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SOME RESULTS ON GRAPHS WITH AT MOST TWO POSITIVE EIGENVALUES

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Abstract. We determine all graphs G such that G and its complementary graph $\bar{G}$ have exactly one (or, respectively, exactly two) positive eigenvalues.

1. Introduction

We consider only finite, undirected graphs without loops or multiple edges. The spectrum of a graph is the spectrum of its 0-1 adjacency matrix. Relation $H \subseteq G$ will always mean that H is an induced subgraph of a graph G.

Generally speaking, for every graph theoretical property a problem can be posed of finding all graphs G such that both G and $\bar{G}$ possess it. This topics was treated by several authors in the past ten years. The problem we consider is to find all graphs which together with their complementary graphs have exactly one (respectively, exactly two) positive eigenvalues.

We say that a graph G is *p-positive* ($p \geq 1$) if it has exactly p positive eigenvalues (including multiplicities). A graph G is *double p-positive* if both G and $\bar{G}$ have exactly p positive eigenvalues. In this paper we determine all double p-positive graphs for $p = 1$ and $p = 2$.

It is well known that complete multipartite graphs are the only connected 1-positive graphs [**5**]. A similar characterization of connected p-positive graphs ($p \geq 2$) is still an unsolved problem.

In the sequel we give some basic definitions and lemmas.

Lemma 1 [**5**]. *A graph without isolated vertices is not a complete multipartite graph if and only if it contains any graph from Fig. 1 as an induced subgraph.* $\square$

Next, let X and Y be two disjoint subsets of the vertex set $V(G)$ of a graph G. We say that subsets X and Y are completely adjacent in G if each vertex from X is

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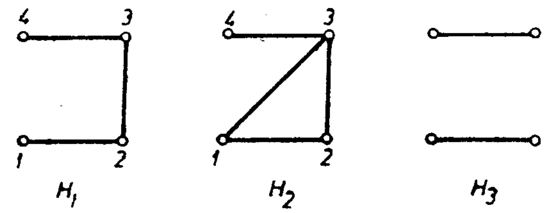

Fig. 1

adjacent to each vertex from Y. Similarly, we say that these subsets are completely nonadjacent if no vertex from X is adjacent to any vertex from Y.

Now, we define two binary relations ρ_1 and ρ_2 on the vertex set $V(G)$ of a graph G in the following way:

1° the vertices x and y are in the relation ρ_1 if they have the same neighbours in $V(G)$;

2° the vertices x and y are in the relation ρ_2 if they are adjacent and have the same neighbours in the set $V(G) \setminus \{x, y\}$.

The relation ρ_1 is obviously an equivalence relation on the vertex set $V(G)$. Let $\{N_1, \dots, N_k\}$ be the corresponding quotient set and let $|N_i| = n_i$ $(i = 1, \dots, k)$. The subsets $N_1, \dots, N_k$ (characteristic subsets of G) have the following property: any two vertices from the same subset are not adjacent and any two subsets are completely adjacent or completely nonadjacent in G. The corresponding quotient graph of G is called the canonical graph of G and is denoted by g and obviously $g \subseteq G$. For instance, if G is a complete s-partite graph, then its canonical graph is the complete graph K_s. Of course, the canonical graph of the complete graph K_n is K_n itself.

Next, let $n^+(G)$ and $n^-(G)$ be the numbers of positive and negative eigenvalues of G, respectively.

The following lemma is an easy consequence of the Interlacing theorem [**1**, p. 19] and the fact that adding a vertex related to one already present increases the nullity by 1.

LEMMA 2 [**2**]. *Let g be the canonical graph of a graph G. Then* $n^+(G) = n^+(g)$, $n^-(G) = n^-(g)$. $\square$

The relation ρ_2 is symmetric and transitive. By this relation the vertex set $V(G)$ can be divided into certain disjoint subsets $C_1, \dots, C_p$ such that, for each $i = 1, \dots, p$, the graph induced by the set C_i is a complete graph. Two subsets C_i and C_j $(i \neq j)$ are always completely adjacent or completely nonadjacent in G.

For the subset N_i $(1 \leq i \leq k)$, relation $x \ \rho_1 \ N_i$ will mean that $x \ \rho_1 \ y$ holds for each vertex $y \in N_i$. Similarly, for the subset C_j $(1 \leq j \leq p)$, relation $x \ \rho_2 \ C_j$ will mean that $x \ \rho_2 \ y$ holds for each vertex $y \in C_j$ $(x \neq y)$. If $|N_i| > 1$ and $|C_j| > 1$ then obviously $N_i \cap C_j = \varnothing$.

Next, denote by a white circle ○ any graph without edges, and by a black circle ● any complete graph. The line between two circles will mean that all possible edges between the corresponding graphs are present.

