega,\cdot,x(\cdot))}$. Then observe that

$$d\big(z,G(\omega,x)\big)=\inf_{n\ge1}\|z-r(\omega,f_n)\|_\infty\Rightarrow\ \omega\to d(z,G(\omega,x))\text{ is measurable,}$$

and thus the claim follows (see section 2).

Next we will show that $x\to G(\omega,x)$ is h-continuous. So let $x,x'\in C(T,X)$ and let $y'\in G(\omega,x')$. Then for any $f\in S^1_{\widehat{F}(\omega,\cdot,x(\cdot))}$ we have:

$$\begin{aligned}&\Big\|y'(t)-p(\omega,t)-\int_0^t K(\omega,t,s)f(s)\,ds\Big\|\\ &=\Big\|p(\omega,t)+\int_0^t K(\omega,t,s)f'(s)\,ds-p(\omega,t)-\int_0^t K(\omega,t,s)f(s)\,ds\Big\|\end{aligned}$$

for some $f'\in S^1_{\widehat{F}(\omega,\cdot,x'(\cdot))}$. So

$$\begin{aligned}
&\left\| y'(t) - p(\omega,t) - \int_0^t K(\omega,t,s) f(s)\,ds \right\| \\
&\leq \int_0^t \|K(\omega,t,s)\| \cdot \|f'(s) - f(s)\|\,ds \leq L(\omega) \int_0^t \|f'(s) - f(s)\|\,ds \\
&\Rightarrow\ \|y' - y\|_\infty \leq L(\omega)\|f' - f\|_1 \ \text{ for any } y \in G(\omega,x).
\end{aligned}$$

Hence

$$\begin{aligned}
&d(y', G(\omega,x)) \leq L(\omega) d\big(f', S^1_{\widehat{F}(\omega,\cdot,x(\cdot))}\big) \\
&\Rightarrow\ h^*(G(\omega,x'), G(\omega,x)) \leq L(\omega) h^*\big(S^1_{(\widehat{F}(\omega,\cdot.x'(\cdot))}, S^1_{\widehat{F}(\omega,\cdot,x(\cdot))}\big).
\end{aligned}$$

By interchanging the roles of x and x' we have

$$h^*(G(\omega,x), G(\omega,x')) \leq L(\omega) h^*\big(S^1_{\widehat{F}(\omega,\cdot,x(\cdot))}, S^1_{\widehat{F}(\omega,\cdot,x'(\cdot))}\big).$$

So finally we get

$$h(G(\omega,x'), G(\omega,x)) \leq L(\omega) h\big(S^1_{\widehat{F}(\omega,\cdot,x'(\cdot))}, S^1_{\widehat{F}(\omega,\cdot,x(\cdot))}\big).$$

Also from the proof of theorem 4.5 in [**16**] we know that

$$h\Big(S^1_{\widehat{F}(\omega,\cdot,x'(\cdot))}, S^1_{\widehat{F}(\omega,\cdot,x(\cdot))}\Big) = \int_0^b h\big(\widehat{F}(\omega,t,x'(t)), \widehat{F}(\omega,t,x(t))\big)\,dt.$$

From this through the dominated convergence theorem, we easily get that $G(\omega,\cdot)$ is indeed h-continuous. Now let $W(\omega) \subseteq C(T,X)$ be defined by

$$\begin{aligned}
W(\omega) = \Big\{ & z \in C(T,X) \colon z(t) = p(\omega,t) \\
& + \int_0^t K(\omega,t,s) g(s)\,ds,\ t \in T,\ \|g(t)\| \leq \psi(\omega,t) \text{ a.e.} \Big\},
\end{aligned}$$

with $\psi(\omega,t) = \psi_1(\omega,t) + \psi_2(\omega,t) M(\omega)$.

An easy application of the Arzela-Ascoli theorem tells us that for all $\omega \in \Omega$, $W(\omega)$ is compact in $C(T,X)$. Because of the a priori bound obtained in the beginning of the proof, we see that for all $(\omega,x) \in \Omega \times C(T,X)$, we have $G(\omega,x) \subseteq W(\omega)$. Next let $H(\omega) = \{x \in C(T,X) \colon x \in G(\omega,x)\}$. From the Kakutani-Ky Fan fixed point theorem, we deduce that $H(\omega) \neq \varnothing$ for all $\omega \in \Omega$. Also note that

$$\operatorname{Gr} H = \{(\omega,x) \in \Omega \times C(T,X) \colon d\big(x, G(\omega,x)\big) = 0\}.$$

From the properties of $G(\cdot,\cdot)$ proved above and since the distance function is Lipschitz, we have that $(\omega,x) \to d\big(x, G(\omega,x)\big)$ is Caratheodory, hence jointly measurable. Therefore $\operatorname{Gr} H \in \Sigma \times B\big(C(T,X)\big)$. Once again Aumann's selection theorem gives us $h \colon \Omega \to C(T,X)$ measurable s.t. $h(\omega) \in H(\omega)$ for all $\omega \in \Omega$. Set

