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ON THE TOPOLOGY OF RIEMANN SPACES OF QUASI-CONSTANT CURVATURE

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Abstract. We compute Betti numbers of certain Riemann spaces of q-constant curvature such as conformally flat hypersurfaces of elliptic-space-forms, subprojective spaces and para-Sasakian manifolds.

1. Harmonic forms on manifolds of quasi-constant curvature

Let (M^n, g_{ij}) be an $n-$dimensional space and V a unit vector field on M^n, with (local) components v^i. Let $v_i = g_{ij}v^j$; if (M^n, g_{ij}) is conformally flat and for some real valued smooth functions $a, b \in C^\infty(M^n)$ the curvature of (M^n, g_{ij}) is expressed by:

$$
\begin{aligned}
R^h_{kji} &= a\{\delta^h_k g_{ji} - \delta^h_j g_{ki}\} \\
&\quad + b\{(\delta^h_k v_j - \delta^h_j v_k)v_i + (v_k g_{ji} - v_j g_{ki})v^h\}
\end{aligned} \tag{1.1}
$$

then (M^n, g_{ij}) is said to be a Riemann space of q-constant curvature, cf. [**7**], [**8**]. We suppose throughout that $b \neq 0$, (for $b = 0$, M^n falls into nothing but a real space-form). Further contraction of indices in (1.1) gives the Ricci form:

$$
R_{ji} = \{(n-1)a + b\}g_{ji} + b(n-2)v_j v_i. \tag{1.2}
$$

If $\alpha_{i_1\dots i_p}$, $\beta_{j_1\dots j_p}$ are two p-forms on M^n, we put as usual

$$
\langle \alpha, \beta \rangle = \alpha^{(i_1\dots i_p)}\beta_{(i_1\dots i_p)}\,,
$$

$|\alpha|^2 = \langle \alpha, \alpha \rangle$. We proceed by establishing the following:

THEOREM 1. *Let (M^n, g_{ij}) be a compact orientable Riemann space of q-constant curvature. The following relation holds:*

$$
\begin{aligned}
&\int_{M^n} \{p[p!\{(n-p)a + b\}|\alpha|^2 + \\
&+(p-1)!(n-2p)b|i(V)\alpha|^2] + \nabla_j \alpha_{i_1\dots i_p}\nabla^j\alpha^{i_1\dots i_p}\} * 1 = 0
\end{aligned} \tag{1.3}
$$

AMS Subject Classification (1980): Primary 53 C 20, 53 C 40, 58 A 10, 58 A 14 *for each harmonic p-form α on M^n.*

Proof. Let $(p!)^{-1}\alpha_{i_1\ldots i_p}dx^{i_1}\wedge\ldots\wedge dx^{i_p}$ be a differential p-form on M^n. Note that:

$$v_i\alpha^{ii_2\ldots i_p} = (i(V)\alpha)^{i_2\ldots i_p}, \tag{1.4}$$

where $i(V)$ denotes the interior product with V. Taking into account (1.4), (1.2) one obtains:

$$R_{ji}\alpha^{ji_2\ldots i_p}\alpha^i_{i_2\ldots i_p} = p!\{(n-1)a+b\}|\alpha|^2 + (p-1)!(n-2)b\,|i(V)\alpha|^2. \tag{1.5}$$

Using $R_{ijks} = g_{ih}R^h_{ksj}$ and (1.1) one derives:

$$R_{ijkh}\alpha^{iji_3\ldots i_p}\alpha^{kh}_{i_3\cdots i_p} = 2ap!\,|\alpha|^2 + 4b(p-1)!\,|i(V)\alpha|^2 \tag{1.6}$$

By (3.2.9) in [**6**, p. 88], if $\Delta\alpha = 0$ then

$$\int_{M^n}\{pF_p(\alpha) + \nabla_j\alpha_{i_1\ldots i_p}\nabla^j\alpha^{i_1\ldots i_p}\} * 1 = 0, \tag{1.7}$$

where the quadratic from $F_p(\alpha)$ is given by (3.2.10) in [**6**]. Finally (1.5)–(1.6) furnish:

$$F_p(\alpha) = p!\{(n-p)a+b\}|\alpha|^2 + (p-1)!(n-2p)b|i(V)\alpha|^2 \tag{1.8}$$

and (1.7) leads to our (1.3), QED.

