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RELATIONS WITH I-STRUCTURE IN CATEGORIES WITH PULLBACKS

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Abstract. Theory of relations in both set-theoretical and in categorical approach, rarely is concerned with a possible existing structure between objects on which relations are defined. The aim of this paper is to give one model of relations having in mind a specific structure, the so-called I-structure, between objects in the domain of considered relations and to consider some properties of such category of relations.

Introduction

An n-ary relation R on sets $A_1, A_2, \dots, A_n$ is usually defined as a subset of $A_1 \times A_2 \times \cdots \times A_n$. In categories with pullbacks relations are defined by certain collections of morphisms. In both, common set-theoretical and in more general categorical approach possible existing relations between objects on which relations are defined, are rarely considered. The aim of this paper is to define one kind of (abstract) relational structure and to consider corresponding relations in a category K with pullbacks.

A relational structure I is defined as a kind of a free graph — category with arrows corresponding to existing connections between objects on which relations are considered. Relations with such kind of structure are taken as objects of a specific subcategory of the comma category $(K^I \downarrow D)$ where D is a functor ("domain-functor"), $D : I \longrightarrow K$. Objects of that subcategory $R_K(I, D)$ are natural transformations, namely those functors $R : I \longrightarrow K$ for which there exists a natural transformation (extension) $e : R \longrightarrow D$. Accordingly, some properties and operations are considered. Among other results, let us emphasize one that gives necessary and sufficient conditions for respecting certain limits by extensions. Those conditions enable us to recognize when a relation with I-structure decomposed by "projections" may be recomposed (by functional joins) into the primary one.

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Many authors considered relations for different categories, among others Y. Kawahara [**7**] and L. Coppey and R. Davar-Panah [**6**]. Binary relations, defined by pairs of morphisms in categories with pullbacks, have been studied in Kawahara's paper and decompositions and categories of relations are considered in [**6**]. Some views on different relational models are given by this author in [**2**] and [**3**]. The idea of considering abstract relational structure has come from paper by J. Rissanen [**10**]. Necessary categorical preliminaries may be found in both S. Mac Lane's [**8**] and E. Manes' book [**9**].

1. Relational structure

1.1. Let $(E, \leq)$ be an order. A *trivial relational structure* T is a category defined as follows. For an element X of E let X be an object of T; for X and Y objects of T, let the set of arrows $T(X,Y)$ consist of one arrow $X \longrightarrow Y$, whenever $Y \leq X$, otherwise let $T(X,Y) = \varnothing$.

1.2. A trivial relational structure T is a well-defined category. It defines a graph $G' = \bigcup T$, with the same objects as $\mathrm{Ob}(T)$ (as knots) forgetting which arrows are composition and which are identities. This graph is dual to one usually induced and directed by the given order.

Example 1. Let $M = (X, Y, Z)$ and $E = P(M)$. Consider $(P(M), \subset)$. A trivial relational structure contains among others the following arrows $M \longrightarrow \{X\}$, $\{X, Y\} \longrightarrow \{Y\}$, $X \longrightarrow \varnothing$, ... and one possible interpretation of an arrow in T is "... has more information than..."

1.3. Let $(E, \leq, \sup)$ be a sup-complete semi-lattice and $G' = \bigcup T$. Let N denote a new collection of arrows between some knots of the graph G' (not adding any new knots) and let $G = G' \cup N$. A *relational structure* $I := I(G)$ *induced by a graph* G is a category constructed in the following three steps:

(i) Enlarge the graph G by one new arrow $A \longrightarrow XY := \sup\{X, Y\}$ whenever G already contains two arrows $A \longrightarrow X$ and $A \longrightarrow Y$, $X \neq Y$, and apply this rule as long as new arrows may be produced. (Identify XX with X for $X \in E$.)

(ii) Construct a category whose objects are those of G and whose arrows are finite strings $A_1 \longrightarrow A_2 \longrightarrow \ldots \longrightarrow A_n$ composed of $n-1$ arrows $f_i : A_i \longrightarrow A_{i+1}$ of G, and regard that string as an arrow $A_1 \longrightarrow A_n$. The composition of these arrows is defined by juxtaposition of strings (therefore, associative) and the identity arrows are strings A_n of length 1.

(iii) If X and Y are objects of I, identify all arrows that belong to $\hom(X, Y)$. Then $\hom(X, Y)$ is either empty or consists of only one arrow.

