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ON HAAR MEASURE ON $SL(N, \mathbf{R})$

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Abstract. We found an explicit form of Haar measure on the group $SL(n, \mathbf{R})$, i.e. its density with respect to Hausdorff (n^2-1)-measure on $GL(n, \mathbf{R})$ and some interesting parametrizations of the group $SL(n, \mathbf{R})$.

1. Introduction and notations. Dealing with Lie groups one usually use differential forms techniques, but we here prefer classical techniques which gives easier and transparent proofs. The main result (see Theorem 8 bellow) can be proved by differential forms techniques developed e.g. in [**5**, pp. 153–155]. The other reason is that the theory developed here is applicable not only to manifolds but to surfaces in $\mathbf{R}^n$ as well.

We use the standard basis $\{e_1, \dots, e_n\}$ of $\mathbf{R}^n$ and $x \in \mathbf{R}^n$ is respented as a column. Norm in $\mathbf{R}^n$ is Euclidean

$$\|x\|^2 = (x \mid x) = \sum_{i=1}^n x_i^2, \qquad x = \sum_{i=1}^n x_i e_i.$$

The derivative of a function $f : \mathbf{R}^n \to \mathbf{R}$ is represented as a row. By $gl(n, \mathbf{R})$ we denote the Lie algebra of $GL(n, \mathbf{R})$. Norm on $gl(n, \mathbf{R})$ is Euclidean $\|A\|^2 = (A \mid A) = \operatorname{tr} AA^\tau$, $A \in gl(n, \mathbf{R})$. We also need the group $SL^*(n, \mathbf{R}) = \{A \in GL(n, \mathbf{R}); \det A = \pm 1\}$. If $A \in gl(n, \mathbf{R})$, by A^+ we denote the matrix of algebraic complements. If $\operatorname{rank}(A) = 1$ then there exist $x, y \in \mathbf{R}^n$ such that $A = xy^\tau$ i.e. $Az = (y \mid z)x$, $z \in \mathbf{R}^n$.

We need the following elementary lemma from linear algebra.

2. Lemma. *Let $x, y \in \mathbf{R}^n$. Then*

(1) $\det(I + xy^\tau) = 1 + (x \mid y)$;

(2) $(I + xy^\tau)^+ = \big(1 + (x \mid y)\big)I - yx^\tau$;

(3) $(I + xy^\tau)^{-1} = I - xy^\tau / \big(1 + (x \mid y)\big)$, $(x \mid y) \neq -1$;

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(4) *Let B be an $n\times(n-1)$-matrix and $A\in gl(n,\mathbf{R})$. Then there exists $a\in\mathbf{R}^n$, $\|a\|=1$, such that $a^\tau B=0$ and*

$$\det\bigl((AB)^\tau AB\bigr)=\|A^+a\|^2\det(B^\tau B).$$

The vector a is unique up to a sign if $\operatorname{rank}(B)=n-1$ *and* $\det A\neq 0$.

3. Theorem (Change-of-variable-theorem for hypersurfaces). *Let $M_1,M_2\subset\mathbf{R}^n$ be hypersurfaces, smooth by parts, h Hausdorff $(n-1)$-measure on $\mathbf{R}^n$, $n(x)$ unit normal at $x\in M_1$ existing h-a.e. on M_1, U open neighbourhood of M_1 and $f:U\to\mathbf{R}^n$ such that*:

(1) $f(M_1)=M_2$;

(2) *f is differentiable on M_1 h-a.e. and $f'(x)^+\neq 0$ h-a.e. on M_1;*

(3) $\operatorname{card} f^{-1}(x)=1$, *$h$-a.e. on M_2.*

Then for every $F\in L_1(M_2,h)$

$$\int_{M_2}F\,dh=\int_{M_1}F(f(x))\|f'(x)^+n(x)\|\,dh(x).$$

Proof. Let $\sigma:\mathbf{R}^{n-1}\to M_1$ be a parametrization of M_1. Then $f\circ\sigma$ is a parametrization of M_2 and $(f\circ\sigma)'(t)=f'(\sigma(t))\sigma'(t)$, $t\in\mathbf{R}^{n-1}$. By the above lemma with $A=f'(x)$, $B=\sigma'(t)$, $x=\sigma(t)$, $a=n(x)$ we have

$$\begin{aligned}\det(f\circ\sigma)'(t)^\tau(f\circ\sigma)'(t)&=\det(f'(x)\sigma'(t))^\tau f'(x)\sigma'(t)\\&=\|f'(x)^+n(x)\|\det\sigma'(t)^\tau\sigma'(t).\end{aligned}$$

Let $F\in L_1(M_2,h)$. Then

$$\begin{aligned}\int_{M_2}F\,dh&=\int_{\mathbf{R}^{n-1}}F(f(\sigma(t)))|\det(f\circ\sigma)'(t)^\tau(f\circ\sigma)'(t)|^{1/2}\,dt\\&=\int_{\mathbf{R}^{n-1}}F(f(\sigma(t)))\|f'(\sigma(t))^+n(\sigma(t))\|\cdot|\det\sigma'(t)^\tau\sigma'(t)|^{1/2}\,dt\\&=\int_{M_1}F(f(x))\|f'(x)^+n(x)\|\,dh(x).\end{aligned}$$

4. *Definition*. Let $H(n)$ be the set of all continuous functions $\varphi:\mathbf{R}^n\to[0,\infty)$ such that

(1) $\varphi(tx)=t\varphi(x)$, $t>0$, $x\in\mathbf{R}^n$;

(2) φ is differentiable a.e. and $\varphi'(x)\neq 0$ a.e.

