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SEMIPRIME IDEALS OF SKEW POLYNOMIAL RINGS

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Abstract. We study relations among the semiprime ideals of a ring R and those of a skew polynomial ring S_n over R, in connection with results obtained in [**11**] for prime ideals of R and S_n.

Notice that results on the prime ideal sructure of a skew polynomial ring $R[x, f, d]$ have been obtained only under additional assumptions; Goldie and Michler [**1**] assume that R is right Noetherian and $d = 0$, Jordan [**4**] assumes that R is right Noetherian and f is the identity map of R, while Irving [**2** and **3**] assumes that R is a commutative ring. In [**11**] we have studied what happens when f is a non trivial automorphism of R and d is a non zero f-derivation of R and we have extended these results for skew polynomial rings in finitely many variables over R.

1. Preliminaries

All the rings considered in this paper are with identities. Let R be a ring, let $H = \{f_1, \dots, f_n\}$ be a finite set of automorphisms of R and let $D = \{d_1, \dots, d_n\}$ be a finite set of mappings from R to R, such that d_i is a f_i-derivation of R, for all $i = 1, \dots, n$ (i.e. $d_i(a + b) = d_i(a) + d_i(b)$ and $d_i(ab) = ad_i(b) + d_i(a)f_i(b)$, for all a, b in R). Then an ideal I of R is called an H-ideal if $f_i(I) = I$ for each i. Also I is called a D-ideal if $d_i(I) \subseteq I$, for each i. An ideal I of R which is both an H-ideal and a D-ideal is called for brevity an (H, D)-ideal of R. In the special case where $H = \{f\}$ and $D = \{d\}$, I is called an (f, d)-ideal of R. We recall that an ideal P of R is called a semiprime ideal if, given any ideal A of R such that $A^k \subseteq P$ for some non negative integer k, one has $A \subseteq P$. A ring is called semiprime if 0 is a semiprime ideal. From the previous definition it becomes clear that R is a semiprime ring if and only if R has no non zero nilpotent ideals. Now an (H, D)-ideal of R is called an (H, D)-semiprime ideal if, given any (H, D)-ideal A of R such that $A^k \subseteq I$ for some non negative integer k, one has $A \subseteq I$; and R is called an (H, D)-semiprime ring if 0 is an (H, D)-semiprime ideal of R. The notions

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of an H-semiprime and a D-semiprime ideal of R can be also defined in the obvious way. Assume next that $d_i \circ d_j = d_j \circ d_i$, $f_i \circ f_j = f_j \circ f_i$ and $d_i \circ f_j = f_j \circ d_i$ for all $i, j = 1, \ldots, n$ and consider the set S_n all polynomials in n variables $x_1, \ldots, x_n$ over R. Define in S_n addition in the usual way and multiplication by the relations $x_i r = f_i(r)x_i + d_i(r)$ and $x_i x_j = x_j x_i$, for all r in R and all $i, j = 1, \ldots, n$. Then S_i is an Ore extension over S_{i-1} (cf. [**7**]) for all $i = 1, \ldots, n$, where $S_0 = R$ (cf. Theorem 2.4 of [**9**]). We call the ring S_n a skew polynomial ring in n variables over R and we denote it by $S_n = R[x_1, f_1, d_1] \ldots [x_n, f_n, d_n]$. Notice that under these conditions one can extend f_i to an automorphism and d_i to an f_i-derivation of S_n by putting $f_i(x_j) = x_j$ and $d_i(x_j) = 0$ for all $i, j = 1, \ldots, n$ (cf. Theorems 2.2 and 2.3 of [**9**]). In the special case where f_i is the identity for all f_i in H we get the skew polynomial ring $S_n^* = R[x_1, d_1] \ldots [x_n, d_n]$, while if $d_i = 0$ for all d_i in D we get the skew polynomial ring $S'_n = R[x_1, f_1] \ldots [x_n, f_n]$. When R is right Noetherian the usual proof of the Hilbert's Basis Theorem adapts easily to show, together with induction on n, that S_n is a right Noetherian ring too (this is not true if we take H to be any set of monomorphisms of R, (cf. [**6**]).