Let G_i $(i = 1, \dots, s)$ be a sequence of disjoint graphs, i.e. such that $V(G_i) \cap V(G_j) = \varnothing$ $(i \neq j)$. Denote by $P(G_1, \dots, G_n)$ the graph obtained from the direct sum $G_1 \dotplus \cdots \dotplus G_n$ by joining every vertex of G_i with every vertex of G_{i+1} $(i = 1, \dots, s-1)$. Next, denote by $Q(G_1, G_2, G_3, G_4, G_5)$ the graph obtained from $P(G_1, G_2, G_3, G_4)$ and G_5 by joining every vertex of G_5 with every vertex of G_2 and G_3. The graphs $P(K_m, K_n, K_p, K_q)$, $\overline{P(\bar{K}_m, \bar{K}_n, \bar{K}_p, \bar{K}_q) \dotplus \bar{K}_r}$, $P(K_1, K_m, \bar{K}_n, K_p, K_1)$, $\overline{P(K_1, K_m, \bar{K}_n, K_p, K_1) \dotplus \bar{K}_q}$ and $Q(\bar{K}_m, K_n, K_p, \bar{K}_q, K_1)$, depicted in Figure 2, will be essential for our purposes.

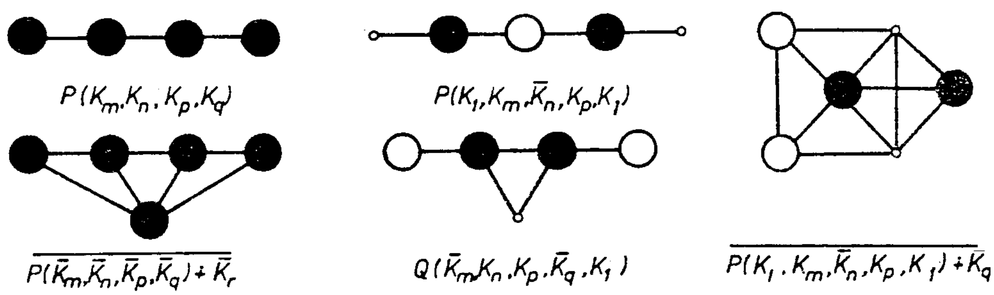

Fig. 2

First, we give some necessary and sufficient conditions under which any of the graphs $P(K_m, K_n, K_p, K_q)$, $\overline{P(\bar{K}_m, \bar{K}_n, \bar{K}_p, \bar{K}_q) \dotplus \bar{K}_r}$ is a 2-positive graph.

Lemma 3. *The graph $P(K_m, K_n, K_p, K_q)$ i a 2-positive graph if and only if the relation*

$$mnpq + m + n + p + q \leq mnp + npq + mp + mq + nq + 1 \tag{1}$$

holds.

Proof. Positive eigenvalues of the graph $P(K_m, K_n, K_p, K_q)$ are determined by the equation

$$\begin{aligned} D(\lambda) = (\lambda+1)^4 - (m+n+p+q)(\lambda+1)^3 + (mp+mq+nq)(\lambda+1)^2 \\ + (mnp+npq)(\lambda+1) - mnpq = 0. \end{aligned}$$

It is a matter of routine to see that this equation has exactly two positive roots if and only if $D(0) \geq 0$, i.e. if and only if the relation (1) holds. □

Lemma 4. *The graph $\overline{P(\bar{K}_m, \bar{K}_n, \bar{K}_p, \bar{K}_q) \dotplus \bar{K}_r}$ is a 2-positive graph if and only if the relation*

$$mnpqr + mnpq + m + n + p + q + r \leq mnq + mnr + mpq + npr + pqr + mn + np + pq + 1 \tag{2}$$

holds.

Proof. Positive eigenvalues of the graph $\overline{P(\bar{K}_m, \bar{K}_n, \bar{K}_p, \bar{K}_q) \dot{+} \bar{K}_r}$ are determined by the equation

$$D(\lambda) = (\lambda+1)^5 - (m+n+p+q+r)(\lambda+1)^4 + (mn+np+pq)(\lambda+1)^3$$
$$+ (mnq+mnr+mpq+npr+pqr)(\lambda+1)^2 - mnpq(\lambda+1) - mnpqr = 0.$$

Similarly as in Lemma 3, the above equation will have two positive roots if and only if $D(0) \geq 0$, i.e. if and only if the relation (2) holds.

LEMMA 5. *For all values of the parameters* $m, n, p, q \in \mathbf{N}$ *the graphs* $P(K_1, K_m, K_1, K_p, K_1)$, $\overline{P(K_1, K_1, \bar{K}_n, K_1, K_1) \dot{+} \bar{K}_q}$, *and* $Q(K_1, K_n, K_p, K_1, K_1)$ *are 2-positive graphs.*

Proof. Positive eigenvalues of the three graphs $P(K_1, K_m, K_1, K_p, K_1)$, $\overline{P(K_1, K_1, \bar{K}_n, K_1, K_1) \dot{+} \bar{K}_q}$, and $Q(K_1, K_n, K_p, K_1, K_1)$ are determined, respectively, by the following equations:

$$\lambda^4 - (m+p+2)\lambda^3 + (mp-3m-3p+1)\lambda^2 + (4mp-2m-2p)\lambda + 3mp = 0,$$
$$\lambda^3 - (n+q+1)\lambda^2 - 2(q+1)\lambda + 2n(q+1) = 0,$$
$$\lambda^4 - (n+p-2)\lambda^3 - (3n+3p-1)\lambda^2 + 2(np-n-p)\lambda + 3np = 0.$$

Since each of the equations above has exactly two positive roots, the statement is proved. □

2. Main results

Theorem 1. *Graph G is double 1-positive if and only if G or $\bar{G}$ is the graph* $P(\bar{K}_m, K_n)$ $(m \geq 2,\ n \geq 1)$.

