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NOTE ON A PAPER BY H. L. MONTGOMERY (OMEGA THEOREMS FOR THE RIEMANN ZETA-FUNCTION)

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Abstract. We study Omega theorems for the expression $E = \mathrm{Re}(e^{i\theta}\log\zeta(\sigma_0 + it_0))$ where $1/2 \le \sigma_0 < 1$ and $0 \le \theta < 2\pi$ (σ_0, θ fixed) as $t_0 \to \infty$. In fact we prove $E \ge C(1-\sigma_0)^{-1}(\log t_0)^{1-\sigma_0}(\log\log t_0)^{-\sigma_0}$ for at least one t_0 in $[T^\varepsilon, T]$ where C is a positive constant. Note that $(1-\sigma_0)^{-1} \to \infty$ as $\sigma_0 \to 1$.

1. Introduction. In [**8**] Montgomery developed a method for studying Ω-theorems for $\mathrm{Re}(e^{-i\theta}\log\zeta(s_0))$, where θ is fixed such that $0 \le \theta < 2\pi$, the real part σ_0 of s_0 is fixed in $[1/2, 1)$, and the imaginary part t_0 tends to infinity. In fact he gets a lower bound for the maximum of this expression in $T^{(\sigma_0-1/2)/3} \le t_0 \le T$, namely

$$\frac{1}{20}(\sigma_0 - 1/2)^{1/2}(\log T)^{1-\sigma_0}(\log\log T)^{-\sigma_0}. \tag{1.1}$$

He obtains by his method that (on Riemann hypothesis)

$$|\zeta(1/2+it)| = \Omega(\exp(1/20(\log t/\log\log t)^{1/2})) \tag{1.2}$$

and that

$$\mathrm{Arg}\,\zeta(1/2+it) = \Omega((\log t/\log\log t)^{1/2}). \tag{1.3}$$

As a passing remark, it should be mentioned that in [**9**] Ramachandra has an alternative method for proving some of these results (sometimes in a stronger and sometimes in a weaker form); for example, for $\sigma \in [1/2, 1)$

$$\log|\zeta(\sigma+it)| = \Omega_\pm((\log t)^{1-\sigma-\varepsilon}). \tag{1.4}$$

He has also other results. (For the advantages of Ramachandra's method see Remark 1 after Main Theorem).

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The aim of this paper is (following the method of Montgomery and making an optimal use of his parameter α) to prove the following main theorem, which is perhaps the limit of his method. To state the main theorem, it is convenient to make a hypothesis. Let $N(\mu, T)$ denote the number of zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ satisfying $\beta \geq \mu$ and $|\gamma| \leq T$.

HYPOTHESIS. *For a fixed μ such that $1/2 < \mu < 1$ and all large T, there exists a $\delta > 0$ such that $N(\mu, T) \ll T^{1-\delta}$, where the implied constant depends on μ and δ.*

Remark. If we assume Riemann hypothesis, it suffices to take $\delta = 1$. However the hypothesis is always satisfied by any $\delta < 1 - 3(1-\mu)/(2-\mu)$, by the well-known density result of A. E. Ingham.

MAIN THEOREM. *Let $1/2 \leq \sigma_0 < 1$, $0 \leq \theta < 2\pi$, $\varepsilon > 0$. Let y be the positive solution of $e^y = 2y + 1$. Let l be an integer constant satisfying $l \geq 6$, $c_2 = 2y/(2y+1)^2$, $0 < c_1 < c_2$. Then for $T \geq T_0$ depending on these constants we have*

$$\mathrm{Re}(e^{-i\theta} \log \zeta(\sigma_0 + it_0)) \geq (1-\sigma_0)^{-1} c_0 c_1 (\log t_0)^{1-\sigma_0} (\log\log t_0)^{-\sigma_0}$$

for at least one t_0 in $T^{\varepsilon} \leq t_0 \leq T$, where $c_0 = (\cos(2\pi/l))(\delta/\log l)^{1-\sigma_0}$. Here $\delta = 1$ if we assume the Riemann hypothesis. Otherwise we have to assume $1/2 < \sigma_0 < 1$ and then we can take $\delta = 1 - 3(1-\sigma_0)/(2-\sigma_0)$.

