PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 50 (64), 1991, 111–122

RADIAL N-TH DERIVATIVES OF BOUNDED ANALYTIC OPERATOR FUNCTIONS

Dušan R. Georgijević

Abstract. We give, roughly, necessary and sufficient conditions, in terms of the Potapov-Ginzburg factorization, for the existence of N-th radial derivatives of bounded analytic operator functions. Our result is a generalization of the result of Ahern and Clark concerning scalar functions [**1**]. For inner matrix functions (in the case N odd) such a result was proved in [**2**].

1. Introduction. Throughout this paper H will be a fixed separable (non-trivial) Hilbert space. We denote by C, S_1 and S_∞, respectively, the spaces of all bounded, nuclear and compact linear operators from H into H. We will denote by $\|\cdot\|$ the norm in C (the uniform norm), and by $\|\cdot\|_1$ the norm in S_1 (the trace norm). The identity operator on H will be denoted by I. By D we denote the unit disc $|z|<1$ in the complex plane. Some operator functions with values in C or in S_1 will be considered. Boundedness, limits, derivatives, analyticity etc. of such functions will be understood in the sense of trace norm, except when it is stated otherwise.

Let $f : D \to C$ be an analytic operator function bounded by 1, in the sense of uniform norm. We will use the following continuation of f: if $|z|>1$ and $f(\overline{z}^{-1})$ is boundedly invertible, then

$$f(z) =: f(\overline{z}^{-1})^{*-1}. \tag{1}$$

The continued function f is analytic at every point z in its domain.

Given a function $f : D \to C$, we will consider the kernel $K(f;w,z) =: (1-\overline{w}z)^{-1}(I-f(w)^*f(z))$, $w,z \in D$. For the sake of shortness we shall write $K^{j,m}(f;w,z)$ instead of $\partial^{j+m}K(f;w,z)/\partial\overline{w}^j\partial z^m$, $j,m \in \mathbf{N}\cup\{0\}$.

Note 1. If f is analytic and bounded by 1, in the sense of uniform norm, then the kernel $K(f;w,z)$ is positive definite [**3**, **4**]. But, by [**5**] $K^{j,m}(f;w,z)$ is also positive definite.

AMS Subject Classification (1985): Primary 30 G 35, Secondary 47 B 38

Key words: analytic operator function, radial derivative, operator-valued kernel

2. The Potapov-Ginzburg factorization. The well known Potapov-Ginzburg factorization of bounded analytic operator functions [**6**], stated as Theorem 1 below, will play an important role in this paper.

Let G be the class of functions $\theta : D \to C$ analytic on D in the uniform norm and such that: (1) $\theta(z)^*\theta(z) \leq I$, $z \in D$; (2) there exists $\theta(0)^{-1} \in C$; (3) $\theta(0) - I \in S_1$.

THEOREM 1 [**6**]. *A necessary and sufficient condition for a function $\theta : D \to C$ to belong to the class G is that for every $z \in D$ its value $\theta(z)$ can be represented in the form*

$$\theta(z) = F(z) \cdot U \cdot B(z), \tag{2}$$

$$B(z) = \overset{\curvearrowright}{\prod}_{j=1}^{p} b_j(z) = \overset{\curvearrowright}{\prod}_{j=1}^{p} \left(\frac{|a_j|(a_j - z)}{a_j(1 - \overline{a}_j z)} P_j + (I - P_j) \right), \tag{3}$$

$$F(z) = \overset{\curvearrowleft}{\int_0^l} \exp\{-v(x,z)\, dE(x)\}, \tag{4}$$

where: $p \leq \infty$; $|a_j| < 1$; P_j are orthoprojectors, $\operatorname{Tr} P_j = \dim P_j H = p_j < \infty$; $\sum_{j=1}^{p} p_j(1-|a_j|) < \infty$; $v(x,z) = (1+e^{-iy(x)}z)(1-e^{-iy(x)}z)^{-1}$, *$y$ is a nondecreasing scalar function $(0 \leq y(x) \leq 2\pi)$; $E : [0,l] \to S_1$ is an Hermitian-increasing operator function satisfying* $\operatorname{Tr} E(x) = x$, $x \in [0,l]$; *U is a unitary operator for which $U - I \in S_1$. Here the partial products converge uniformly on compact subsets of D to the product of Blaschke-Potapov type $B(z)$, and in the same manner the integral products converge to the multiplicative integral $F(z)$.*

The function y in the factorization (2)–(4) can always be chosen to be left continuous and to take the value 2π only at the point $x = l$ or nowhere on $[0,l]$. From now on we will consider y as having these properties.