For $\varphi\in H(n)$ we introduce the sets

$$\begin{aligned}S_\varphi&=\{x\in\mathbf{R}^n;\,\varphi(x)=1\},\\D_\varphi&=\{x\in\mathbf{R}^n;\,\varphi(x)<1\},\\O_\varphi&=\{x\in\mathbf{R}^n;\,\varphi(x)=0\}.\end{aligned}$$

Differentiating (1) with respect to t and x we obtain

(a) $\varphi'(x)x = \varphi(x)$, a.e., $x \in \mathbf{R}^n$;

(b) $\varphi'(tx) = \varphi'(x)$, a.e., $x \in \mathbf{R}^n$, $t > 0$.

5. Lemma. *Let $\varphi, \psi \in H(n)$ and $f : \mathbf{R}^n \setminus (O_\varphi \cup O_\psi) \to \mathbf{R}^n \setminus (O_\varphi \cup O_\psi)$, $f(x) = \big(\varphi(x)/\psi(x)\big)x$. Then*

(1) *f is a homeomorphism and* $f^{-1}(x) = \dfrac{\psi(x)}{\varphi(x)}x$;

(2) $f(S_\varphi \setminus O_\psi) = S_\psi \setminus O_\varphi$;

(3) $\det f'(x) = \varphi(x)^n/\psi(x)^n$, *a.e.*;

(4) $\left\| f'(x)^+ \dfrac{\varphi'(x)^\tau}{\|\varphi'(x)\|} \right\| = \dfrac{\varphi(x)^n}{\psi(x)^n} \cdot \dfrac{\|\psi'(x)\|}{\|\varphi'(x)\|}$, *a.e.* ($\|x^\tau\| = \|x\|$).

Proof. (1) and (2) are evident; (3) and (4) follow from Lemma 1 and

$$f'(x) = \frac{\varphi(x)}{\psi(x)}\big[I + x\big(\varphi'(x)\psi(x) - \psi'(x)\varphi(x)\big)/\big(\varphi(x)\psi(x)\big)\big].$$

6. Corollary. *Let $\varphi, \psi \in H(n)$ and $F \in L_1(S_\varphi, h)$. Then*

(1) $\displaystyle\int_{S_\psi} F\,dh = \int_{S_\varphi} F\left(\frac{x}{\psi(x)}\right)\frac{1}{\psi(x)^n} \cdot \frac{\|\psi'(x)\|}{\|\varphi'(x)\|}\,dh(x)$;

(2) $\displaystyle\int_{S_\psi} \frac{F\,dh}{\|\psi'\|} = \int_{S_\varphi} F\left(\frac{x}{\psi(x)}\right)\frac{1}{\psi(x)^n}\frac{dh(x)}{\|\varphi'(x)\|}$;

(3) $\displaystyle\int_{D_\psi} F(x)\,dx = \int_{D_\varphi} F\left(\frac{\varphi(x)}{\psi(x)}x\right)\frac{\varphi(x)^n}{\psi(x)^n}\,dx.$

Proof follows from Lemma 5 and Theorem 3.

7. Theorem (Polar formula). *Let $\varphi \in H(n)$ and $F \in L_1(\mathbf{R}^n)$. Then the following formula holds*

$$\int_{\mathbf{R}^n} F(x)\,dx = \int_0^\infty \int_{S_\varphi} F(tx)t^{n-1}\,\frac{dt\,dh(x)}{\|\varphi'(x)\|}.$$

Proof. (1) Let first $\varphi(x) = \|x\| = (x \mid x)^{1/2}$. Then $\varphi'(x) = x^\tau/\|x\|$ and $\|\varphi'(x)\| = 1$, $x \neq 0$. Let $\sigma : \mathbf{R}^{n-1} \to S_\varphi$ be a parametrization of the Euclidean sphere S_φ. For $x \in \mathbf{R}^n$ we have $x = \psi(t, y) = t\sigma(y)$, $y \in \mathbf{R}^{n-1}$, $t = \|x\|$, and now $\psi'(t, y) = [\sigma(y), t\sigma'(y)] \in GL(n, \mathbf{R})$. Hence $|\det\psi'(t, y)| = t^{n-1}|\det\sigma'(y)^\tau\sigma'(y)|^{1/2}$ and therefore

$$\begin{aligned}\int_{\mathbf{R}^n} F(x)\,dx &= \int_0^\infty \int_{\mathbf{R}^{n-1}} F(t\sigma(y))t^{n-1}|\det\sigma'(y)^\tau\sigma'(y)|^{1/2}\,dtdy\\ &= \int_0^\infty \int_{S_\varphi} F(tx)t^{n-1}\,dt\,dh(x).\end{aligned}$$

(2) Using now Corollary 6 and Fubini theorem we obtain the polar formula for any $\varphi \in H(n)$.