Remark 1. An advantage of Montgomery's method is that it gives a better Ω-result for $|\zeta(\sigma + it)|$ than the result of Levinson and Ramachandra, namely

$$\log|\zeta(\sigma + it)| = \Omega_{\pm}\big((\log t)^{1-\sigma}/\log\log t\big) \qquad \text{for } 1/2 < \sigma < 1.$$

A disadvantage of Montgomery's method and that of Levinson [**7**] is that it does not work for short intervals. The method of Ramachandra does not assume Riemann hypothesis, works for short intervals, yields a slightly weaker result for $1/2 < \sigma_0 < 1$ than the result of Montgomery. Also Ramachandra's method, as developed by Ramachandra and Balasubramanian [**2**], works for L-series and so on. For example, it gives (without the assumption of Riemann hypothesis, etc.)

$$\log|L(1/2 + it, \chi)| = \Omega_{+}\big((\log t/\log\log t)^{1/2}\big)$$

and other results (see also [**10**]). Continuing [**2**] Balasubramanian [**1**] has shown that the constant can be taken to be $3/4$ for $\zeta(s)$.

Remark 2. The positive root of the equation $e^y = 2y + 1$ is $y = 1.2564312086\ldots$; so $c_2 = 1/4.910814964\ldots$. For example if we take $l = 10$, then $\cos(2\pi/l) \geq 0.809$ and $(\log l)^{1-\sigma_0} \geq 1/1.5175$. So we get, on the assumption of Riemann hypothesis,

$$|\zeta(1/2 + it)| = \Omega\big(\exp(c_3(\log t/\log\log t)^{1/2})\big) \qquad \text{as } t \to \infty,$$

where $c_3 = 1/4.60578\ldots$. This improves Theorem 2 of [**8**].

Remark 3. If σ_0 is very close to 1, then the constant we get, is boosted up very much because of the factor $(1-\sigma_0)^{-1}$.

Remark 4. By the method of this paper, it follows that for $1/2 < \sigma < 1$,

$$|\log\zeta(\sigma_0 + it_0)| \geq 2(1-\sigma_0)^{-1} c_0 c_1 (\log t_0)^{1-\sigma_0} (\log\log t_0)^{-\sigma_0}$$

for some t_0 satisfying $T^{\varepsilon} \leq t_0 \leq T$, where c_0 and c_1 are as in the main theorem.

Remark 5. The method of Montogomery is to start with

$$\begin{aligned}&\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} \big(e^{-i\theta}\log\zeta(s+s_0)\big)\left(\frac{e^{\alpha s}-e^{-\alpha s}}{s}\right)^2 (2 + x^s e^{i\theta} + x^{-s}e^{-i\theta})\,ds\\ &= \sum_{|\log(p/x)|\leq 2\alpha} p^{-s_0}\big(2\alpha - |\log(p/x)|\big) + \text{terms of smaller order,}\end{aligned}$$

break off $|\operatorname{Im} s| \geq \tau = (\log t_0)^2$ and then, move the line of integration in the rest of this integral to $\operatorname{Re} s = 0$. Notice now that $ds/i = dt$ and the quantity multiplying $e^{-i\theta}\log\zeta(s+s_0)$ is non-negative and hence we can take the real part of both sides and take out the maximum of the real part of $e^{-i\theta}\log\zeta(s+s_0)$ in $|t| \leq \tau$. This leads to a lower bound provided the real part of right-hand side is big. This is provided by the Dirichlet box principle. We work out the details in Sections 3 and 4. To get the result of Remark 4, we have to consider

$$\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} \big(\log\zeta(s+s_0)\big)\left(\frac{e^{\alpha s}-e^{-\alpha s}}{s}\right)^2 x^s\,ds$$

and treat it similarly.