Betti numbers of conformally flat (compact, orientable) Riemann spaces are known to vanish (cf. Th. 3.9.1. in [**6**, p. 118]) provided that the Ricci curvature is positive definite. Yet, if M^n is a space of q-constant curvature, by (1.2) one has $R_{ji}v^jv^i = (n-1)(a+b)$, i.e. R_{ji} is degenerate along the distribution generated by V, provided $b = -a$. As an application of Theorem 1 we get

Theorem 2. *Let M^n be a compact orientable connected Riemann space of q-constant curvature, $n > 2$. If $a =$const.> 0 and $db = V(b)v$ then M^n has the homology type of $S^1 \times S^{n-1}$.*

Proof. By a result of Wang and Adati, i.e. Th. 2.3. in [**8**, p. 101], if $a =$const. and $b_j = b_iv^iv_j$, $b_i = \nabla_ib$, then $b = -a$ and V is parallel. Therefore, since $a > 0$, the coefficients in (1.8) are subject to $(n-p)a+b > 0$, $(n-2p)b \geq 0$ iff $n/2 \leq p < n-1$. If n is even, i.e. $n = 2m$, we have $b_p(M^n) = 0$ for all $m \leq p < 2m-1$. By Poincaré duality one obtains also $b_p(M^n) = 0$, $1 < p \leq m$. A similar argument for n odd shows that all Betti numbers of M^n vanish except for $b_1(M^n) = b_{n-1}(M^n)$. Since v_i is parallel, it is harmonic. Thus $b_1(M^n) \geq 1$. Let β be any other harmonic l-form on M^n. By applying the Hodge operator one obtains a harmonic $(n-1)$-form $*\beta$. At this point we may use (1.8) to get

$$F_{n-1}(*\beta) = (n-1)!(a+b)|*\beta|^2 - (n-2)!(n-2)b|i(V)*\beta|^2 \tag{1.9}$$

Since $b = -a$ and $*\beta$ is harmonic, (1.3) or (1.7) furnishes $i(V) * \beta = 0$, which by applying once more the Hodge operator gives $v \wedge \beta = 0$ or $\beta = fv$ for some everywhere non-vanishing $f \in C^\infty(M^n)$. Since β is harmonic it is closed, so that $df \wedge v = 0$ or $df = \lambda v$ for some $\lambda \in C^\infty(M^n)$. But β is coclosed, too, such that $(df, \beta) = (f, \delta\beta) = 0$, by (2.9.3) in [**6**, p. 74], i.e. df and β are orthogonal. Thus $0 = (df, \beta) = \lambda f \mathrm{vol}(M^n)$ yields $\lambda = 0$; since M^n is connected one obtains $f =$const., i.e. $b_1(M^n) = 1$. Consequently, M^n has the same Betti numbers as the product $S^1 \times S^{n-1}$, QED.

2. Conformally flat real hypersurfaces

Let $M^{n+1}(k)$ be a real space-form, i.e. a Riemann space of constant sectional curvature k. Let M^n be a conformally flat hypersurface of $M^{n+1}(k)$. Let h be the second fundamental form of the given immersion of M^n in $M^{n+1}(k)$. By a classical result of J. A. Schouten, $h = \alpha_0 g + \beta_0 v \otimes v$, for some α_0, $\beta_0 \in C^\infty(M^n)$ and some unit tangent vector field $V = \sharp v$. Here $\sharp$ means raising of indices with respect to the induced metric g on M^n. Then, B. Y. Chen and K. Yano, [**4**], have shown (via the Gauss equation, e.g. (2.6) in [**3**, p. 45]) that M^n is a space of q-constant curvature with $a = k + \alpha_0^2$, $b = \alpha_0\beta_0$ and unit vector (V). This allows us to apply Theorem 1 to get

Theorem 3. *Let M^n be a conformally flat compact orientable real hypersurface of the real projective space $IRP^{n+1}(k)$, $k > 0$. Then M^n is a real homology sphere, provided $\alpha_0\beta_0 \geq 0$.*

Proof. Let α be a harmonic p-form on M^n. By (1.8) we obtain $\alpha = 0$ if $(n-p)(k+\alpha_0^2) + \alpha_0\beta_0 > 0$, and $(n-2p)\alpha_0\beta_0 \geq 0$. Under the hypothesis of Theorem 3, it is sufficient to impose $p \leq n/2$. We finally use Poincaré duality to prove that all Betti numbers of M^n vanish, QED.