$x(\omega,t) = h(\omega)(t)$. From the lemma in [**15**], we have that $x(\cdot,\cdot)$ is a Caratheodory process. Since $\|x(\omega,t)\| \le M(\omega)$ and recalling the definition of $\widehat{F}(\omega,t,x)$, we conclude that $x(\cdot,\cdot)$ is the desired random solution of $(**)$. QED

5. A perturbed Volterra integral inclusion

In this section we consider the following perturbed version of $(*)$.

$$x(t) \in G(t,x(t)) + \int_0^t K(t,s)F\big(s,x(s)\big)\,ds, \quad t \in T. \qquad (***)$$

By a *solution* of $(***)$, we understand a continuous function $x: T \to X$ s.t.

$$x(t) = g(t) + \int_0^t K(t,s)f(s)\,ds, \quad t \in T, \quad g \in S^1_{G(\cdot,x(\cdot))}, \quad f \in S^1_{F(\cdot,x(\cdot))}.$$

Again X is finite dimensional.

Theorem 5.1. *If (1) $G: T \times X \to P_{fc}(X)$ is a multifunction s.t.*

(1a) *$(t,x) \to G(t,x)$ is u.s.c.,*

(1b) *$|G(t,x)| \le L$,*

(2) *$F: T \times X \to P_{fc}(X)$ is another multifunction s.t.*

(2a) *$t \to F(t,x)$ is measurable,*

(2b) *$x \to F(t,x)$ is u.s.c.,*

(2c) *$|F(t,x)| \le \psi_1(t) + \psi_2(t)\|x\|$ a.e. with $\psi_1(\cdot), \psi_2(\cdot) \in L^1_+$, and*

(3) *$K: \Delta \to \mathcal{L}(X)$ is a map s.t.*

(3a) *$\lim_{t'-t\to 0}\big[\int_t^{t'} \|K(t',s)\|\psi(s)\,ds + \int_0^t \|K(t',s) - K(t,s)\|\psi(s)\,ds\big] = 0$, where $\psi(s) = \psi_1(s) = \psi_2(s)M$ for some $M > 0$,*

(3b) *$\|K(t,s)\| \le L'$,*

then $(***)$ admits a solution.

Proof: From De Blasi [**4**], proposition 4.1, we know that we can find h-continuous multifunctions $G_n: T \times X \to P_{fc}(X)$ s.t. $|G_n(t,x)| \le L$, $\ldots$ $G_{n+1}(t,x) \subseteq G_n(t,x) \subseteq G_{n-1}(t,x) \subseteq \cdots$ and $G_n(t,x) \xrightarrow{h} F(t,x)$ as $n \to \infty$ for all $(t,x) \in T \times X$. Applying Michael's selection theorem (see Aubin-Cellina [**1**]), we can find $g_n: T \times X \to X$ continuous functions s.t. $g_n(t,x) \in G_n(t,x)$. Let $u(\cdot) \in C(T,X)$ and consider the following integral inclusions:

$$x(t) \in g_n(t,u(t)) + \int_0^t K(t,s)F(s,x(s))\,ds. \qquad (***)^n_u$$

Denote the solution set of $(***)^n_u$ by $S_n(u)$. From theorem 3.1 (see also [**17**]), we know that $S_n(u) \ne \varnothing$ for all $n \ge 1$ and furthermore is compact in

$C(T,X)$. Consider the multifunctions $S_n: C(T,X) \to P_k(C(T,X))$ defined by $u \to S_n(u)$, $n \geq 1$. We claim that S_n is u.s.c. $n \geq 1$. To this end let $W = \{y \in C(T,X): y(t) = v + \int_0^t K(t,s)z(s)\,ds,\ t \in T,\ v \in B_L(0),\ \|z(s)\| \leq \psi_1(s) + \psi_2(s)M = \psi(s) \text{ a.e.}\}$, where M is the a priori bound for the solutions obtained in the beginning of the proof of theorem 3. 1 and $B_L(0)$ is the ball of radius L centered at the origin. An application of the Arzela-Ascoli theorem tells us that W is compact in $C(T,X)$. Note that for all $u(\cdot) \in C(T,X)$ and all $n \geq 1$, $S_n(u) \subseteq W$. So in order to show the upper semicontinuity of $S_n(\cdot)$, $n \geq 1$, it suffices to show that $\operatorname{Gr} S_n$, $n \geq 1$, is closed. Thus let $\{(u_m, x_m)\}_{m \geq 1} \subseteq \operatorname{Gr} S_n$ s.t. $(u_m, x_m) \to (u, x)$ in $C(T,X)$. We have

$$x_m(t) \in g_n\big(t, u_m(t)\big) + \int_0^t K(t,x)F\big(s, x_m(s)\big)\,ds, \quad m \geq 1, \quad t \in T.$$