Remark 6. The extension of the results to zeta functions of algebraic number fields presents no difficulties. It is verbatim the same as that for $\zeta(s)$. Some remarks will be made in Section 5 and the appendix.

2. Notation. Let

(1) $A, B, C, \dots, C_0, C_1, C_2, \dots$ denote positive constants;
(2) $f \ll g$ denote that $|f| \leq Ag$, where A is some positive constant;
(3) $\|\theta\|$ denote $\min_n |\theta - n|$, where n runs through all integers;
(4) $[x]$ denote the integral part of x.

3. Some lemmas. LEMMA 3.1. [**8**] *Let $\theta_1, \theta_2, \dots, \theta_M$ be distinct positive real numbers and suppose that $l \geq 6$, where l is an integer constant. Then there exist at least R integers r_k' such that $1 \leq r_k' \leq J = l^M R$ and $\|r_k'\theta_m\| < 1/l$ for $1 \leq m \leq M$.*

Proof. For the sake of completeness, we give the proof of this lemma. Consider the M-dimensional cube $(0 \leq x_i < 1,\ i = 1, 2, \dots, M)$ in the M-dimensional

Euclidean space. Let $l \geq 6$ be an integer constant. Divide each side of the cube into l equal parts:

$$[0, 1/l), \quad [1/l, 2/l), \quad \ldots, \quad [(l-1)/l, 1).$$

to get l^M small cubes. Put $J_0 = l^M R + 1$. Let $x_{j,r} = r\theta_j - m_{j,r}$ where $m_{j,r} = [r\theta_j]$, $1 \leq j \leq M$ and $1 \leq r \leq J_0$. Then clearly $0 \leq x_{j,r} < 1$. Consider the points $x_r = (x_{1,r}, x_{2,r}, \ldots, x_{M,r})$ in the unit cube. At least one small cube contains $R+1$ such points. Otherwise, since there are l^M small cubes and if each of them contained at most R points, then the total number of points would be $\leq l^M R$, which is a contradiction because of the fact that $1 \leq r \leq J_0$. Choose that particular small cube which contains $R+1$ points.

The set of these points is a subset of $[1, J_0]$. Let the corresponding values of r be $r_1, \ldots, r_{R+1}$ and $r_1 < r_2 < \ldots < r_{R+1}$. We see that there exists an A_j (independent of k) such that $0 \leq A_j \leq l-1$ and $A_j/l \leq x_{j,r_k} < (A_j+1)/l$ for every $k = 1, 2, \ldots, R+1$, $j = 1, 2, \ldots, M$. This implies that for suitable integers m_j^* we have $-1/l < (r_k - r_1)\theta_j - m_j^* < 1/l$ $\forall k = 2, 3, \ldots, R+1$, i.e. $\|(r_k - r_1)\theta_j\| < 1/l$, $\forall k = 2, 3, \ldots, R+1$.

Also we have $1 \leq r_k' \leq l^M R$ where $r_k' = r_k - r_1$, $\forall k = 2, 3, \ldots, R+1$. This proves the lemma.

Lemma 3.2. *Let $\theta_1, \theta_2, \ldots, \theta_M$ of Lemma 3.1 be $P \log p$ where P is a fixed positive integer and p runs through a finite set of primes containing those p satisfying $|\log(p/x)| \leq 2\alpha$, where $x \geq 10$ is fixed and α is a positive constant which will be fixed later. Let $l_{k-1} = r_k' P$ for every $k = 2, 3, \ldots, R+1$. Then we have*

(i) $\cos(2\pi l_k \log p) \geq \cos(2\pi/l)$ $\forall k = 1, 2, \ldots, R$.

(ii) $P \leq l_k \leq l^M RP$, $\forall k = 1, 2, \ldots, R$.

Proof. (i) From Lemma 3.1 we have $\|l_k \log p\| < 1/l$, $\forall k = 1, 2, 3, \ldots, R$. This proves (i).