Note 2. It follows form Theorem 1 that $\theta(z) - I \in S_1$, $z \in D$, and that the function $\theta - I$ is analytic on D.

We denote by $B_m(z)$ the partial products of (3): $B_m(z) = \overset{\curvearrowright}{\prod}_{j=1}^{m} b_j(z)$, $1 \leq m \leq p$; $B_0(z) \equiv I$. We set also $B^m(z) = B(z)B_m(z)^{-1}$, $1 \leq m \leq p$. In connection with (4), we set $F_a^b(z) = \overset{\curvearrowleft}{\int_a^b} \exp\{-v(x,z)\, dE(x)\}$, $0 \leq a < b \leq l$. We write $F^u(z)$ instead of $F_0^u(z)$ and $F_u(z)$ instead of $F_u^l(z)$. Accordingly, we set $\theta^u(z) = F^u(z)UB(z)$.

Note 3. Let $y(x) \neq 0$, $x \neq 0$. It is not hard to see that then the function F_a^b is analytic at the point $z = 1$ and that $K(F_a^b; \cdot, \cdot)$ is analytic at the point $(\overline{w}, z) = (1,1)$, whenever $a \neq 0$ and $b \neq l$.

Note 4. If $\theta \in G$, then $\theta(z) - I \in S_1$ (as it was emphasized in Note 2), which implies that $\det \theta(z)$ exists for every $z \in D$ [**7**]. One can easily see that this determinant can be expressed in terms of the factorization (2)–(4):

$$\det \theta(z) = \prod_{j=1}^{p} \left(\frac{|a_j|(a_j - z)}{a_j(1 - \overline{a_j}z)^{-1}} \right)^{p_j} \cdot \exp\left\{ -\int_0^l v(x,z)\, dx \right\},$$

and that $\det\theta(z)$ is an inner function.

Definition 1. A function $\theta_1 \in G$ is called a (right) divisor of a function $\theta \in G$ if $\theta = \theta_0\theta_1$, $\theta_0 \in G$.

Note 5. It is not hard to see that, if θ_1 is a divisor of θ, then the kernel $K^{j,m}(\theta; w, z) - K^{j,m}(\theta_1; w, z)$ is positive-definite.

Note 6. Let $\theta \in G$ and let $d(z) =: \det\theta(z)$. Then for every divisor d_1 of d there exists a divisor θ_1 of θ such that $\det\theta_1(z) = d_1(z)$, $z \in D$. This divisor θ_1 is unique up to a unitary left multiplicator [**6**].

3. Auxiliary statements. LEMMA 1. *Let f be a function defined on $(0,1)$ and taking values in S_1 and let f possess the N-th derivative on $(0,1)$. Let the limits $\lim_{r\to 1-} f^{(j)}(r)$ $(:= f^{(j)}(1))$, $0 \le j \le N-1$, exist, and let $f^{(N)}(r)$ be bounded as $r \to 1-$. Then*

$$f(r) = \sum_{j=0}^{N-1} \frac{f^{(j)}(1)}{j!} \cdot (r-1)^j + \frac{g(r)}{N!} \cdot (r-1)^N, \qquad r \in (0,1), \tag{5}$$

where g is a function from $(0,1)$ into S_1, bounded as $r \to 1-$.

Proof. The function g is defined by (5), which also implies that $g(r) \in S_1$, $r \in (0,1)$. Let A be an arbitrary operator in S_∞. Applying the Taylor theorem (with remainder in the Lagrange form) to the real and imaginary parts of $\mathrm{Tr}(Af(r))$, we establish that $\mathrm{Tr}(Ag(r))$ can be represented in the form

$$\mathrm{Tr}(Ag(r)) = \mathrm{Re}(\mathrm{Tr}(Af^{(N)}(r_1))) + i\,\mathrm{Im}(\mathrm{Tr}(Af^{(N)}(r_2))),$$

for some points $r_1, r_2 \in (r, 1)$. Therefore we have

$$|\,\mathrm{Tr}(Ag(r))| \le 2\|A\| \sup\{\|f^{(N)}(\rho)\|_1 : r \le \rho < 1\}.$$

Since A is an arbitrary operator in S_∞ and since S_1 is the dual of S_∞ (via the trace duality), it follows that $\|g(r)\|_1 \le 2\sup\{\|f^{(N)}(\rho)\|_1 : r \le \rho < 1\}$ and the boundedness of $g(r)$ as $r \to 1-$ is established.