We have $g_n(t, u_m(t)) \to g_n(t, u(t))$ as $m \to \infty$. Also from corollary 3, p. 632 of Łojasiewicz [**9**], we know that we can find an increasing sequence of closed sets $T_m \subseteq T$ s.t. $\lambda(T \setminus T_m) < 1/m$ and $F|_{T_m \times X}$ is u.s.c.. Define $T_0 = T \setminus \bigcup_{m \geq 1} T_m$, $A_1 = T_1$ and $A_m = T_m \setminus \bigcap_{k=1}^{m-1} T_k$. Clearly $\lambda(T_0) = 0$. Set $\widehat{F}(t,x) = \chi_{T_0}(t)\{0\} + \sum_{m \geq 1} \chi_{A_m}(t)F(t,x)$. Then $(t,x) \to \widehat{F}(t,x)$ is measurable, $x \to \widehat{F}(t,x)$ is u.s.c. and for all $(t,x) \in (T \setminus T_0) \times X$ we have $\widehat{F}(t,x) = F(t,x)$. Thus for all $t \in T$ and all $m \geq 1$: $\int_0^t K(t,s)F(s, x_m(s))\,ds = \int_0^t K(t,s)\widehat{F}(s, x_m(s))\,ds$. Next for $x^* \in X^*$ we have

$$\begin{aligned}
&\sup\left\{(x^*, z): z \in \int_0^t K(t,s)\widehat{F}\big(s, x_m(s)\big)\,ds\right\} \\
&= \sigma\left(x^*, \int_0^t K(t,s)\widehat{F}(s, x_m(s))\,ds\right) = \int_0^t \sigma\big(K(t,s)^* x^*, \widehat{F}(s, x_m(s))\big)\,ds.
\end{aligned}$$

Applying Fatou's lemma and recalling that $\sigma(K(t,s)^* x^*, \widehat{F}(s,\cdot))$ is u.s.c. since $\widehat{F}(s,\cdot)$ is, we get

$$\begin{aligned}
&\overline{\lim} \int_0^t \sigma\big(K(t,s)^* x^*, \widehat{F}(s, x_m(s))\big)\,ds \\
&\leq \int_0^t \overline{\lim}\, \sigma\big(K(t,s)^* x^*, \widehat{F}(s, x_m(s))\big)\,ds \leq \int_0^t \sigma\big(K(t,s)^* x^*, \widehat{F}(s, x(s))\big)\,ds \\
&\Rightarrow\ \overline{\lim}\, \sigma\left(x^*, \int_0^t K(t,s)\widehat{F}(s, x_m(s))\,ds\right) \leq \sigma\left(x^*, \int_0^t K(t,s)\widehat{F}(s, x(s))\,ds\right).
\end{aligned}$$

Invoking proposition 4.1 of [**16**] we get

$$\overline{\lim} \int_0^t K(t,s)\widehat{F}(s, x_m(s))\,ds \subseteq \int_0^t K(t,s)\widehat{F}(s, x(s))\,ds.$$

So in the limit as $m \to \infty$, we have

$$x(t) \in g_n\big(t, u(t)\big) + \int_0^t K(t,s)\widehat{F}\big(s, x(s)\big)\, ds, \quad t \in T$$
$$\Rightarrow \ (u,x) \in \operatorname{Gr} S_n \ \Rightarrow \operatorname{Gr} S_n \text{ is closed i.e. } S_n(\cdot) \text{ is u.s.c.}$$

Applying the Kakutani-Ky Fan fixed point theorem, we can find $u_n(\cdot) \in C(T,X)$ s.t. $u_n \in S(u_n)$. Then we have

$$u_n(t) \in G_n(t, u_n(t)) + \int_0^t K(t,s)\widehat{F}(s, u_n(s))\, ds, \quad t \in T, \quad n \geq 1.$$

Since $\{u_n\}_{n\geq 1} \subseteq W$ and the latter is compact in $C(T,X)$, by passing to a subsequence if necessary, we may assume that $u_n \to u$ in $C(T,X)$. Because the G_n's can be chosen to be locally h-Lipschitz (see remark 4.2 of DeBlasi [**4**]) we have $G_n(t, u_n(t)) \xrightarrow{h} G(t, u(t))$, while as above we can show that

$$\overline{\lim} \int_0^t K(t,s)\widehat{F}(s, u_n(s))\, ds \subseteq \int_0^t K(t,s)\widehat{F}(s, u(s))\, ds$$
$$\Rightarrow \ u(t) \in G(t, u(t)) + \int_0^t K(t,s)\widehat{F}(s, u(s))\, ds$$
$$\Rightarrow \ u(\cdot) \text{ is the desired solution of } (***). \text{ QED}$$

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