(ii) From Lemma 3.1 we have $1 \leq r_k' \leq l^M R$, $\forall k = 2, 3, \ldots, R+1$. This implies that $P \leq l_k \leq l^M RP$, $\forall k = 1, 2, \ldots, R$.

Lemma 3.3. *For $1/2 \leq \sigma_0 < 1$, we have*

$$\sum_{|\log(p/x)| \leq 2\alpha} p^{-\sigma_0} \left(2\alpha - \left| \log\left(\frac{p}{x}\right) \right| \right) \sim \left(\frac{2\sinh(\alpha(1-\sigma_0))}{1-\sigma_0} \right)^2 \frac{x^{1-\sigma_0}}{\log x}.$$

Proof. The proof uses the fact that $\pi(x) \sim x/\log x$ and also Stieltjes integral and integration by parts.

Lemma 3.4. *If $x > 0$, $c > 0$, then we have*

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^s}{s^2}\, ds = \begin{cases} \log x, & \text{if } x \geq 1, \\ 0, & \text{if } 0 < x \leq 1. \end{cases}$$

Proof. The proof follows by moving the line of integration to the extreme left and the extreme right respectively.

LEMMA 3.5. *If $\alpha > 0$, $x > 0$ and $c > 0$, then we have*

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{e^{\alpha s}-e^{-\alpha s}}{2}\right)^2 x^s\,ds = \begin{cases} 2\alpha - |\log x| & \text{if } |\log x| \le 2\alpha, \\ 0 & \text{if } |\log x| \ge 2\alpha. \end{cases}$$

Proof. Follows from Lemma 3.4.

LEMMA 3.6. *Let $0 \le \theta < 2\pi$, $\alpha > 0$ and $1/2 \le \sigma_0 < 1$ be constants and let $s = \sigma + it$, $s_0 = \sigma_0 + it_0$. Then for all x, with $10 \le x \ll (\log t_0)\log\log t_0$ we have*

$$\begin{aligned}&\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty}\big(e^{-i\theta}\log\zeta(s+s_0)\big)\left(\frac{e^{\alpha s}-e^{-\alpha s}}{s}\right)^2(2+x^s e^{i\theta}+x^{-s}e^{-i\theta})\,ds\\ &= \sum_{|\log(p/x)|\le 2\alpha} p^{-s_0}\big(2\alpha - |\log(p/x)|\big) + O((\log x)^2).\end{aligned}$$

Proof. Follows from Lemma 3.5.

LEMMA 3.7. *Let $\tau = (\log t_0)^2$. If $\{\sigma > 0, |t| \le 2\tau\}$ is zero free for $\zeta(s+s_0)$, then in $0 < \sigma \le 1$ we have $\log\zeta(s+s_0) = O((\log t_0)(\log(2/\sigma)))$.*

Proof. See Theorem 9.6 (B) of [**12**].

LEMMA 3.8. *Let θ, α, σ_0 and t_0 be as in Lemma 3.6. The contribution of $|t| \ge (\log t_0)^2$ to the integral in Lemma 3.6 is $O((\log x)^2)$. Also the contributions from the integrals over $[i\tau, 1+i\tau]$ and $[-i\tau, 1-i\tau]$ are $O((\log x)^2)$.*

Proof. Follows from Lemma 3.7.

LEMMA 3.9. *With $\tau = (\log t_0)^2$, we have,*

$$\begin{aligned}&\mathrm{Re}\left\{\frac{1}{2\pi i}\int_{-i\tau}^{i\tau}\big(e^{-i\theta}\log\zeta(s+s_0)\big)\left(\frac{e^{\alpha s}-e^{-\alpha s}}{s}\right)^2(2+x^s e^{i\theta}+x^{-s}e^{-i\theta})\,ds\right\}\\ &= \sum_{|\log(p/x)|\le 2\alpha} p^{-\sigma_0}\cos(t_0\log p)(2\alpha - |\log(p/x)|) + O((\log x)^2)\end{aligned}$$

Proof. Follows from lemmas 3.6, 3.7 and 3.8.