Proof. As it is known, at least one of the graphs G and $\bar{G}$ must be connected. Without loss of generality, we can assume that G is a connected graph, since in the opposite case the proof is quite similar. Also it is known that complete multipartite graphs $K_{n_1,\dots,n_s}$ are the only connected 1-positive graphs ([**5**]). The proof is now an easy consequence of the relation $\bar{K}_{n_1,\dots,n_s} = K_{n_1} \dot{+} \cdots \dot{+} K_{n_s}$ and the fact that $K_{n_1} \dot{+} \cdots \dot{+} K_{n_s}$ is a 1-positive graph if and only if exactly one of the parameters $n_1, \dots, n_s$ is greater than 1. □

THEOREM 2. *Graph G is double 2-positive if and only if G or $\bar{G}$ is one of the following* 18 (*families of*) *graphs*:

1° $Q(\overline{K}_m, K_n, K_p, \overline{K}_q, K_1)$ $(m, n, p, q \geq 1)$;
2° $P(K_1, K_m, \overline{K}_n, K_p, K_1) \dot{+} \overline{K}_q$ $(m, n, p \geq 1;\ q \geq 0)$;
3° $P(K_m, K_n, K_p, \overline{K}_q) \dot{+} \overline{K}_r$ $(m, n, p, q \geq 1;\ r \geq 0)$;
4° $P(\overline{K}_m, \overline{K}_n, K_p, K_q) \dot{+} \overline{K}_r$ $(m, n, p, q \geq 1;\ r \geq 0)$;
5° $P(K_m, \overline{K}_n, \overline{K}_p, K_q) \dot{+} \overline{K}_r$ $(m, n, p, q \geq 1;\ r \geq 0)$;

6°	$P(K_m, K_n, K_p, K_q) \dotplus \overline{K}_r$	$(m, n, p, q \geq 1;\ r \geq 0;\ mnpq + m + n + p + q \leq mnp + npq + mp + mq + nq + 1)$
7°	$P(K_m \overline{K}_n, K_p, K_q) \dotplus \overline{K}_r$	$(m, n, p, q \geq 1;\ r \geq 0;\ mpq + m + p \leq 2mp + mq + pq)$;
8°	$P(\overline{K}_m, K_n, K_p, \overline{K}_q) \dotplus \overline{K}_r$	$(m, n, p, q \geq 1;\ r \geq 0;\ mqr \leq mq + mr + qr)$;
9°	$P(K_m, \overline{K}_n, K_p, \overline{K}_q) \dotplus \overline{K}_r$	$(m, n, p, q \geq 1;\ r \geq 0;\ nqr + 1 \leq 2nr + qr + n + q)$;
10°	$P(\overline{K}_m, \overline{K}_n, K_p, \overline{K}_q) \dotplus \overline{K}_r$	$(m, n, p, q \geq 1;\ r \geq 0;\ mnqr + m + r \leq mnr + mn + mq + nr + qr)$;
11°	$P(K_m, \overline{K}_n, \overline{K}_p, \overline{K}_q) \dotplus \overline{K}_r$	$(m, n, p, q \geq 1;\ r \geq 0;\ npqr + npq + p + q + r \leq npr + pqr + np + nq + nr + 2pq)$;
12°	$P(\overline{K}_m, \overline{K}_n, \overline{K}_p, \overline{K}_q) \dotplus \overline{K}_r$	$(m, n, p, q \geq 1;\ r \geq 0;\ mnpqr + mnpq + m + n + p + q + r \leq mnq + mnr + mpq + npr + pqr + mn + np + pq + 1)$;
13°	$P(K_m, K_n, K_p) \dotplus \overline{K}_q$	$(m, n, q \geq 1;\ p \geq 2)$;
14°	$P(\overline{K}_m, K_n, K_p) \dotplus \overline{K}_q$	$(m, n, q \geq 1;\ p \geq 2)$;
15°	$P(K_m, \overline{K}_n, K_p) \dotplus \overline{K}_q$	$(m, n, q \geq 1;\ p \geq 2)$;
16°	$P(\overline{K}_m, \overline{K}_n, K_p) \dotplus \overline{K}_q$	$(m, n, q \geq 1;\ p \geq 2)$;
17°	$P(\overline{K}_m, K_n) \dotplus K_p \dotplus \overline{K}_q$	$(m, p \geq 2;\ n \geq 1;\ q \geq 0)$;
18°	$P(\overline{K}_m, \overline{K}_n) \dotplus K_p \dotplus \overline{K}_q$	$(m, p \geq 2;\ n \geq 1;\ q \geq 0)$.