Example 2. $(P(M), \subset, \cup)$, $M = \{X, Y, Z\}$, nontrivial arrows $\{Y\} \longrightarrow \{X\}$, $\{Z\} \rightarrow \{X\}$. A relational structure I consists of all (trivial) T-arrows, nontrivial arrows, $\{Y\} \rightarrow \{X\}$, $\{Z\} \longrightarrow \{X\}$ and new-constructed arrows: $\{Y\} \longrightarrow \{X, Y\}$, $\{Y, Z\} \longrightarrow \{X\}$.

1.4. Proposition. *A relational structure* $I := I(G)$ *is a well defined category and*

(a) *A trivial relational structure* T *is a subcategory of a corresponding relational structure* I,

(b) I *has finite products,* $X \times Y = \sup\{X, Y\}$ (*Note*: $XX = X$)

(c) I *has finite pullbacks*: *for a pair of arrows* $X \longrightarrow A$, $Y \longrightarrow A$ *pullback is* $XY \longrightarrow A$; *and*

(d) I *has an initial object* (*namely* $\sup E = 1$).

(e) *If* $(E, \leq)$ *is a complete lattice,* I *has initial and terminal objects.*

1.5. Let $(F, U) : \mathbf{Graph} \longrightarrow \mathbf{Kat}$ be a pair of adjoint functors between the category of (small) graphs and the category of (small) categories. FG is a free category constructed over a graph G of **Graph** and UK is a graph-like category under the forgetful functor U.

Proposition. *There exists a quotient category* $FG/_{=}$ *and a functor* Q, $Q : FG \longrightarrow FG/_{=}$ *such that:*

(a) *if* $f_1 : X \longrightarrow Y$ *and* $f_2 : X \longrightarrow Y$ *are arrows of* FG, *then* $Qf_1 = Qf_2$;

(b) *if* $H : FG \longrightarrow K$ *is any functor, satisfying* $Hf_1 = Hf_2$, *for all* $f_1, f_2 : X \longrightarrow Y$, *then there exists a unique functor* $H' : FG/_{=} \longrightarrow K$ *such that* $H'Q = H$;

(c) *there is an isomorphism between categories* $FG/_{=}$ *and* $I(G)$.

Proof. Functor Q is a bijection on objects and it maps all arrows from X to Y to a unique arrow $X \longrightarrow Y$ of $FG/_{=}$, so that for any X, Y objects of $FG/_{=}$ the set of arrows with domain X and codomain Y contains at most one element. The isomorphism between $FG/_{=}$and $I(G)$ exists by the construction of $I(G)$.

1.6. A morphism $M : I_1 \longrightarrow I_2$, of relational structures is a covariant functor which respects (preserves) products. A composition of morphisms of relational structures is the usual composition of covariant functors.

1.7. Proposition. Relational structures together with morphisms of relational structures form a category.

1.8. Proposition. *Arrows of a relational structure* I *possess the following properties*:

(a) $X \longrightarrow Y$, $X \longrightarrow Z$ *if and only if* $X \longrightarrow YZ$,

(b) *If* $X \longrightarrow Y$ *and* $V \longrightarrow W$ *then* $XV \longrightarrow YW$,

(c) *If* $X \longrightarrow Y$ *and* $YV \longrightarrow Z$ *then* $XV \longrightarrow Z$,

(d) $X \longrightarrow Y$ *if and only if* $X \longrightarrow XY$.

The proof is a consequence of well known lattice properties and the construction of products (universal) in relational structure I.

1.9. Let X be an arbitrary object of I and let $z(X)$ denote a collection of I-objects defined by the following:

(i) X is an object of $z(X)$,

(ii) If Y is an object of $z(X)$ and there exists in I an arrow $Y \longrightarrow V$, then V is an object of $z(X)$-collection, and

(iii) all objects of $z(X)$ are given by (i) and (ii).

Consider $z(X)$. If a relational structure has products, one may define a functor (endofunctor) $\mathrm{Cl} : I \longrightarrow I$, where $\mathrm{Cl}(X)$ is the product of all objects of $z(X)$, for any object X in I, and on arrows $\mathrm{Cl}(f) : \mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(Y)$ is induced by $z(X) \longrightarrow z(Y)$, for any arrow $f : X \longrightarrow Y$ in I.

Let $h : \mathrm{Cl} \longrightarrow 1$ and $k : \mathrm{Cl} \longrightarrow \mathrm{Cl}^2$ be two natural transformations defined on components by $h_X : \mathrm{Cl}(X) \longrightarrow X$ and $k_X : \mathrm{Cl} \longrightarrow \mathrm{Cl}(\mathrm{Cl}(X))$.