2. Main results

2.1. Theorem. *Let P be an H-semiprime ideal of S_n; then $P \cap R$ is a (H, D)-semiprime ideal of R.*

Proof. Since P is a H-ideal of S_n, $P \cap R$ is an H-ideal of R. On the other hand, for all r in $P \cap R$ and each i, $d_i(r) = x_i r - f_i(r)x_i$, is in $P \cap R$, therefore $P \cap R$ is an (H, D)-ideal of R. Now, let A be any (H, D)-ideal of R, such that $A^k \subseteq P \cap R$ for some non negative integer k. Then AS_n is an ideal of S_n, because $x_i A \subseteq f_i(A)x_i + d_i(A) \subseteq Ax_i + A \subseteq AS_n$ for each i. It is also clear that AS_n is an H-ideal of S_n. But $(AS_n)^2 = A(S_n AS_n) \subseteq A^2 S_n$ and therefore an easy induction shows that $(AS_n)^k \subseteq A^k S_n$. Thus $(AS_n)^k \subseteq (P \cap R)S_n \subseteq P$. Then, by our hypothesis, $AS_n \subseteq P$ and therefore $A \subseteq AS_n \cap R \subseteq P \cap R$, and this finishes the proof.

We record two obvious corollaries.

2.2. Corollary. *Let P be a semiprime ideal of S_n^*, then $P \cap R$ is a D-semiprime ideal of R.*

2.3. Corollary. *Let P be an H-semiprime ideal of S'_n, then $P \cap R$ is an H-semiprime ideal of R.*

2.4. Theorem. *Let R be a right Noetherian ring and let P be a semiprime ideal of S'_n none of whose minimal primes contains x_i, for each i. Then $P \cap R$ is an H-semiprime ideal of R.*

Proof. It is enough to show that that P is an H-ideal of S'_n and then to apply Corollary 2.3.

For this, since S'_n is right Noetherian, there exist finitely many prime ideals of S'_n, say $P_1, \ldots, P_k$, such that $P_1 \ldots P_k \subseteq P$ and $P_1, \ldots, P_k \supseteq P$. Thus $P_1 \cap P_2 \cap \ldots \cap P_k \supseteq P$. But, $(P_1 \cap P_2 \cap \ldots \cap P_k)^k \subseteq P_1 P_2 \ldots P_k \subseteq P$, therefore $P_1 \cap P_2 \cap \ldots \cap P_k \subseteq P$ and so $P_1 \cap P_2 \cap \ldots \cap P_k = P$.

Choose P' to be any of the above mentioned prime ideals of S'_n and let g be in P. Then g is also in P' and therefore $f_i(g)x_i = x_i g$, is in P', for each i. Thus $f_i(g)S'_n x_i = f_i(g)x_i S'_n \subseteq P'$. But x_i is not in P', therefore $f_i(g)$ is in P' and so $f_i(g)$ is in P. Thus $f_i(P) \subseteq P$.

Now we have the ascending chain of ideals $P \subseteq f_i^{-1}(P) \subseteq f_i^{-2}(P) \subseteq \ldots$, which by the Noetherian property becomes stable after a finite number of steps, say m. Then $f_i^{-m}(P) = f_i^{-m-1}(P)$, therefore $f_i(P) = P$ and this finishes the proof.

Jordan [**5**] has shown that if I is a d-prime ideal of R, then IS_1^* is a prime ideal of S_1^*. The following theorem shows that the same result holds if we replace the term "prime" with the term "semiprime" in the more general situation of the skew polynomial ring S_1.

2.5. THEOREM. *Let I be a (f_1, d_1)-semiprime ideal of R, then IS_1 is an f_1-semiprime ideal of S_1.*

Proof. It is easy to check that IS_1 is an f_1-ideal of S_1 (cf. the proof of Theorem 2.1 for the ideal AS_n). Next let A be an f_1-ideal of S_1 such that $A^k \subseteq IS_1$ for some non negative integer k. Denote by $T(A)$ the set of all leading coefficients of the elements of A, which are polynomials in x_1 with coefficients in R. We are going to show that $T(A)$ is an (f_1, d_1)-ideal of R. For this let g and h be any elements of A with degrees m and l and leading coeficients a and b respectively. Without loss of the generality we may assume that $m \geq l$. Then $a \pm b$ is either zero or the leading coefficient of $g \pm hx_1^{m-1}$ which is in A, therefore $a \pm b$ is in $T(A)$. Moreover, for any r in R, ra is either zero or the leading coefficient of rg. On the other hand $gf_1^{-m}(r) = ax_1^m f_1^{-m}(r)$ + terms of lower degree $= arx_1^m$ + terms of lower degree, therefore ar is either zero or the leading coefficient of $gf_1^{-m}(r)$. Thus ar is in $T(A)$ and therefore $T(A)$ is an ideal of R. Furthermore $f_1(a)$ and $f_1^{-1}(a)$ are the leading coefficients of $f_1(g)$ and $f_1^{-1}(g)$ respectively. Finally, if $g = \sum_{i=0}^m a_i x_i^i$, with $a_m = a$, then