3. Subprojective spaces

Cf. T. Adati, [**1**], if M^n, $n \geq 3$, is conformally flat and the tensor field:

$$L_{ji} = -\frac{1}{n-2}R_{ji} + \frac{R}{2(n-1)(n-2)}g_{ji} \tag{3.1}$$

is expressed by:

$$L_{ji} = R\rho g_{ji} + \rho_j\sigma_i, \tag{3.2}$$

where $\rho_j = \nabla_j\rho$, $\sigma_i = \nabla_i\sigma$ and σ is a function of ρ, $\rho \neq 0$, then M^n is said to be a subprojective space. Here R denotes the scalar curvature of (M^n, g_{ij}) cf. [**8**, p. 96], M^n follows to be a manifold of q-constant curvature with $a = -2$, $b = -K$, $v_i = \gamma^{-1}\rho_i$, where $\gamma = |d\rho|$. Also if $\sigma = f(\rho)$, for some C^2-smooth function f of one variable, then (3.2) gives $K = \gamma^2 f'(\rho)$. Using (1.3) we obtain:

Theorem 4. *Let $M^n, n \geq 3$ be a compact orientable subprojective space with $\rho < 0$ and $\sigma = f(\rho)$ for some C^2-smooth decreasing function $f\colon (-\infty, 0) \to \mathbf{R}$. Then M^n is a real homology sphere.*

Proof. Using (1.3) for the harmonic p-form α on M^n we note that

$$2(n-p)\rho + \gamma^2 f'(\rho) < 0, \quad (2p-n)\gamma 2 f'(\rho) \geq 0,$$

provided that $p \geq n/2$. As in Th. 3, by separately analysing the cases $n =$ odd and $n =$ even, and by Poincaré duality one obtains $b_p(M^n) = 0$, for all p, QED.

4. Special para-Sasakian manifolds with vanishing D-concircular tensor

Let (M^n, g_{ij}) be a Riemann space, $n \geq 3$, and ξ^i a given unit vector field with $\nabla_j \xi^i = \varepsilon(-\delta^i_j + \eta_j \xi^i)$, $\eta_j = g_{ji}\xi^i$, $\varepsilon = \pm 1$. Then M^n carries a para-contact metric structure $(\phi^i_j, \xi^i, \eta_j, g_{ij})$, $\phi^i_j = \nabla_j \xi^i$, [**2**]. On the other hand, cf. G. Chuman, [**5**], one may consider the D-concircular tensor:

$$U^h_{kji} = R^h_{kji} + \frac{R + 2(n-1)}{(n-1)(n-2)}\{g_{ki}\delta^h_j - g_{ji}\delta^h_k\} - \\ - \frac{R + n(n-1)}{(n-1)(n-2)}\{g_{ki}\eta_j\xi^h - g_{ji}\eta_k\xi^h + \eta_k\eta_i\delta^h_j - \eta_j\eta_i\delta^h_k\}. \tag{4.1}$$

Consequently, if $U^h_{kji} = 0$ then the curvature of M^n is expresed by (1.1) with $a = \dfrac{R + 2(n-1)}{(n-1)(n-2)}$, $b = -\dfrac{R + n(n-1)}{(n-1)(n-2)}$, $v_i = \eta_i$. Using (1.3) one obtains:

Theorem 5. *Let M^n be a compact orientable special para-Sasakian manifold with a vanishing D-concircular tensor. Then there exists $R_0 > 0$ such that if $R \geq R_0$ then $b_p(M^n) = 0$, $1 < p < n-1$.*

Proof. Let α be a harmonic p-form on M^n. By (1.8) if $(n-p)(R + 2(n-1)) - (R + n(n-1)) > 0$, $(2p-n)(R + n(n-1)) \geq 0$ then $F_p(\alpha)$ is positive definite. Suppose $R \geq 0$. Define $f_n(R) = n - \dfrac{R + n(n-1)}{R + 2(n-1)}$. Then f_n is strictly increasing, $f_n(0) = n/2$, $\lim_{R \to +\infty} f_n(R) = n - 1$. This makes clear the contents of the assumption on the scalar curvature in Theorem 5. Indeed, as $R \to +\infty$, i.e. if $R \geq R_0$ for some constant $R_0 > 0$, the condition $n/2 \leq p < n-1$ is equivalent to $n/2 \leq p < f_n(R)$, QED.

Note that M^n in Theorem 5 is not necessarily conformally flat, but the form (1.1) is sufficient for our purpose; by [**8**, p. 98], the additional condition $R =$const. implies M^n is a space of q-constant curvature.

If M^n is a space of q-constant curvature with $R =$ const. $\neq 0$ and at the same time an S-manifold (i.e. $\nabla_m \nabla_s R^h_{kji} = \nabla_s \nabla_m Rkji^h$) then $b_1(M^n) \geq 1$, since from Theorem 3.3. in [**8**, p. 102] it follows that v_i is parallel, and thus harmonic. Nevertheless, this situation may not be used in our Theorem 5. Indeed, by a result of Wang, [**7**], a space of q-constant curvature is an S-manifold iff $a+b=0$ (while any special para-Sasakian manifold with $U_{kji}^{h}=0$ has $a+b=-1$).

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