LEMMA 2. *Let $f : D \to C$ be an operator function bounded and analytic on D, in the sense of uniform operator convergence, and let $t \in \partial D$. If the radial limit $\lim_{r\to 1-} f(rt)$ exists in the sense of uniform operator convergence, then the non-tangential limit $\lim_{z\to t\,(\text{n.-t.})} f(z)$ also exists, again in the sense of uniform operator convergence.*

Proof. Assume that the radial limit is equal to 0, which does not affect generality. Suppose that $\|f(rt)\| \le \varepsilon$, $r \ge r_0$ $(r < 1)$ and take an angle α, $0 < \alpha < \pi$, with vertex at the point t and halved by the radius of the disc D ending at t. For any $a, b \in H$ the function $g(z) = \langle f(z)a, b\rangle$, $z \in D$, is a scalar function bounded and analytic on D. Moreover, for $r \ge r_0$ the following inequality $|g(rt)| \le \varepsilon\|a\|\,\|b\|$

holds. According to the classical Lindelöf theorem (see the proof of Theorem 3–5 in [**8**]), for each $z \in \alpha$ satisfying $|z| \geq r_1 = (r_0+1)/2$ the following holds: $|g(z)| \leq (\varepsilon\|a\|\,\|b\|)^{\lambda}$, where $\lambda > 0$ depends only on the angle α (and not on a and b). But, since $\|f(z)\| = \sup\{|\langle f(z)a, b\rangle| : a, b \in H,\ \|a\| = \|b\| = 1\}$, it follows that $\|f(z)\| \leq \varepsilon^{\lambda}$, $z \in \alpha$, $|z| \geq r_1$. This means that $\lim_{z \to t\,(\text{n.-t.})} f(z) = 0$ in the sense of uniform convergence, q.e.d.

4. Main result. THEOREM 2. *Let $\theta \in G$ and $t \in \partial D$.*

(i) *If N is an odd natural number, then the following are equivalent*:

(a_1) *The limit*

$$\lim_{r \to 1} \theta^{(j)}(rt) \quad (:= \theta^{(j)}(t)) \tag{6}$$

exists for $0 \leq j \leq N-1$, and the N-th derivative $\theta^{(N)}(rt)$ remains bounded as $r \to 1-$.

(b_1) *The derivative $f^{(N)}(rt)$ remains bounded as $r \to 1-$ and the limit $\lim_{r\to 1} f(rt)$ exists for $f = \theta$ and every divisor of θ.*

(c_1) *The limit* (6) *exists for all j, $0 \leq j \leq N$.*

(d_1) *The mixed partial derivative $K^{j,m}(\theta; rt, rt)$ remains bounded as $r \to 1-$, for $0 \leq j + m \leq N-1$.*

(e_1)

$$R_{N+1}(\theta) =: \sum_{j=1}^{p} |1 - \overline{a}_j t|^{-N-1}(1 - |a_j|)p_j + \int_0^1 |1 - e^{-iy(x)}t|^{-N-1}\,dx < \infty \tag{7}$$

(*with the notation introduced in Theorem* 1).

(ii) *If N is an even natural number, then the following are equivalent*:

(b_1).

(a_2) *The limit*

$$\lim_{r \to 1} f^{(j)}(rt) \tag{8}$$

exists for $0 \leq j \leq N$, for $f = \theta$ and every divisor of θ.

(b_2) *The mixed partial derivative $K^{j,m}(f; rt, rt)$ remains bounded as $r \to 1-$, for $0 \leq j+m \leq N-1$, for $f = \theta$ and every divisor of θ.*

(e_1).

We will begin the procedure of proving this theorem with the proof that (7) implies (a_2) for every nonnegative integer N, which we will give as a separate lemma. Actually, the lemma will contain slightly more, in accordance with what is needed in the course of the proof.

LEMMA 3. *In Theorem 2 condition* (7) *implies* (a_2), *for every nonnegative integer N. Even more, if* (7) *is satisfied, then there exist numbers $M_N > 0$ and*

$r_0 > 1$, *such that* $\|f^{(j)}(rt)\|_1 \leq M_N$, $r \in [0, r_0]$, $0 \leq j \leq N$, *for* $f = \theta$ *and every divisor of* θ.