$$x_1 g - f_1(g)x_1 = \sum_{i=0}^{m} [x_1 a_i - f_1(a_i)x_1]x_1^i = \sum_{i=0}^{m} d(a_i)x_1^i,$$

therefore $d(a)$ is either zero or the leading coefficient $x_1 g - f_1(g)x_1$ which is in A. Thus $T(A)$ is in fact an (f_1, d_1)-ideal of R. Also, since $ax_1^m f_1^{-m}(b)x_1^l = abx_1^{m+1}$ + terms of lower degree, ab is either zero or the leading coefficient of $gf_1^{-m}(h)$. Thus ab is in $T(A^2)$ and therefore $[T(A)]^2 \subseteq T(A^2)$. Then an easy induction shows that $[T(A)]^k \subseteq T(A^k)$, for any non negative integer k. Thus $[T(A)]^k \subseteq T(IS_1) = I$ and therefore, by our hypothesis,

$$T(A) \subseteq I. \tag{1}$$

Now we may as well assume that $A \supseteq IS_1$, otherwise we can use $A + IS_1$ instead of A. This, together with the relation (1), shows that if $g = \sum_{i=0}^{m} a_i x_1^i$ is in A, then $a_m x_1^m$ is in A and therefore $g - a_m x_1^m = \sum_{i=0}^{m-1} a_i x_1 i$ is in A. Thus a_{m-1} is in $T(A) \subseteq I$. Repeating this argument we finally get that a_i is in I for each $i = 0, 1, \dots, m$ and therefore $A \subseteq IS_1$, as required.

We are going now to show that an analogous result holds for the skew polynomial ring S_n. For this we need the following lemma.

2.6. Lemma. *Let A be an H-ideal of S_n and let $T(A)$ be the set of all leading coefficients of the elements of A written as polynomials in x_n with coefficients in S_{n-1}. Put $T_i(A) = T[T_{i+1}(A)]$ in S_i, for each $i = 0, 1, ..., n-1$, where $S_0 = R$ and $T_n(A) = A$. Then $T_0(A)$ is an (H, D)-ideal of R and $[T_0(A)]^k \subseteq T_0(A^k)$ for any non negative integer k.*

Proof. Since $T_{n-1}(A) = T[T_{n-1}(A)] = T(A)$, $T_{n-1}(A)$ is an (f_n, d_n)-ideal of S_{n-1} (cf. the proof of the previous theorem). Similarly $T_{n-2}(A) = T[T_{n-1}(A)]$ is an (f_{n-1}, d_{n-1})-ideal of S_{n-2}. Now let $r = r(x_1, \dots, x_{n-2})$ be any element of $T_{n-2}(A)$; then there exists s in $T_{n-1}(A)$ with leading coefficient r with respect to x_{n-1}. Then $f_n(s)$, $f_n^{-1}(s)$ and $d_n(s)$ are all in $T_{n-1}(A)$, while $f_n(x_{n-1}) = x_{n-1}$ and $d_n(x_{n-1}) = 0$; therefore $f_n(r)$, $f_n^{-1}(r)$ and $d_n(r)$ are all in $T_{n-2}(A)$. Thus $T_{n-2}(A)$ is also an (f_n, d_n)-ideal of S_{n-2}. Repeating this procedure we shall eventually find that $T_0(A)$ is a (H, D)-ideal of R, as required. In the proof of Theorem 2.5 we have seen that $[T(A)]^k \subseteq T(A^k)$, therefore $[T_{n-1}(A)]^k \subseteq T_{n-1}(A^k)$. Next, applying induction on m, we assume that $[T_{n-m}(A)]^k \subseteq T_{n-m}(A^k)$. Then

$$[T_{n-m-1}(A)]^k = \left[T[T_{n-m}(A)]\right]^k \subseteq \left[[T_{n-m}(A)]^k\right] \subseteq T[T_{n-m}(A^k)] = T_{n-m-1}(A^k).$$

Thus, if we put $m = n$, we get that $T_0(A)^k \subseteq T_0(A^k)$ and this finishes the proof.