1.10. PROPOSITION. (Cl, h, k) *is a comonad in a relational structure* I *with products. The corresponding* Cl-*coalgebras are those objects of* I *for which* $\mathrm{Cl}(X) = X$.

2. Relations of the given relational structure

Any relational structure I defines, in an abstract manner, relations that may be defined starting form I and corresponding to I-objects suitable domains and subobjects of products of domains, having in mind already existing arrows between objects.

2.1. Let K be a category (base-category) with the following pairs of adjoint functors: $(\Delta, T_K) : K \rightharpoonup 1$ and $(\Delta, P') : K \rightharpoonup K^{\rightarrow\cdot\leftarrow}$ (i.e. K possesses a terminal object and pullbacks) and let $D : I \longrightarrow K$ be a covariant functor that respects products for different knots from I (domain-functor). A comma category $(K^I \downarrow D)$ defined for the following pair of functors $(\mathrm{id} : K^I \longrightarrow K^I,\ D : 1 \longrightarrow K^I)$ has as objects all those functors $S : I \longrightarrow K$ for which there exists a natural transformation $e^S : S \longrightarrow D$ and as morphisms all arrows (natural transformations) $\alpha : S \to S'$ such that $e^{S'}\alpha = e^S$ where $S' : I \longrightarrow K$.

2.2. Arrows of DT may be considered as "projections" and therefore the existence of the following morphisms in K is obvious:

(i) If $f : X \longrightarrow U$ and $g : Y \longrightarrow V$ are arrows of I there exists a unique arrow $r : DX \times DY \longrightarrow DU \times DV$ in category K such that $(p_U, p_V)r = (Dfp_X, Dgp_Y)$, where $p_X : DX \times DY \longrightarrow DX$, $p_Y : DX \times DY \longrightarrow DY$, $p_U : DU \times DV \longrightarrow DU$, $p_V : DU \times DV \longrightarrow DV$

(ii) If $X \longrightarrow Y$ and $YV \longrightarrow Z$ are I-arrows then there exists a unique arrow $s : DX \times DV \longrightarrow DZ$ such that $s = DgD(f, 1_V) \simeq Dg(Df, D1_V)$.

2.3. Natural transformations $e : R \longrightarrow D$ of $(K^I \downarrow D)$ with all components $e_X : RX \longrightarrow DX$ (mono) subobjects are called *extensions*.

For $X \in \mathrm{Ob}(I)$, the following diagram is commutative

$$\begin{array}{ccc} E & \quad & \begin{tikzcd} RE \arrow[r, "e_E"] \arrow[d, "q_X"'] & DE \arrow[d, "p_X"] \\ RX \arrow[r, "e_X"] & DX \end{tikzcd} \\ \downarrow & & \\ X & & \end{array}$$

Since $e : R \longrightarrow D$ is an extension, for a collection $\{X_i \mid i = 1, 2, \dots\}$ of I-objects, there is a morphism

$$(Dp)e_{\sup Xi} = e_{X_i} Rp : R(\sup X_i) \longrightarrow DX, \quad \text{where} \quad p : \sup X_i \longrightarrow X_i.$$

A monic arrow $e_{\sup X_i} : R(\sup X_i) \longrightarrow D(\sup X_i)$ defines $R(\sup X_i)$ as a subobject, of $D(\sup X_i)$.

2.4. A functor $R : I \longrightarrow K$ is a *relation with I-structure* (*I-relation*) whenever there exists in $(K^I \downarrow D)$ an extension $e : R \longrightarrow D$, D being a domain functor. Let $R, S : I \longrightarrow K$ be two I-relations. A morphism between two I-relations R and S is a natural transformation $t : R \longrightarrow S$ such that $te^R = e^S$ where $e^R : R \longrightarrow D$ and $e^S : S \longrightarrow D$ are extensions.

Example 3. (a) One simple interpretation of the Example 2 is the following: Let $DX = \{\text{addresses}\}$; $DY = \{\text{cities}\}$; $DZ = \{\text{phone numbers}\}$ and consider $R(XYZ)$ as a relation (in $DX \times DY \times DZ$) for a restricted number of cities (for example, in one state) and the corresponding addresses and phone numbers.