In standard monographs [**1**], [**2**] and [**3**] there is no explicit form of the Haar measure on the group $SL(n, \mathbf{R})$. Let us apply above theory to find it.

8. Theorem. *Let h be the Hausdorf (n^2-1)-measure on $gl(n, \mathbf{R})$ and let ω be a measure on $SL^*(n, \mathbf{R})$ defined by*

$$d\omega(X) = n\, dh(X)/\|X^{-1}\|, \qquad X \in SL^*(n, \mathbf{R}).$$

Then ω is the Haar measure on $SL^*(n, \mathbf{R})$.

Proof. Let $A, B \in gl(n, \mathbf{R})$, and $\mathcal{A}_{A,B} : gl(n, \mathbf{R}) \to gl(n, \mathbf{R})$, $\mathcal{A}_{A,B}X = AXB$. Then $\det \mathcal{A}_{A,B} = (\det AB)^n$. Because of $\det'(A)B = (A^+ \mid B) = \operatorname{tr} A^+ B^\tau$ the unit normal on the group $SL^*(n, \mathbf{R})$ at $X \in SL^*(n, \mathbf{R})$ is $n(X) = X^+/\|X^{-1}\|$. Let $Y \in SL^*(n, \mathbf{R})$. Then $\mathcal{A}_{Y,I} SL^*(n, \mathbf{R}) = SL^*(n, \mathbf{R})$ and we can apply Theorem 3 on $f = \mathcal{A}_{Y,I}$. We have

$$\begin{aligned} f'(X)^+ &= \mathcal{A}_{Y,I}{}^+ = (\det Y)^n \mathcal{A}_{Y^{-1\tau}, I}, \\ \|f'(X)^+ n(X)\| &= \|X^{-1}Y^{-1}\|/\|X^{-1}\|. \end{aligned}$$

Let $\lambda(X)$ be the density of the Haar measure with respect to h. Then by Theorem 3 we have

$$\begin{aligned} \int_{SL^*(n,\mathbf{R})} f(X)\lambda(X)\, dh(X) &= \int_{SL^*(n,\mathbf{R})} F(YX)\lambda(X)\, dh(X) \\ &= \int_{SL^*(n,\mathbf{R})} f(X)\lambda(Y^{-1}X)\|X^{-1}Y\|\, dh(X)/\|X^{-1}\|. \end{aligned}$$

Therefore $\lambda(X) = \lambda(Y^{-1}X)\|X^{-1}Y\|/\|X^{-1}\|$, $X, Y \in SL^*(n, \mathbf{R})$, h-a.e. For $X = I$ we get $\lambda(Y^{-1}) = \sqrt{n}\lambda(I)/\|Y\|$ or $\lambda(Y) = \sqrt{n}\lambda(I)/\|Y^{-1}\|$, where $\lambda(I)$ is a positive number. Taking $\lambda(I) = \sqrt{n}$ we finish the proof.

9. Corollary. *For any integrable function F on $GL(n, \mathbf{R})$ we have*

$$\int_{GL(n,\mathbf{R})} F(X) \frac{dX}{|\det X|^n} = n \int_0^\infty \int_{SL^*(n,\mathbf{R})} F(tX)\, \frac{dt}{t}\, \frac{dh(X)}{\|X^{-1}\|}.$$

Proof. Let $\varphi \in H(n^2)$, $\varphi(X) = |\det X|^{1/n}$. Then $S_\varphi = SL^*(n, \mathbf{R})$ and by the polar formula we have

$$\begin{aligned} \int_{gl(n,\mathbf{R})} F(X)\, dX &= \int_0^\infty \int_{SL^*(n,\mathbf{R})} F(tX) t^{n^2-1}\, \frac{dt\, dh(X)}{\|\varphi'(X)\|} \\ &= \int_0^\infty \int_{SL^*(n,\mathbf{R})} F(tX) n t^{n^2-1}\, \frac{dt\, dh(X)}{\|X^{-1}\|} \end{aligned}$$

$$= n \int_0^\infty \int_{SL^*(n,\mathbf{R})} F(tX) t^{n^2-1} \frac{dt\, dh(X)}{\|X^{-1}\|}.$$

If we take $F(X)/|\det X|^n$ in place of $F(X)$ we obtain the formula.

This formula represents desintegration of the Haar measure on $GL(n,\mathbf{R})$ with respect to the Haar measure on $SL^*(n,\mathbf{R})$, according to the action of $\mathbf{R}_+$ on $GL(n,\mathbf{R})$ defined by $(t,X) \mapsto tX$.

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