Proof. Let G or $\overline{G}$ be one of the graphs 1°–12° with corresponding values of the respective parameters. Then, with regard to Lemmas 2, 3, 4 and 5, we have $n^+(G) = n^+(\overline{G}) = 2$. For instance, if $G = Q(\overline{K}_m, K_n, K_p, \overline{K}_q, K_1)$ $(m, n, p, q \geq 1)$, then $\overline{G} = Q(\overline{K}_p, K_m, K_q, \overline{K}_n, K_1)$ and $n^+(G) = n^+(Q(K_1, K_1, K_n, K_p, K_1)) = 2$, $n^+(\overline{G}) = n^+(Q(K_1, K_1, K_m, K_q, K_1)) = 2$.

Now, let G or $\overline{G}$ be one of the graphs 13°–16°. Canonical graphs of these graphs are of the type $P(K_m, K_n, K_p) \dotplus K_1$ and canonical graphs of their complementary graphs are of the type $P(K_m, K_n, K_p)$ $(p \geq 2)$. Since, $P(K_m, K_n, K_p) \subseteq P(K_m, K_n, K_p) \dotplus K_1 \subseteq P(K_m, K_n, K_p, K_1) \dotplus K_1$ and $n^+(P(K_m, K_n, K_p, K_1)) = 2$, we have by Lemma 2 that $n^+(G) = n^+(\overline{G}) = 2$.

Next, let G or $\overline{G}$ be one of the graphs 17°–18°. Canonical graphs of these graphs are of the type $K_n \dotplus K_p \dotplus K_1$ $(n, p \geq 2)$ and canonical graphs of their complementary graphs are of the type $P(K_m, K_n, K_p)$ $(m \geq 2)$. Since $n^+(K_n \dotplus K_p \dotplus K_1) = 2$ and $n^+(P(K_m, K_n, K_p)) = 2$, we conclude by Lemma 2 that $n^+(G) = n^+(\overline{G}) = 2$.

This completes one part of the proof.

Now, assume that G is a double 2-positive graph. As it is known, at least one of the graphs G and $\overline{G}$ must be connected. Without loss of generality, we can assume that G is a connected graph, since the proof is quite similar otherwise.

Next, note that there is exactly one graph with 5 vertices and exactly 56 graphs with 6 vertices such that they or their complementary graphs are 3-positive graphs. 27 such graphs with at most 7 edges are depicted in Fig. 3. The mentioned 27 graphs suffice for our purposes. By the Interlacing theorem, any of the graphs G and $\overline{G}$ does not contain any of the above 57 graphs as an induced subgraph.

Since G is connected and is not a complete multipartite graph, by Lemma 1 it contains one of the graphs H_1, H_2 from Fig. 1 as an induced subgraph. We distinguish the following two cases:

(A) G contains the graph H_1 as an induced subgraph;

(B) G contains the graph H_2 and does not contain the graph H_1 as an induced subgraph.

Note that both graphs H_1 and H_2 are labelled, and the vertex set is $\{1, 2, 3, 4\}$ in both cases.