(b) Consider Example 1 with nontrivial arrows $X \longrightarrow Y$, $Y \longrightarrow Z$, $Z \longrightarrow X$ and let $D : I \longrightarrow \mathbf{Set}$ (preserving products for different knots) be given by $DX = DY = DZ = [0,1]$ and let $R : I \longrightarrow \mathbf{Set}$ be defined by $R(XYZ) = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$. Relation $e : R \longrightarrow D$ is determined by an embedding $e_{XYZ} : R(XYZ) \longrightarrow D(XYZ)$ and the corresponding projections: e_X, e_Y, e_Z, e_{XY}, e_{XZ}, e_{YZ}, e_{XYZ}.

2.5. An I-morphism $f : X \longrightarrow Y$ is *embedded into a relation* R if and only if $R(f)R(t_X) = R(t_Y)$ where $t_X : 1 \longrightarrow X$, $t_Y : 1 \longrightarrow Y$, $(1 = \sup E)$ are T-arrows.

2.6. Proposition. *I-relations and morphisms between them, in base category K and with the domain functor $D : I \longrightarrow K$, form a subcategory* $\mathrm{Rel} := \mathrm{Rel}_K(I, D)$ *of the comma category $(K^I \downarrow D)$ and the following properties are valid*:

(i) *For any I-relation R, $e : R \longrightarrow D$ and for any I-object X, $e_X Rt_X = Dt_X e_1$ where $Rt_X : R1 \longrightarrow RX$, $Dt_X : D1 \longrightarrow DX$ and $1 = \sup E$.*

(ii) *Let R be an* Rel-*object, $e : R \longrightarrow D$. Any I-morphism $f : X \longrightarrow Y$ is embedded into R and $(Df)e_X = e_Y Rf$.*

(iii) *Extensions of T-arrows are (mono-) restrictions of projections.*

Proof. (iii) If $t : X \longrightarrow Y$ is a T-arrow, then $\sup\{X, Y\} = X$ and hence $t : \sup\{X, Y\} \longrightarrow Y$. Further, $DX \times DY \simeq DXY \xrightarrow{Dt} DY$ is a projection and Rt is a restriction of the projection Dt.

2.7. PROPOSITION. *For any I-relation R from the category* Rel*,*

(a) *A morphism $(e_X, e_Y) : RX \times RY \longrightarrow DX \times DY$ is an embedding (monomorphism).*

(b) *There exists a unique monomorphism $m : R(XY) \longrightarrow RX \times RY$ such that $i(e_X, e_Y)m = e_{XY}$ (where $e_{XY} : R(XY) \longrightarrow D(XY)$, $i : DX \times DY \simeq D(XY)$).*

Proof. (a) By a standard categorical argument.

(b) Since XY is a product in I and $e : R \longrightarrow D$ is a natural transformation, $e_X Rt_X = p_X e_{XY}$ and $e_Y Rt_Y = p_Y e_{XY}$ where $e_{XY} : R(XY) \longrightarrow D(XY) \simeq DX \times DY$ is an XY-component of e. Since $DX \times DY$ is a product, (e_X, e_Y) is a unique morphism such that $i(e_X, e_Y)m = e_{XY}$. Also, since e_{XY} is monic, m is monic. If m is not unique, let $m, m_1 : R(XY) \rightrightarrows RX \times RY$, $m \neq m_1$. Then, $i(e_X, e_Y)m = e_{XY} = i(e_X, e_Y)m_1$ and since $i(e_X, e_Y)$ is monic, $m = m_1$.

2.8. LEMMA. *Let $\alpha : R \longrightarrow S$ be a morphism between two I-relations. An I-morphism $f : X \longrightarrow Y$ is embedded in both R and S, and the following connections are valid: $S(f)\alpha_X = \alpha_Y R(f)$, $(e^S)_X \alpha_X = (e^R)_X$ and $(e^S)_Y \alpha_Y = (e^R)_Y$.*

2.9. PROPOSITION. *A relation R from the category $\mathrm{Rel}_K(I, D)$ is determined in a unique way by a graph-morphism $h : G \longrightarrow UK$.*

Proof. An adjoint pair of functors (F, U) : **Graph** $\rightharpoonup$ **Kat** extends a morphism $h : G \longrightarrow UK$ to a unique functor $H : FG \longrightarrow UK$ and then by Proposition 1.4. it extends a functor H to a unique $H' : FG/_{=} \longrightarrow K$ so that $H'Q = H$.