Proof. We assume that $t = 1$, without loss of generality. It can be easily seen that (7) implies that for every angle $\alpha < \pi$ with vertex at the point 1, halved by the radius of the disc D ending at 1, there exists a disc of radius $r_1 < 1$ centered at the point 1, such that the intersection of this disc and the angle α does not contain any point a_j, i.e. that we have $\det \theta(z) \neq 0$ there. We assume that $\alpha = \pi/3$ and set $r_0 = 2/(2 - r_1)$.

The case $\theta(z) \equiv B(z)$. From (2) and (1) it follows that

$$B(r) - I = \sum_{m=1}^{p} (b_m(r) - I) B_{m-1}(r), \tag{9}$$

for $r \in [0, 1) \cup (1, r_0]$. Since

$$b_m(r) - I = (|a_m| - 1)(1 - \bar{a}_m r)^{-1} (|a_m|/a_m \cdot r + 1) P_m, \qquad r \in [0, 1) \cup (1, r_0], \tag{10}$$

and

$$|1 - \bar{a}_m r| > |1 - \bar{a}_m|/2, \qquad r \in [0, 1), \tag{11}$$

it follows that

$$\|b_m(r) - I\|_1 \leq 4 |1 - \bar{a}_m|^{-1} (1 - |a_m|) p_m, \qquad r \in [0, 1). \tag{12}$$

Taking into account that $\|B_{m-1}(r)\| \leq 1$, $r \in [0, 1)$, we see that the series (9) can be majorized by a convergent numerical series and that we have $\|B(r) - I\|_1 \leq 4R_1(B)$, $r \in [0, 1)$. From the uniform convergence just established of the series (9) on $[0, 1)$ it follows that

$$\lim_{r \to 1-} B(r) = \overset{\curvearrowleft}{\prod}_{j=1}^{p} b_j(1). \tag{13}$$

In order to establish analogous facts for $r > 1$, first let a_m be outside the angle α. Since then $|1 - \bar{a}_m r| \cdot |1 - \bar{a}_m|^{-1} > \sin(\alpha/2)$, it follows that for a_m outside the angle the inequality (11) is true also if $r > 1$, which means, according to (10), that (12) is satisfied for $r \in (1, r_0]$, with $2(r_0 + 1)$ instead of 4. For the remaining a_m's we must have $|1 - a_m| \geq r_1$ and therefore

$$|1 - \bar{a}_m r| \geq 1 - (1 - r_1) r_0 = r_0 - 1, \qquad r \in (1, r_0]. \tag{14}$$

In this case instead of (12) we have

$$\|b_m(r) - I\|_1 \leq (r_0 + 1)(r_0 - 1)^{-1} (1 - |a_m|) p_m, \qquad r \in (1, r_0]. \tag{15}$$

We also have to establish the boundedness of $B_{m-1}(r)$. Applying (13) to the scalar function $\det B(z)$ (the case $\dim H = 1$), we see that $S =: \sup\{|\det B(r)| : r \in (1, r_0]\} < \infty$. Now for $r \in (1, r_0]$ it follows that

$$\|B_{m-1}(r)\| \leq |\det B_{m-1}(r)| \leq |\det B(r)| \leq S, \tag{16}$$

and the boundedness is established. As S does not depend on m, the series (9) is majorized by a convergent numerical series and thus $\|B(r)-I\|_1 \leq 2(r_0+1)SR_1(B)+(r_0+1)(r_0-1)^{-1}SR_0(B)$. From the uniform convergence of the series (9) on $(1,r_0]$ it follows that (13) is also satisfied as $r\to 1+$, so that the limit $\lim_{r\to 1} B(r)$ exists.

It is clear that in the considerations above an arbitrary divisor of B can be put instead of B. (We note that the same r_0 can serve for all divisors.) This proves the statement for $N=0$.