LEMMA 3.10. *We have*

$$\max_{|t|\le\tau}\big(\mathrm{Re}\, e^{-i\theta}\log\zeta(s+s_0)\big)\frac{1}{2\pi i}\int_{|t|\le\tau,\,\sigma=0}\left(\frac{e^{\alpha s}-e^{-\alpha s}}{s}\right)^2(2+x^s e^{i\theta}+x^{-s}e^{-i\theta})\,ds$$

$$\geq \sum_{|\log(p/x)|\leq 2\alpha} p^{-\sigma_0}\cos(t_0\log p)(2\alpha - |\log(p/x)|) + O((\log x)^2).$$

Proof. Follows from Lemma 3.9.

LEMMA 3.11. *For $\tau = (\log t_0)^2$ and $2\alpha \leq |\log x|$ we have*

$$\frac{1}{2\pi i}\int_{|t|<\tau,\,\sigma=0}\left(\frac{e^{\alpha s}-e^{-\alpha s}}{s}\right)^2(2+x^s e^{i\theta}+x^{-s}e^{-i\theta})\,ds = 4\alpha + O(1/\tau).$$

Proof. Since the integrand is $O(t^{-2})$ for $|t| \geq \tau$, we have

$$\frac{1}{2\pi i}\int_{|t|\leq\tau,\,\sigma=0}\left(\frac{e^{\alpha s}-e^{-\alpha s}}{s}\right)^2(2+x^s e^{i\theta}+x^{-s}e^{-i\theta})\,ds$$
$$= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\left(\frac{e^{\alpha s}-e^{-\alpha s}}{s}\right)^2(2+x^s e^{i\theta}+x^{-s}e^{-i\theta})\,ds + O\left(\frac{1}{\tau}\right) = 4\alpha + O\left(\frac{1}{\tau}\right)$$

by using Lemma 3.5 and continuity.

LEMMA 3.12. *Let c be a positive constant to be chosen later. Let p be the set of primes satisfying*

$$ce^{-2\alpha}(\log P\log\log P) \leq p \leq ce^{2\alpha}(\log T\log\log T)$$

where we now refer to Lemma 3.2 and put $T = l^M RP$. M will be greater than or equal to the number of primes satisfying the inequalities just stated. We put $M = [(ce^{2\alpha}+\varepsilon)\log T]$ where $\varepsilon > 0$ is an arbitrary but fixed constant. Let $x = c\log t_0\log\log t_0$ where c is a small positive constant and $t_0 = 2\pi l_k$ $(k = 1, 2, \ldots, R)$ for any k. Then for all primes p satisfying $|\log(p/x)| \leq 2\alpha$, we have $\cos(t_0\log p) \geq \cos(2\pi/l)$. Thus

$$\sum_{|\log(p/x)|\leq 2\alpha} p^{-\sigma_0}\cos(t_0\log p)(2\alpha - |\log(p/x)|)$$
$$\geq \cos\left(\frac{2\pi}{l}\right)c^{1-\sigma_0}\left(\frac{2\sinh(\alpha(1-\sigma_0))}{1-\sigma_0}\right)^2\frac{(\log t_0)^{1-\sigma_0}}{(\log\log t_0)^{\sigma_0}}.$$

Proof. The lemma follows from Lemma 3.3.