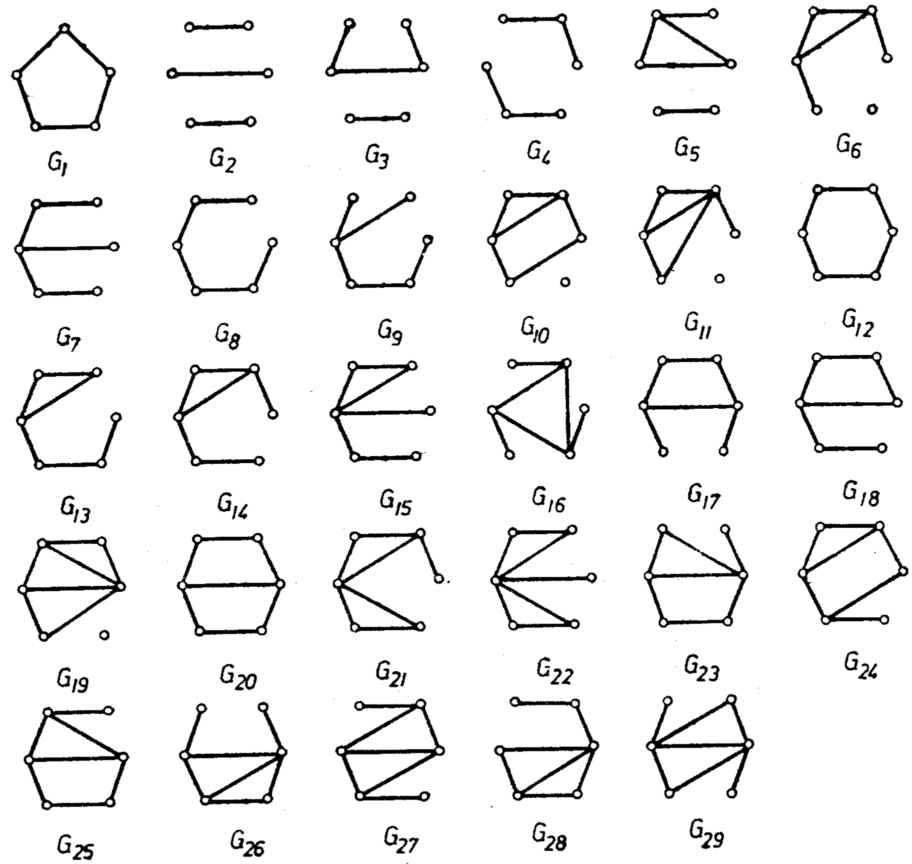

Fig. 3

Case A. Denote by $T_{i_1 \dots i_k}$ $(1 \le i_1 < \dots < i_k \le 4;\ 1 \le k \le 4)$ the set of all vertices in $V(G) \setminus V(H_1)$ which are adjacent exactly to the vertices $i_1, \dots, i_k$ of the graph H_1. Next, let T_0 be the set of all vertices in $V(G) \setminus V(H_1)$ which are nonadjacent to any vertex of the graph H_1.

The set T_{14} is empty (in the opposite case, we would have the contradiction $G_1 \subseteq G^1$). The set T_0 is also empty ($G_7 \subseteq G$ or $G_8 \subseteq G$ or $G_{14} \subseteq G$ or $G_{16} \subseteq G$ or $G_{17} \subseteq G$ or $G_{23} \subseteq G$ or $G_{26} \subseteq G$ or $G_{19} \subseteq \overline{G}$).

	T_1	T_4	T_{23}	T_{124}	T_{134}	T_2	T_3	T_{12}	T_{13}	T_{24}	T_{34}	T_{123}	T_{234}	T_{1234}
T_1	0_1	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	1	0	$\varnothing$	$\varnothing$	$\varnothing$	0	$\varnothing$
T_4		0_1	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	0	1	0	$\varnothing$	$\varnothing$
T_{23}			0_1	$\varnothing$	$\varnothing$	0	0	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	1	1	$\varnothing$
T_{124}				0_1	$\varnothing$	$\varnothing$	1	1	$\varnothing$	0	$\varnothing$	$\varnothing$	$\varnothing$	1
T_{134}					0_1	1	$\varnothing$	$\varnothing$	0	$\varnothing$	1	$\varnothing$	$\varnothing$	1
T_2						0	0	$\varnothing$	1	0	0	1	0	1
T_3							0	0	0	1	$\varnothing$	0	1	1
T_{12}								1	1	0	0	1	0	1
T_{13}									0	1	0	$\varnothing$	1	1
T_{24}										0	1	1	$\varnothing$	1
T_{34}											1	0	1	1
T_{123}												1	1	1
T_{234}													1	1
T_{1234}														1

Table 1

The adjacency relations between the sets T_1, T_2, T_3, T_4, T_{12}, T_{13}, T_{23}, T_{24}, T_{34}, T_{123}, T_{124}, T_{134}, T_{234} and T_{1234} in G are obtained by direct checking. They are presented in Table 1. The fact that corresponding sets are completely adjacent, completely nonadjacent or noncoexistent is denoted by the symbols 1, 0 and $\varnothing$, respectively. For example, we have that the sets T_1 and T_{12} are completely adjacent ($G_{14} \subseteq G$), while the sets T_1 and T_4 are noncoexistent, i.e. they cannot be nonempty at same time in the graph G ($G_8 \subseteq G$ or $G_{12} \subseteq G$).

In the same table adjacency relations in each of the mentioned sets are presented. So, we have that the graphs induced by the sets T_1, T_2, T_3, T_4, T_{13}, T_{23}, T_{24}, T_{124} and T_{134} have no edges whereas the graphs induced by the sets T_{12}, T_{34}, T_{123}, T_{234} and T_{1234} are complete. Besides, each of the sets T_1, T_4, T_{23}, T_{124} and T_{134} has at most one vertex, which is indicated by the symbol 0_1.

From Table 1 we conclude that the sets T_2 and T_{12} are noncoexistent, and that $1\ \rho_1\ T_2$ and $1\ \rho_2\ T_{12}$. Also, the sets T_3 and T_{34}, T_{13} and T_{123}, T_{24} and T_{234}

[1] We shall often simply say "$G_1 \subseteq G$".

are noncoexistent, and we have 4 ρ_1 T_3, 4 ρ_2 T_{34}, 2 ρ_1 T_{13}, 2 ρ_2 T_{123}, 3 ρ_1 T_{24} and 3 ρ_2 T_{234}.