3. Operations

3.1. *Projections* of a relation R with an extension $e : R \longrightarrow D$ are e-images of the corresponding projections in a relational structure.

Clearly, for a trivial arrow $t : XY \longrightarrow X$ the commutativity $(Dt)e_{XY} = e_X(Rt)$ illustrates the presence of one possible projection.

3.2. The *product* of two I-relations R and S, with extensions e^R and e^S, denoted by $e^R \times e^S : R \times S \longrightarrow D$, is defined by components

$$(e^R \times e^S)(X) := ((e^R)_X, (e^S)_X) : RX \times SX \longrightarrow DX.$$

3.3. For any given pair of arrows from the relational structure I, $f : Y \longrightarrow X$, $g : Z \longrightarrow X$, the *functional join of Rf and Rg* is a pullback of a pair of morphisms (Rf, Rg) in the base category K. It is denoted by $R(Y) \circ_X R(Z)$.

3.4. A *functional join of relations* R and S (with extensions e^R and e^S) is a pullback of a pair of morphisms (components) $(e^R)_X$, $(e^S)_X$ defined by $(R \circ S)(X) := RX \circ_{DX} SX$ and denoted by $e^R \circ e^R : R \circ S \longrightarrow D$. Actually, if RX and SX are treated as subobjects of DX, $(R \circ S)X$ is the intersection of the subobjects $(e^R)_X$ and $(e^S)_X$ in the partially ordered set of all subobjects of DX.

3.5. In the category **Set**, a composition of relations $R(XY)$ and $R(XZ)$ is defined in the usual way, by $R(XY) \circ R(XZ) := \{(x, y, z) \mid (x, y) \in R(XY), (x, z) \in R(XZ)\}$.

3.6. Proposition. *In the category of sets, a composition of relations $R(XY)$ and $R(XZ)$ is exactly the functional join of a pair of projections $p_X : R(XY) \longrightarrow RX$, $q_X : R(XZ) \longrightarrow RX$.*

3.7. Lemma. *Let $f : X \longrightarrow Y$ be an arrow of a relational structure I and R an I-relation. Then, $RX \circ_X RY \simeq RXY$.*

Proof. It is enough to prove that a morphism $r : RXY \longrightarrow RX \circ_X RY$, defined by the universal construction of a pullback for a pair of arrows $(R\,\mathrm{id}_X, Rf)$ is an isomorphism. Obviously, $p_Y r = \mathrm{id}_{RY}$ where $p_Y : RX \circ_X RY \longrightarrow RY$. Further, p_Y is a monomorphism. For let $r_1, r_2 : N \rightrightarrows RX \circ_X RY$ be a pair of different arrows in the base category K, with $p_Y r_1 = p_Y r_2$. Hence, $R(f)p_Y r_1 = R(f)p_Y r_2$ and since $R(f)p_Y = p_X$, it would be $p_X r_1 = p_X r_2$. Now, a pair of arrows $p_X r_1 = p_X r_2 : N \longrightarrow RX$, $p_Y r_1 = p_Y r_2 : N \longrightarrow RY$, together with id_{RX} and $R(f)$ forms a commutative square. By the universal property of pullback square, there exists a unique arrow $r_1 = r_2 : N \longrightarrow RX \circ_X RY$. Therefore, p_Y is a monomorphism and hence r is an isomorphism.

3.8. Corollary. *Let I be a relational structure with terminal object 0 and let R be an I-relation. Then,*

$$\text{(a)}\quad RX \circ_X RX \simeq RX, \qquad \text{(b)}\quad RX \circ_X R0 \simeq R0, \qquad \text{(c)}\quad R1 \circ_X RY \simeq RY.$$

The following proposition describes (existing) K-arrow between some objects — $R(YZ)$, $RY \circ_X RZ$, $RY \times RZ$.

3.9. Proposition. *There are unique K-monomorphisms $g : R(YZ) \longrightarrow RY \circ_X RZ$, $k : R(YZ) \longrightarrow RY \times RZ$, and $h : RY \circ_X RZ \longrightarrow RY \times RZ$ such that $hg = k$.*

Proof. Consider commutative diagrams (3.9) and the corresponding universal arrows:

$$\begin{tikzcd}
R(YZ) \arrow[r, "g"] \arrow[dr, "k"'] & RY \circ_X RZ \arrow[d, "h"] & \\
RY \arrow[r, "r_1"] \arrow[dr] & RY \times RZ \arrow[r, "r_2"] & RZ \arrow[dl] \\
& RX &
\end{tikzcd}$$