We proceed by induction over N. Differentiating the equality

$$B'(z)=\sum_{m=1}^{p} B^m(z)b'_m(z)B_{m-1}(z) \tag{17}$$

$N-1$ times at the point $z=r$, we obtain

$$B^{(N)}(r)=\sum_{j=0}^{N-1}\binom{N-1}{j}\sum_{u=0}^{j}\binom{j}{u}\sum_{m=1}^{p}(B^m)^{(N-1-j)}(r)b_m^{(u+1)}(r)B_{m-1}^{(j-u)}(r), \qquad r\in[0,1)\cup(1,r_0].$$

By the induction hypothesis it suffices to show that the series

$$\sum_{m=1}^{p} B^m(r)b_m^{(N)}(r)B_{m-1}(r) \tag{18}$$

can be majorized by a convergent numerical series for $r\in[0,1)\cup(1,r_0]$. But, since

$$b_m^{(N)}(r)=-|a_m|/a_m\cdot(1-|a_m|^2)N!\bar{a}_m^{N-1}(1-\bar{a}_m r)^{-N-1}P_m, \tag{19}$$

it follows, according to (11), that for $r\in[0,1)$ we have

$$\|b_m^{(N)}(r)\|_1\leq 2^{N+2}N!|1-\bar{a}_m|^{-N-1}(1-|a_m|)p_m, \tag{20}$$

and the same is also true for $r\in(1,r_0]$ if a_m lies outside the angle α. For the remaining a_m's, according to (14), it follows that

$$\|b_m^{(N)}(r)\|_1\leq 2(r_0-1)^{-N-1}N!(1-|a_m|)p_m. \tag{21}$$

Now, the conclusion we needed about the series (18) follows easily. The same reasoning holds also for an arbitrary divisor of B.

The case $\theta(z)\equiv F(z)$. It is clear that (7) implies that $y(x)\neq 0$, $x\in(0,l]$. We will apply a reasoning analogous to that in the preceding case. In this case one can set $r_0=2$. The following equality is an analogue of (9):

$$F_a^b(r)-I=-\int_a^b v(u,r)\,dE(u)F_a^u(r), \qquad r\in[0,1)\cup(1,2],\ 0<a<b<l.$$

Instead of (11) and (14) we have now $|1-e^{-iy(u)}r| \geq |1-e^{-iy(u)}|/2$, $r \in [0,1) \cup (1,2]$. Instead of (12) and (15) the following estimate holds: $|v(u,r)| \leq 6|1-e^{-iy(u)}|^{-1}$, $r \in [0,1) \cup (1,2]$, and instead of (16) the following estimate:

$$\|F_a^u(r)\| \leq \exp\{6R_1(F)\}, \qquad r \in [0,1) \cup (1,2].$$

From these estimates it follows that

$$\|F_a^b(r) - I\|_1 \leq 6\exp\{6R_1(F)\} \int_a^b |1-e^{-iy(u)}|^{-1}\, du, \\ r \in [0,1) \cup (1,2], \quad 0 < a < b < l,$$

which shows that $F_\varepsilon^{l-\varepsilon}(r) \to F(r)$, as $\varepsilon \to 0+$, uniformly in $r \in [0,1) \cup (1,2]$. Hence it follows that the statement is true for $N = 0$, taking into account the fact that the function F_a^u is analytic at the point $z = 1$, for $0 < a < u < l$ (Note 3).

Further, the analogue of (17) is the relation

$$F'(z) = -\int_0^l F_u(z)[\partial v(u,z)/\partial z]\, dE(u) F^u(z).$$

As for the analogues of (19), (20) and (21), we will have now

$$\partial^N v(u,r)/\partial z^N = 2e^{-iNy(u)} N!(1-e^{-iy(u)}r)^{-N-1},$$

and hence

$$|\partial^N v(u,r)/\partial z^N| \leq 2^{N+2} N! |1-e^{-iy(u)}|^{-N-1}, \qquad r \in [0,1) \cup (1,2].$$

The rest is clear.

The general case. The statement in the general case follows now from the factorization (2) and from the statements already proved for the previous cases.

Proof of Theorem 2. We can assume that $t = 1$, without loss of generality.

The case $N = 1$. It is clear that in this case the implications $(c_1) \Rightarrow (b_1)$ and $(b_1) \Rightarrow (a_1)$ are true.

$(a_1) \Rightarrow (d_1)$. Existence of the limit (6) for $j = 0$ means that $\lim_{r \to 1-} \theta(r) = \lim_{r \to 1+} \theta(r) = \theta(1)$, where, according to (1), we must have $\theta(1)^{-1} = \theta(1)^*$. By Lemma 1, we have $\theta(r) = \theta(1) + (r-1)g(r)$, $r \in (0,1)$, where g is a bounded operator function. Therefore $K(\theta; r, r) = (r+1)^{-1}(2\operatorname{Re}(\theta(1)^* g(r)) + (r-1)g(r)^* g(r))$, and hence the boundedness of $K(\theta; r, r)$, as $r \to 1-$, follows immediately.