Now, taking into account symmetry and excluding isomorphic graphs, we distinguish the following subcases:

(A.1) $T_1 \neq \varnothing$;

(A.2) $T_{23} \neq \varnothing$;

(A.3) $T_{124} \neq \varnothing$;

(A.4) $T_1 = T_4 = T_{23} = T_{124} = T_{134} = \varnothing$.

Ad (A.1). In this case the set of vertices $V(G)$ is a subset of the set $V(H_1) \cup T_1 \cup T_{12} \cup T_{13} \cup T_{234}$. Hence, the graph G is of the type $P(K_1, K_m, \overline{K}_n, K_p, K_1)$, and the graph $\overline{G}$ is of the type $P(K_1, K_m, \overline{K}_n, K_p, K_1)$.

The canonical graph of the graph G is of the type $P(K_1, K_m, K_1, K_p, K_1)$, while the canonical graph of the graph $\overline{G}$ is of the type $P(K_1, K_1, \overline{K}_n, K_1, K_1)$. By Lemmas 2 and 5 we conclude that $n^+(G) = n^+(\overline{G}) = 2$. Thus, the graph G is of the type 2° (with $q = 0$).

Ad (A.2). In this case the set of vertices $V(G)$ is a subset of the set $V(H_1) \cup T_2 \cup T_3 \cup T_{23} \cup T_{123} \cup T_{234}$, so that G is of the type $Q(\overline{K}_m, K_n, K_p, \overline{K}_q, K_1)$. Consequently, the graph $\overline{G}$ is of the type $Q(\overline{K}_n, K_q, K_m, \overline{K}_p, K_1)$.

The canonical graphs of the graphs G and $\overline{G}$ are respectively of types $Q(K_1, K_n, K_p, K_1, K_1)$ and $Q(K_1, K_q, K_m, K_1, K_1)$. By Lemmas 2 and 5 we get $n^+(G) = n^+(\overline{G}) = 2$, which means that G is of the type 1°.

Ad (A.3). In this case the set of vertices $V(G)$ is a subset of the set $V(H_1) \cup T_3 \cup T_{12} \cup T_{24} \cup T_{124} \cup T_{1234}$. Consequently, the graph G is of the type $\overline{P(K_1, K_m, \overline{K}_n, K_p, K_1) \dot{+} \overline{K}_q}$ and the graph $\overline{G}$ is of the type $P(K_1, K_m, \overline{K}_n, K_p, K_1) \dot{+} \overline{K}_q$.

The canonical graphs of the graphs G and $\overline{G}$ are respectively of the types $\overline{P(K_1, K_1, \overline{K}_n, K_1, K_1) \dot{+} \overline{K}_q}$ and $P(K_1, K_m, K_1, K_p, K_1) \dot{+} K_1$. By Lemmas 2 and 5 we have that $n^+(G) = n^+(\overline{G}) = 2$, which means that $\overline{G}$ is of the type 2°.

Ad (A.4). In this case, by Table 1, we have that the set of vertices $V(G)$ is a subset of one of the following 10 sets: $V(H_1) \cup T_2 \cup \cup T_3 \cup T_{13} \cup T_{234} \cup T_{1234}$, $V(H_1) \cup T_2 \cup T_{24} \cup T_{34} \cup T_{123} \cup T_{1234}$, $V(H_1) \cup T_{12} \cup T_{13} \cup T_{24} \cup T_{34} \cup T_{1234}$, $V(H_1) \cup T_2 \cup T_3 \cup T_{13} \cup T_{24} \cup T_{1234}$, $V(H_1) \cup T_2 \cup T_{13} \cup T_{24} \cup T_{34} \cup T_{1234}$, $V(H_1) \cup T_2 \cup T_3 \cup T_{123} \cup T_{234} \cup T_{1234}$, $V(H_1) \cup T_2 \cup T_{13} \cup T_{34} \cup T_{234} \cup T_{1234}$, $V(H_1) \cup T_2 \cup T_{34} \cup T_{123} \cup T_{234} \cup T_{1234}$, $V(H_1) \cup T_{12} \cup T_{13} \cup T_{34} \cup T_{234} \cup T_{1234}$ and $V(H_1) \cup T_{12} \cup T_{34} \cup T_{123} \cup T_{234} \cup T_{1234}$.

If $V(G)$ is a subset of the set $V(H_1) \cup T_2 \cup T_3 \cup T_{13} \cup T_{234} \cup T_{1234}$ then the graph G is of the type $\overline{P(K_m, K_n, K_p, \overline{K}_q) \dot{+} \overline{K}_r}$ and the graph $\overline{G}$ is of the type $P(K_m, K_n, K_p, \overline{K}_q) \dot{+} \overline{K}_r$. The canonical graphs of the graphs G and $\overline{G}$ are respectively of types $\overline{P(K_1, K_1, K_1, \overline{K}_q) \dot{+} \overline{K}_r}$ and $P(K_m, K_n, K_p, K_1) \dot{+} K_1$. By Lemmas 2, 3 and 4, we conclude that $n^+(G) = n^+(\overline{G}) = 2$, for all values of parameters $m, n, p, q \geq 1$, $r \geq 0$. Hence, the graph $\overline{G}$ is of the type 3°.