3.10. Proposition. *Let $Y \longrightarrow A \longleftarrow Z$ and $a : A \longrightarrow B$ be arrows of a relational structure I. Then, the chain of arrows $1 \longrightarrow A \longrightarrow B \longrightarrow 0$ induces, for* *any I-relation R, a chain of K-arrows*

$$R1 \xrightarrow{R(t)} R(YZ) \xrightarrow{gA} RY \circ_A RZ \xrightarrow{m} RY \circ_B RZ \longrightarrow RY \times RZ,$$

with the following equalities: $mg_A = g_B$, $g_B h_B = k = g_A h_A$, $g_A R(t) = t_A$, $g_B R(t) = t_B$, *where* $t_A : R1 \longrightarrow RY \circ_A RZ$, $t_B : R1 \longrightarrow RY \circ_B RZ$.

3.11. THEOREM. *For any I-relation in base category K and with the domain functor D, $R(XYZ) \simeq R(XY) \circ_X R(XZ)$ if and only if there exists in a relational structure I either an arrow $X \longrightarrow Y$ or $X \longrightarrow Z$.*

Proof. Without loss of generality, suppose $X \longrightarrow Y$ is an I-arrow. This arrow yields a unique arrow $X \longrightarrow XY$ and hence $X \simeq XY$. Then, since R is a functor, and by 3.7. $R(XY) \circ_X R(XZ) \simeq R(X) \circ_X R(XZ) \simeq R(XZ)$. On the other hand, the arrow $X \longrightarrow XY$ yields an (unique) arrow $XZ \longrightarrow XYZ$ and therefore, $R(XYZ) \simeq R(XZ)$. Hence $R(XYZ) \simeq R(XY) \circ_X R(XZ)$.

The converse is obvious by the following example: Let R be an I-relation in $\mathrm{Rel}_{\mathrm{Set}}(I, D)$ where G is a graph with three objects and no arrows and let $R(XYZ) = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$. Then $R(XY) \circ_X R(XZ) \not\simeq R(XYZ)$.

3.12. COROLLARY. *A functional join operation (whenever defined) has the following properties*:

(a) $RX \circ_1 RY \simeq RX \times RY$,

(b) $RX \circ_X RX \simeq RX$,

(c) $RX \circ_A RY \simeq RY \circ_A RX$,

(d) $(RX \circ_A RY) \circ_A RZ \simeq RX \circ_A (RY \circ_A RZ)$,

(e) $(RX \circ_A RY) \circ_B RZ \simeq RX \circ_A (RY \circ_B RZ)$,

(f) $(RX \circ_A RY) \circ_Y RZ \simeq RX \circ_A RZ$,

(g) $(RX \times RY) \circ_A RZ \simeq (RX \circ_A RZ) \times RY$.

4. Decompositions of relations

Corollary 3.12 suggests a generalization of a functional join operation to a successive join operation and, as its special case, multiple functional join.

4.1. For any given collection $W = \{(f_i, g_i) : \operatorname{dom} g_i = \operatorname{dom} f_{i+1},\ \operatorname{cod} g_i = \operatorname{cod} f_i,\ i = 1, 2, \dots\}$ of I-arrows, *R-successive functional join* is a limit for the diagram scheme $W_R := \{(Rf_i, Rg_i) \mid (f_i, g_i) \in W\}$. It is a K-object $\circ W_R$ together with a sequence of K-arrows $r_i : \circ W_R \longrightarrow R(\operatorname{dom} f_i)$, $i = 1, 2, \dots$ with the corresponding universal property: First, for any $i = 1, 2, \dots$ $R(g_i) r_{i+1} = R(f_i) r_i$, and second, for given collection of arrows $m_i : M \longrightarrow R(\operatorname{dom} f_i)$, $i = 1, 2, \dots$ for which $R(f_{i+1}) m_{i+1} = R(f_i) m_i$ there exists a unique arrow $t : M \longrightarrow \circ W_R$ such that $r_i t = m_i$ for $i = 1, 2, \dots$.

For a given collection of I-arrows $f_i : X_i \longrightarrow A_i$, $i = 1, 2, \dots$ a *multiple functional join* is an R-successive join for a collection $W = \{(f_i, f_{i+1}) \mid i = 1, 2, \dots\}$.