We are ready now to prove

2.7. Theorem. *Let I be an (H, D)-semiprime ideal of R; then IS_n is an H-semiprime ideal of S_n.*

Proof. Proceeding as in the proof of Theorem 2.1 for the ideal AS_n one can show that IS_n is an H-ideal of S_n. Next let A be an H-ideal of S_n such that $A^k \subseteq IS_n$. Then, by Lemma 2.6, $T_0(A)$ is an (H, D)-ideal of R and $[T_0(A)]^k \subseteq T_0(A^k) \subseteq T_0(IS_n) = I$; therefore $T_0(A) \subseteq I$. Without loss of generality we may assume that $A \supseteq IS_n$, otherwise we may use $A + IS_n$ instead of A. Let $g = \sum_{i=0}^{m_1} a_i x_n^i$ be a polynomial of A with coefficients in S_{n-1}; then a_{m_1} is in $T_{n-1}(A)$. We write $a_{m_1} = \sum_{i=0}^{m_2} b_i x_{n-1}^i$. Then b_{m_2} is in $T_{n-2}(A)$ and $a_{m_1} x_n^{m_1} = b_{m_2} x_n^{m_1} x_{n-2}^{m_2} + \sum_{i=0}^{m_2-1} b_i x_{n-1} x_n^{m_1}$. We proceed the same way until we find some r in $T_0(A)$, such that $h = r x_m^{m_1} x_{n-1}^{m_2} \dots x_1^{m_n}$ is term of $a_m x_n^{m_1}$. Then r is in I, therefore h is in $IS_n \subseteq A$. Thus $g - h$ is in A. Repeating the same argument for $g - h$ and keeping going in the same way we eventually find that $a_{m_1} x_n^{m_1}$ is in $IS_n \subseteq A$. Thus $\tilde{g} = g - a_{m_1} x_n^{m_1} = \sum_{i=0}^{m_1-1} a_i x_n^i$ is in A. If we apply the same argument for $\tilde{g}$ and we keep going in the same way we finally find that g is in IS_n, which was to be proved.

The following are straightforward corollaries of Theorems 2.1 and 2.7.

2.8. Corollary. *The skew polynomial ring S_n is H-semiprime if and only if R is an (H, D)-semiprime ring.*

2.9. Corollary. *If I is a D-semiprime ideal of R, then IS_n^* is a semiprime ideal of S_n^*, therefore S_n^* is a semiprime ring if and only if R is a D-semiprime ring.*

2.10. Corollary. *If I is an H-semiprime ideal of R, then IS'_n is an H-semiprime ideal of S'_n, therefore S'_n is an H-semiprime ring if, and only if R is an H-semiprime ring.*

3. Remarks

(1) An (H, D)-ideal I of R is called an (H, D)-prime ideal if, given any two (H, D)-ideals A and B of R such that $AB \subseteq I$, one has either $A \subseteq I$ or $B \subseteq I$. The notions of an H-prime and of a D-prime ideal of R can be also defined in the obvious way.

The statements of all results obtained in the previous section remain true if we replace the term "semiprime", whenever it appears, with the term "prime" (cf. [**11**]).

(2) The statement of Theorem 2.4 shoud be restated as follows: "Let P be a prime ideal of S'_n such that x_i is not in P, for each i. Then $P \cap R$ is an H-prime ideal of R".

That is, the ring R need not be right Noetherian in this case. But the hypothesis that x_i is not in P for each i is not superflous (cf. [**1**, Example 3]).

(3) If R is a right Noetherian ring and I is an H-prime ideal of R, then IS'_n is a prime ideal of S'_n (cf. [**11**, Theorem 2.9]). Attempts to prove that the above mentioned result remains true for the ring R which is not right Noetherian, i.e. that we can replace the term "prime" with the term "semiprime" and thus to produce a result stronger than Corollary 2.10, have proved unsuccessful.

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