Similarly, we can show that in the remaining cases, the graph $\overline{G}$ is of the type 4°–12°, where the parameters $m, n, p, q \geq 1$, $r \geq 0$, satisfy the corresponding necessary conditions from Lemmas 3 and 4.

Case B. We distinguish the following three subcases:

(B.1) The graph G contains graph H_4 from Fig. 4 as an induced subgraph;

(B.2) The graph G contains graph H_5 and does not contain graph H_4 from Fig. 4 as an induced subgraph;

(B.3) The graph G does not contain graphs H_4 and H_5 from Fig. 4 as induced subgraphs.

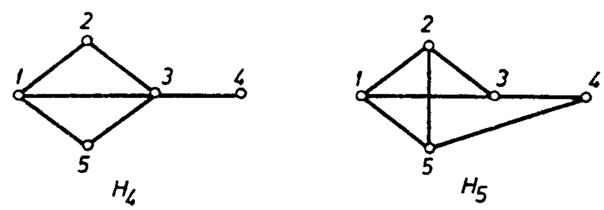

Fig. 4

Let us pay attention to the labelling of graphs H_4, H_5 from Fig. 4.

Ad (B.1). Let $T_{i_1 \dots i_k}$ and T_0 have the same meanings as in case (A), but with respect to the graph H_4 from Fig. 4.

In this case we have $T_1 = T_2 = T_4 = T_5 = T_{12} = T_{14} = T_{15} = T_{23} = T_{24} = T_{25} = T_{35} = T_{45} = T_{124} = T_{125} = T_{134} = T_{145} = T_{234} = T_{245} = T_{345} = T_{1234} = T_{1345} = T_{2345} = \varnothing$ $(H_1 \subseteq G)$, $T_{123} = T_{135} = \varnothing$ $(G_{11} \subseteq \overline{G})$, $T_{1245} = \varnothing$ $(G_5 \subseteq \overline{G})$ and $T_0 = \varnothing$ $(G_{11} \subseteq G)$.

Adjacency relations between vertices of the sets T_3, T_{13}, T_{34}, T_{235}, T_{1235} and T_{12345} in G are presented in Table 2. In the same table, adjacency relations in all of the mentioned sets are also indicated. Graphs induced by the sets T_3, T_{13} and T_{235} have no edges, while graphs induced by the sets T_{34}, T_{1235} and T_{12345} are complete graphs.

From this table we conclude that the sets T_3 and T_{34} are noncoexistent, and we have $4\ \rho_1\ T_3$ and $4\ \rho_2\ T_{34}$. The sets T_{235} and T_{1235} are also noncoexistent, and we have $1\ \rho_1\ T_{235}$ and $1\ \rho_2\ T_{1235}$. Besides, we have $2\ \rho_1\ T_{13}$, $5\ \rho_1\ T_{13}$ and $3\ \rho_2\ T_{12345}$.

Consequently, the set $V(G)$ is a subset of one of the following 4 sets: $V(H_4) \cup T_3 \cup T_{13} \cup T_{235} \cup T_{12345}$, $V(H_4) \cup T_3 \cup T_{13} \cup T_{1235} \cup T_{12345}$, $V(H_4) \cup T_{13} \cup T_{34} \cup T_{235} \cup T_{12345}$ and $V(H_4) \cup T_{13} \cup T_{34} \cup T_{1235} \cup T_{12345}$. It follows that G is one of the graphs $\overline{P(K_m, K_n, K_p) \dotplus \overline{K}_q}$, $\overline{P(\overline{K}_m, K_n, K_p) \dotplus \overline{K}_q}$, $\overline{P(K_m, \overline{K}_n, K_p) \dotplus \overline{K}_q}$ and $\overline{P(\overline{K}_m, \overline{K}_n, K_p) \dotplus \overline{K}_q}$, where $p \geq 2$. Thus $\overline{G}$ is one of the graphs $P(K_m, K_n, K_p) \dotplus \overline{K}_q$, $P(\overline{K}_m, K_n, K_p) \dotplus \overline{K}_q$, $P(K_m, \overline{K}_n, K_p) \dotplus \overline{K}_q$

	T_3	T_{13}	T_{34}	T_{235}	T_{1235}	T_{12345}
T_3	0	0	∅	0	0	1
T_{13}		0	0	1	1	1
T_{34}			1	0	0	1
T_{235}				0	∅	1
T_{1235}					1	1
T_{12345}						1

Table 2

and $P(\overline{K}_m, \overline{K}_n, K_p) \dotplus \overline{K}_q$, respectively. The graphs from these classes have the property $n^+(G) = n^+(\overline{G}) = 2$ for all values of parameters $m, n, q \geq 1$, $p \geq 2$.

We conclude that in this case the graph $\overline{G}$ is one of the graphs 13°–16°.