4.2. An extension $e : R \longrightarrow D$ from a category of relations $\mathrm{Rel}_K(I, D)$ *preserves* (*respects*) *the limit* of a functor $V : J \longrightarrow I$ whenever the following conditions are satisfied:

(L1) If $u : \Delta \lim V \longrightarrow V$ is the limit of a functor V, then $Ru : \Delta R \lim V \longrightarrow RV$ is the limit of the composition $RV : J \longrightarrow K$,

(L2) There exists a mono-natural transformation $\Delta : \lim RV \longrightarrow DV$, such that

(L3) $(eV)(Ru) = (Du)(\Delta e_{\lim V})$.

4.3. Let $Q(x, y)$ denote any diagram of the form $x \to \cdot \leftarrow y$, and let $V : Q(x, y) \longrightarrow I$ be a functor from the diagram category $Q(x, y)$ into a relational structure I such that the middle object in VQ cannot be the initial object.

Proposition. *An extension $e : R \longrightarrow D$ preserves the limit of a functor $V : Q(x, y) \longrightarrow I$ (i.e. binary functional join) whenever there exists in I either $V(\cdot) \longrightarrow V(x)$ or $V(\cdot) \longrightarrow V(y)$.*

The proof follows immediately from 3.11.

4.4. Corollary. *Let M' and M'' be such finite collections of I-arrows for which an extension $e : R \longrightarrow D$ preserves a successive functional join. If there exists a functor $V_1 : Q(x, y) \longrightarrow I$ such that $V_1(x) = \bigcup\{\mathrm{dom}\, f \mid f \in M'\}$ and $V_1(y) = \bigcup\{\mathrm{dom}\, f \mid f \in M''\}$ and e preserves the limit of a functor V_1 then, $e : R \longrightarrow D$ preserves a successive functional join for a collection of I-arrows $M = M' \cup M''$.*

Proof. By induction, from 4.3.

4.5. Proposition. *An extension $e : R \longrightarrow D$ preserves limit of $V : G \longrightarrow I$ for those and only those subfamilies of objects in I satisfying the following conditions*:

(i) *For any subdiagram $Q(x, y)$ from G for which* $\inf\{V(x), V(y)\} \neq 0$ *there exist a* $\lim(V\mid_Q)$, *and*

(ii) *there is no finite subdiagram Z of G with* $\mathrm{Ob}(VZ) = \{Z_1, Z_2, Y_1, Y_2, \dots, Y_n\}$ *for which*

(a) *an extension e preserves the limit of $V\mid_{Q(y_i, y_j)}$ for each $i, j = 1, 2, \dots, n$, but*

(b) *e preserves neither the limit of $V\mid_{Q(y_i, z_k)}$ nor $V\mid_{Q_{z_k, z_m}}$ for $i, j = 1, 2, \dots, n$ and $k, m = 1, 2$.*

To prove this Proposition one needs two following lemmas.

4.6. Lemma. *Let $G_\star$ be a diagram satisfying condition* (i) *of Proposition* 4.5 *and let Z be a finite subdiagram of $G_\star$ of the form* (ii) *in* 4.5. *Then, there is no extension $e : R \longrightarrow D$ preserving the limit of $V : G_\star \longrightarrow I$.*

Proof. It is enough to prove that for each $e : R \longrightarrow D$, $\lim RV(G_\star) \not\simeq R \lim V(G_\star)$. An image $V(G_\star)$ generates a subcategory of the relational structure

I. Let $C = \mathrm{Cl}(Y_1, \dots, Y_m)$. If Z_1 is an object in C, there are arrows $K \longrightarrow Y_1 \longrightarrow C \longrightarrow Z_1$ and hence $K \longrightarrow Z_1$, and therefore e preserves the limit of $V\ |_{Q(y_1,z_1)}$ which contradicts the assumption of this lemma. Similarly, for $Q(y_n, z_2)$. Therefore, neither Z_1 nor Z_2 does belong to the considered closure. On the other hand, since neither $Q(z_1, z_2)$ nor $Q(z_2, z_1)$ are subdiagrams of $G_\star$, for which e preserves $\lim(V\ |_Q)$ only I-arrows between Z_1 and Z_2 are $Z_1 \longrightarrow 1$ and $Z_2 \longrightarrow 1$, and hence $R(Z_1) *_1 R(Z_2) \simeq R(Z_1) \times R(Z_2)$. Now, $\lim R(V) \simeq R(Z_1) \times R(Z_2) * R(Y_1 Y_2 \dots Y_n)$, but $R(\lim V) \simeq R(Z_1 Z_2 Y_1 Y_2 \dots Y_n)$ and obviously, $R(\lim V) \not\simeq R(\lim V)$.