4. Proof of the theorem. Consider the rectangles $\sigma \geq \sigma_0$, $|t_j - t| \leq 2(\log t_0)^2$, $(j = 1, 2, \ldots, R)$. These rectangles are disjoint and their number is R. If $R > DT^{1-\delta}+2$, where D is the constant coming from the hypothesis, then at least two of these rectangles are zero-free. We select the rectangle for which $t_0 + \tau \leq T$ (T to be defined) and we fix $P = T^{\varepsilon_1}$, $R = T^{1-\delta+\varepsilon_2}$, where ε_1, ε_2 are small positive constants. Then we put $M = [(ce^{2\alpha}+\varepsilon)\log T]$ and $l^M RP = T$. If we choose $c = \dfrac{\delta}{e^{2\alpha}\log l} - \dfrac{\varepsilon_3}{e^{2\alpha}\log l}$ for a small positive constant ε_3, then from lemmas 3.10, 3.11 and 3.12 we get (by putting $2\alpha(1-\sigma_0) = \beta$)

$$
\begin{aligned}
&\max_{|t|\le\tau} \mathrm{Re}(e^{-i\theta}\log\zeta(s_0+it)) \\
&\ge \frac{1}{4\alpha}\cos\left(\frac{2\pi}{l}\right)(\log l)^{-(1-\sigma_0)}\frac{\delta^{1-\sigma_0}}{e^{2\alpha(1-\sigma_0)}}\left(\frac{2\sinh(\alpha(1-\sigma_0))}{1-\sigma_0}\right)^2\frac{(\log t_0)^{1-\sigma_0}}{(\log\log t_0)^{\sigma_0}} \\
&= \frac{1}{2}\frac{\cos(2\pi/l)\delta^{1-\sigma_0}}{(\log l)^{1-\sigma_0}(1-\sigma_0)}\left(\frac{1-e^{-\beta}}{\sqrt{\beta}}\right)^2\frac{(\log t_0)^{1-\sigma_0}}{(\log\log t_0)^{\sigma_0}}.
\end{aligned}
$$

By choosing $\beta > 0$ such that $(1-e^{-\beta})/\sqrt{\beta}$ is maximum we see that this expression becomes

$$\frac{\cos(2\pi/l)}{(\log l)^{1-\sigma_0}}\cdot\frac{\delta^{1-\sigma_0}}{1-\sigma_0}\cdot c_1\cdot\frac{(\log t_0)^{1-\sigma_0}}{(\log\log t_0)^{\sigma_0}}$$

where c_1 is a positive constant independent of δ, l and σ_0, and $c_2 = 2y/(2y+1)^2 > c_1$, where y is the root of the equation $e^y = 2y+1$. This proves the theorem.

5. Generalizations. Let K be an algebraic number field. The Dedekind zeta-function of K is defined for $\mathrm{Re}\, s > 1$ by $\zeta_K(s) = \sum_{\mathfrak{A}\ne 0}(N\mathfrak{A})^{-s}$, where $N\mathfrak{A}$ denotes the norm of the ideal $\mathfrak{A}$ and the sum is extended over all non-zero integral ideals of the ring of integers of the field K. We know that $\zeta_K(s)$ can be continued analytically in $\sigma \ge 0$ and there we have $|(s-1)\zeta_K(s)| \ll (|t|+4)^{D_0}$, where D_0 is some positive constant.

Let $\log\zeta_K(s) = \sum_{n=1}^{\infty} e_n n^{-s}$ for $\sigma \ge 2$. We notice that $e_n \ge 0$, $\forall n$. Also it is well-known that

$$\sum_{n\le x} e_n \sim \sum_{p\le x} e_p \sim \sum_{N\mathfrak{P}\le x} 1 \sim \frac{x}{\log x} \tag{5.1}$$

holds, where the third sum from the left of (5.1) is extended over prime ideals. Let $N_K(\mu, T)$ denote the number of zeros $\rho = \beta + i\gamma$ of $\zeta_K(s)$ where $\beta \ge \mu$ and $|\gamma| \le T$. We make the following hypothesis.