Ad (B.2). Let $T_{i_1 \dots i_k}$ and T_0 have the same meanings as in the case (A), but with respect to the graph H_5 from Fig. 4.

In this case we have $T_1 = T_2 = T_3 = T_4 = T_5 = T_{12} = T_{13} = T_{14} = T_{15} = T_{23} = T_{24} = T_{25} = T_{34} = T_{45} = T_{123} = T_{125} = T_{134} = T_{145} = T_{234} = T_{245} = T_{1345} = T_{2345} = \varnothing$ ($H_1 \subseteq G$), $T_{135} = T_{235} = \varnothing$ ($G_5 \subseteq \overline{G}$), $T_{1234} = T_{1245} = \varnothing$ ($G_4 \subseteq \overline{G}$) and $T_0 = \varnothing$ ($H_1 \subseteq G$ or $G_{11} \subseteq \overline{G}$).

The adjacency relations in the sets T_{35}, T_{124}, T_{345}, T_{1235} and T_{12345} and between these sets in the graph G are presented in Table 3. In particular, graphs induced by the sets T_{35} and T_{124} have no edges, while the graphs induced by the sets T_{345}, T_{1235} and T_{12345} are complete.

From this table we conclude that the sets T_{35} and T_{345} are noncoexistent, and we have 4 ρ_1 T_{35} and 4 ρ_1 T_{345}. We also find that 1 ρ_2 T_{1235}, 2 ρ_2 T_{1235}, 3 ρ_1 T_{124} and 5 ρ_1 T_{124}.

	T_{35}	T_{124}	T_{345}	T_{1235}	T_{12345}
T_{35}	0	1	∅	0	1
T_{124}		0	1	1	1
T_{345}			1	0	1
T_{1235}				1	1
T_{12345}					1

Table 3

Hence, the set of vertices $V(G)$ is a subset of one of the following 2 sets: $V(H_5) \cup T_{35} \cup T_{124} \cup T_{1235} \cup T_{12345}$, $V(H_5) \cup T_{124} \cup T_{345} \cup T_{1235} \cup T_{12345}$. It follows that G is one of the graphs $\overline{P(\overline{K}_m, K_n) \dotplus K_p \dotplus \overline{K}q}$ and $\overline{P(\overline{K}_m, \overline{K}_n) \dotplus K_p \dotplus \overline{K}_q}$ $(m, p \geq 2)$, and the graph $\overline{G}$ is one of the graphs $P(\overline{K}_m, K_n) \dot{+} K_p \dot{+} \overline{K}_q$ and $P(\overline{K}_m, \overline{K}_n) \dot{+} K_p \dot{+} \overline{K}_q$, respectively. The graphs from these classes have the property $n^+(G) = n^+(\overline{G}) = 2$ for all values of parameters $m, p \geq 2$, $n \geq 1$, $q \geq 0$.

We conclude that in this case graph $\overline{G}$ is one of the graphs 17° and 18°.

Ad (B.3). Let $T_{i_1 \ldots i_k}$ and T_0 have the same meanings as in the case (A), but with respect to the graph H_2 from Fig. 1.

Then we have $T_1 = T_2 = T_4 = T_{12} = T_{14} = T_{24} = T_{134} = T_{234} = \varnothing$ ($H_1 \subseteq G$), $T_{13} = T_{23} = \varnothing$ ($H_4 \subseteq G$), $T_{124} = \varnothing$ ($H_5 \subseteq G$) and $T_0 = \varnothing$ ($H_1 \subseteq G$ or $G_{11} \subseteq \overline{G}$).

Adjacency relations in the sets T_3, T_{34}, T_{123} and T_{1234} of G and between these sets are presented in Table 4. In particular, the graph induced by the set T_3 has no edges, while graphs induced by the sets T_{34}, T_{123} and T_{1234} are complete.

	T_3	T_{34}	T_{123}	T_{1234}
T_3	0	∅	0	1
T_{34}		1	0	1
T_{123}			1	1
T_{1234}				1

Table 4

From Table 4 we conclude that the sets T_3 and T_{34} are noncoexistent, and we have 4 ρ_1 T_3 and 4 ρ_2 T_{34}. We also have 1 ρ_2 T_{123}, 2 ρ_2 T_{123} and 3 ρ_2 T_{1234}. Consequently, the set $V(G)$ is a subset of one of the following two sets: $V(H_2) \cup T_3 \cup T_{123} \cup T_{1234}$ and $V(H_2) \cup T_{34} \cup T_{123} \cup T_{1234}$. Hence, the graph $\overline{G}$ is one of the types $P(\overline{K}_m, K_n) \dot{+} \overline{K}_p$ and $P(\overline{K}_m, \overline{K}_n) \dot{+} \overline{K}_p$, and the canonical graph of $\overline{G}$ is of the type $K_n \dot{+} K_1$ ($n \geq 2$). Since $n^+(K_n \dot{+} K_1) = 1$, we conclude that in this case there is no double 2-positive graph.

This completes the proof. □

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