4.7. Lemma. *Under assumptions of the Proposition* 4.5, *for any two I-objects A and B from $V(G)$ there exists an object C from the relational structure I and subcollections $A = A_0, A_1, \dots, A_n = C$ and $C = B_0, B_1, \dots, B_m = B$, such that e preserves* $\lim V\ |_Q$ *for $Q(a_i, a_{i-1})$, and $Q(b_j, b_{j-1})$, ($i = 0, 1, \dots, n-1$ and $j = 0, 1, \dots, m-1$) where $Q(a_i, a_{i-1})$, and $Q(b_j, b_{j-1})$ are subdiagrams of G.*

The proof goes by induction on the number of objects between a and b, i.e. on the least number of objects $a = y_0, y_1, \dots, y_k = b$ such that $Q(y_i, y_j)$ is subdiagram of G for all $i, j = 1, 2, \dots, k$ (and e preserves $\lim(V\ |_Q)$).

$$
\begin{tikzcd}
\lim V \\
W \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
\lim V \arrow[r] & V \\
W \arrow[u] \arrow[ur]
\end{tikzcd}
\qquad
\begin{tikzcd}
\Delta D \lim V \arrow[r, "Du"] & DV \\
\Delta \lim RV = \Delta R(\lim V) \arrow[u, "\Delta e_{\lim V}"'] \arrow[r, "Ru"] & RV \arrow[u, "eV"'] \\
S \arrow[u] \arrow[ur]
\end{tikzcd}
\tag{4.5}
$$

$$I \qquad\qquad I^G \qquad\qquad K^I$$

Proof of the Proposition 4.5. Let $VG_\star$ be a collection of objects from the relational structure, satisfying (i). By (ii), there exists an I-object Z_1 from $V(G_\star)$ such that $VQ(z_1, a)$ is a subdiagram of $G_\star$ and e preserves $\lim(V\ |_Q)$ for each $V(a) = A$ from $G_\star$. If that is not true, let z_1 be such that $Q(z_1, a)$ is a subdiagram of $G_\star$ for maximal number of elements a from $G_\star$ and let Z_2 be an I-object for which $Q(z_1, z_2)$ is not a subdiagram of $G_\star$. By the Lemma 4.7 there exists an object $Z = V(z)$ with both $Q(z, z_1)$ and $Q(z, z_2)$ subdiagrams of $VG_\star$. But, that contradicts the maximality of a's, i.e. the maximality of A's. Hence, $VG_\star$ may be ordered as a sequence of I-objects $X_1, X_2, \dots, X_n, \dots$ such that $Q(x_i, x_j)$ for all $i < j$ is a subdiagram of $G_\star$ and e preserves $\lim(V\ |_Q)$. Let the limiting cone of V be given by a natural transformation $u : \Delta \lim V \longrightarrow V$. By 4.3 a limit of a functor $RV : G_\star \longrightarrow K$ is defined by $Ru : \Delta R(\lim V) = \Delta RV(G_\star) = \Delta \lim RV \longrightarrow RV$

and therefore, the proof follows by induction on the number i. Since $e : R \longrightarrow D$ is a mono-natural transformation, for each I-object B, e_B is monic in K. Since $\lim V$ is also an I-object, there exists a monomorphism $e'_{\lim V} : R(\lim V) \longrightarrow DV$ and hence (L2) of 4.2 is satisfied.

Under conditions given in 4.5, (L1) and (L2) of 4.2 are valid and for any $I^{G_\star}$ morphism $\Delta W \longrightarrow V$ and K^I-morphism $\Delta S \longrightarrow RV$ the diagrams labeled by (4.5) are commutative. Therefore, the condition (L3) of 4.2 is satisfied $(eV)(Ru) = (Du)\Delta e_{\lim V} = \lim eV$.

Conversely, if an extension $e : R \longrightarrow D$ of relation R from Rel respects either successive or multiple pullback for a subdiagram $G_\star$ of the relational structure I, we shall show that conditions (i) and (ii) are satisfied. If condition (i) doesn't hold, extension e doesn't respect pullbacks by the Corollary 4.5. In case (ii) doesn't hold, extension e doesn't preserve pullbacks by the Lemma 4.7.

Remark. A simple graph-like version of the question considered in this paragraph may be found in [**1**].

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