$(d_1) \Rightarrow (e_1)$. According to Note 5, and to factorization (2), the boundedness of $K(\theta; r, r)$ as $r \to 1-$ implies that we have, for some $M > 0$ and some r_0, $0 < r_0 < 1$,

$$\operatorname{Tr} K(B; r, r) \leq M, \qquad r \in [r_0, 1), \tag{22}$$

$$\operatorname{Tr}(B(r)^* U^* K(F; r, r) U B(r)) \leq M, \qquad r \in [r_0, 1). \tag{23}$$

It is easy to see that

$$K(B;w,z)=\sum_{m=1}^{p}k_m(w,z),\qquad w,z\in D, \tag{24}$$

where $k_m(w,z)=B_{m-1}(w)^*K(b_m;w,z)B_{m-1}(z)$. From (22) and (24) it follows that

$$\sum_{m=1}^{p}\operatorname{Tr}(k_m(r,r))\le M,\qquad r\in[r_0,1). \tag{25}$$

But, as $\operatorname{Tr}(k_m(r,r)) = |1-\bar{a}_m r|^{-2}(1-|a_m|^2)\operatorname{Tr}(B_{m-1}(r)^*P_mB_{m-1}(r))$ and $\operatorname{Tr}(B_{m-1}(1)^*P_mB_{m-1}(1))=\operatorname{Tr}P_m$, we obtain from (25), letting $r\to 1-$, that

$$R_2(B)\le M. \tag{26}$$

In order to establish that such an inequality is valid also for F, we will apply a reasoning analogous to the above. Now instead of (24) we have

$$\begin{aligned}&B(w)^*U^*K(F;w,z)UB(z)\\ &\quad=(1-\overline{w}z)^{-1}\int_0^l\theta^u(w)^*(\overline{v(u,w)}+v(u,z))\,dE(u)\theta^u(z),\qquad w,z\in D.\end{aligned} \tag{27}$$

Instead of (25) we have, by (23),

$$(1-r^2)^{-1}\int_0^l 2\operatorname{Re}v(u,r)\operatorname{Tr}(\theta^u(r)^*dE(u)\theta^u(r))\le M,\qquad r\in[r_0,1). \tag{28}$$

It is clear by Note 5 that (22) remains valid also if B is replaced by an arbitrary divisor θ_1 of θ. Hence it follows that for every $h\in H$ satisfying $\|h\|=1$ we have

$$1-\|\theta_1(r)h\|^2\le M(1-r_0^2),\qquad r\in[r_0,1). \tag{29}$$

Here we may assume that $M(1-r_0^2)<1$ in which case (29) implies

$$\|\theta_1(r)^{-1}\|\le\bigl(1-M(1-r_0^2)\bigr)^{-1/2}\ (:=S),\qquad r\in[r_0,1). \tag{30}$$

Putting $\theta^u(r)$ instead of $\theta_1(r)$ in (30) we can easily establish the following inequality: $\operatorname{Tr}(\theta^u(r)^*dE(u)\theta^u(r))\ge S^{-2}\,du$, $r\in[r_0,1)$, $u\in[0,l]$. With this inequality in hand, according to the fact that $(1-r^2)^{-1}\operatorname{Re}v(u,r)=|1-e^{-iy(u)}r|^{-2}$, we let $r\to 1-$ in (28), and so we obtain

$$2S^{-2}R_2(F)\le M. \tag{31}$$

Since it is $R_2(\theta)=R_2(B)+R_2(F)$, the statement follows from (26) and (31).

$(\mathrm{e}_1)\Rightarrow(\mathrm{c}_1)$. Established in Lemma 3.

We proceed by induction over N. Of course, we separate the case N even and the case N odd.

The case $N = 2n$, $n \in \mathbf{N}$. It is clear that $(\mathrm{a}_2) \Rightarrow (\mathrm{b}_1)$ is true.