Hypothesis $(*)$. *For a fixed μ such that $1 > \mu > 1/2$, there exists a $\delta > 0$ satisfying $N_K(\mu, T) \ll T^{1-\delta}$, where the constant depends on K, μ and δ.*

Assuming $(*)$ we can prove

Theorem. *Let $\mu < \sigma_0 < 1$, $T > T(\sigma_0)$. For any real θ, there is a t_0 such that $T^{\varepsilon} \le t_0 \le T$ and*

$$\mathrm{Re}\big(e^{-i\theta}\log\zeta_K(s_0)\big) \ge \frac{\cos(2\pi/l)}{(\log l)^{1-\sigma_0}}\cdot\frac{\delta^{1-\sigma_0}}{1-\sigma_0}\cdot c_1\cdot\frac{(\log t_0)^{1-\sigma_0}}{(\log\log t_0)^{\sigma_0}}$$

where c_1 is a positive constant independent of δ, l and σ_0 and $c_2 = 2y/(2y+1)^2 > c_1$, where y is the positive root of the equation $e^y = 2y+1$.

Remark. For the special cases see the appendix. The proof of this theorem is verbatim the same as that for $\zeta(s)$.

Appendix. Let K be an algebraic number field abelian over K'. Let the degrees of K and K' be n and k respectively. Then $\zeta_K(s)$ splits in the following way into abelian L-functions of K

$$\zeta_K(s) = L_1(s) \cdot \ldots \cdot L_j(s), \tag{A1}$$

where $j = n/k$. Let $N(L_i, \sigma, T)$ denote the number of zeros $\rho = \beta + i\gamma$ of $L_i(s)$ in (A1) such that $\beta \geq \sigma$ and $|\gamma| \leq T$. We distinguish the following two cases.

Case i. Suppose for every $L_i(s)$ in (A1) we have

$$|L_i(1/2 + it)| \ll t^{k/6+\varepsilon}, \qquad t \geq 1.$$

(This has been proved by Peter Sohne following the method of [**5**]. We are thankful to Professors D. R. Heath-Brown and W. Schaal for this information). In this case, it is not hard to prove by the standard methods that $N(L_i, \sigma, T) \ll T^{\lambda(1-\sigma)+\varepsilon}$ where $\lambda = (2k+6)/3$. So, we get $N(L_i, \sigma, T) \ll T^{1-\delta}$ if $\sigma > 1 - 3/(2k+6)$. Since the number of zeros is additive, we get

$$N_K(\sigma, T) \ll T^{1-\delta} \qquad \text{if} \qquad \sigma > 1 - 3/(2k+6).$$

In our hypothesis $(*)$, we can take any $\mu > 1 - 3/(2k+6)$.

Case ii. Let μ' be the smallest real number such that

$$\int_0^T |L_i(\mu' + it)|^2 \, dt \ll_\varepsilon T^{1+\varepsilon}.$$

(a) $\mu' = 1/2$ happens when $K' = \mathbf{Q}$ or $\mathbf{Q}(\sqrt{d})$, where d is an integer, not a perfect square. Then by the standard methods (see [**4**] or [**6**] or [**11**]), it follows that

$$N(L_i, \sigma, T) \ll T^{\frac{4(1-\sigma)}{3-2\sigma}+\varepsilon}.$$

In fact this is known uniformly for ordinary L-functions to the modulus $q \leq T$ (see [**4**]). Since the number of zeros is additive, we have

$$N_K(\sigma, T) \ll T^{1-\delta} \qquad \text{for} \qquad \sigma > 1/2.$$

In our hypothesis $(*)$, we can take any $\mu > 1/2$.

(b) If $\mu' > 1/2$ then by standard methods (see [**4**] or [**6**] or [**11**]), we get

$$N(L_i, \sigma, T) \ll T^{1-\delta} \qquad \text{for} \quad \sigma > \mu'.$$

Since the number of zeros is additive, it follows that

$$N_K(\sigma, T) \ll T^{1-\delta} \qquad \text{for} \quad \sigma > \mu'.$$

In our hypothesis $(*)$, we can take any $\mu > \mu'$.

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Added in proof. Independently of us J.-P. Kahane has some results in his paper: *Produits de Riesz et séries de Dirichlet*, in: *Analysis and Partial Differential Equations*, Marcel Dekker, New York and Basel, 1990, p.p. 231–238. However, his point of view is different and his methods are also different. We proved our result earlier.

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