$(\mathrm{b}_1) \Rightarrow (\mathrm{b}_2)$. By the induction hypothesis, we may assume that (a_1) is satisfied, for θ and also for every divisor of θ. Existence of the limit (8) for $j = 0$ implies the possibility of analytic continuation of the function θ to some segment $z = r$, $1 < r < r_0$. Hence it follows that $K(\theta; w, z)$ is analytic in $\overline{w}$ and z at every point $(\overline{w}, z) = (\rho, r)$, $\rho, r \in [0, 1) \cup (1, r_0]$. Let $L(w, z) = (1 - \overline{w}z)^{j+m+1} K^{j,m}(\theta; w, z)$. Assume that $w = z = r$, $r \in (0, 1)$. By Lemma 1 $\theta^{(\nu)}(r) = \sum_{u=0}^{N-1-\nu} (u!)^{-1} \theta^{(\nu+u)}(1)(r-1)^u + [(N-\nu)!]^{-1} g_\nu(r)(r-1)^{N-\nu}$, $r \in (0,1)$, where the function g_ν remains bounded as $r \to 1-$, for $\nu = 0, 1, \ldots, N-1$. Substituting this into $L(r, r)$ we can derive the formula (5) for the function $L(r, r)$, with $j + m + 1$ instead of N. We will show that here the coefficients by $(r-1)^u$ for $u \leq j + m$ must vanish. The coefficient by $(r-1)^u$ equals to the expression

$$(\partial/\partial\overline{w} + \partial/\partial z)^{(u)} L(1, 1), \tag{32}$$

where all the derivatives $\theta^{(\nu)}(1)$ for $\nu \geq N$ are replaced by 0. As the function $L(\rho, r)$ is "divisible" by $(1 - \rho r)^{j+m+1}$, it follows that at every point $(\rho, r) = (r_1^{-1}, r_1)$ ($r_1 \in (r_0^{-1}, 1) \cup (1, r_0)$) all its partial derivatives of order less than $j + m + 1$ must vanish, so that $((\partial/\partial\overline{w}) \cdot r_1^{-1} + (\partial/\partial z) \cdot r_1)^{(u)} L(r_1^{-1}, r_1) = 0$ for $u \leq j + m$. It is clear that derivatives of θ of order grater than $N - 1$ do not enter in the expression above. Now, letting $r_1 \to 1$, we establish that (32) vanishes for $u \leq j + m$. Thus, the formula (5) for $L(r, r)$ reduces to: $L(r, r) = [(j + m + 1)!]^{-1} g(r)(r-1)^{j+m+1}$, where $g(r)$ stays bounded as $r \to 1-$. Here it is shown that $K^{j,m}(\theta; r, r) = (1 - r^2)^{-(j+m+1)} L(r, r)$ stays bounded as $r \to 1-$, for $0 \leq j + m \leq N - 1$. Clearly, in the reasoning above an arbitrary divisor of θ can stay instead of θ.

$(\mathrm{b}_2) \Rightarrow (\mathrm{e}_1)$. By the induction hypothesis and Lemma 3, we may assume that $R_N(f) < \infty$ and $\|f^{(j)}(r)\|_1 \leq M_{N-1}$, $r < 1$, $0 \leq j \leq N - 1$, for $f = \theta$ and every divisor of θ, and also that the limit $\lim_{r \to 1} f(r) := f(1)$ exists and that $f(1)$ is a unitary operator for $f = \theta$ and every divisor of θ.

First let $\theta(z) \equiv B(z)$ and $\operatorname{Im} a_j > 0$, all j, or $\operatorname{Im} a_j < 0$, all j, and $a_j \notin \alpha$, all j, where α is the angle introduced at the beginning of the proof of Lemma 3. By differentiating (24) for $w = z = r$ we can obtain

$$K^{n-1,n}(B; r, r) = \sum_{m=1}^{p} \frac{\partial^{2n-1} k_m(r, r)}{\partial \overline{w}^{\,n-1} \partial z^n}. \tag{33}$$

The boundedness of the right-hand side as $r \to 1-$ and the induction hypothesis together with the definition of k_m imply the boundedness as $r \to 1-$ of the expression

$$\sum_{m=1}^{p} B_{m-1}(r)^* K^{n-1,n}(b_m; r, r) B_{m-1}(r). \tag{34}$$

But since

$$K^{n-1,n}(b_m; r, r) = (n-1)!n! \frac{|a_m|^{2n-2} \bar{a}_m (1 - |a_m|^2)}{|1 - \bar{a}_m r|^{2n} (1 - \bar{a}_m r)} P_m,$$

since

$$\operatorname{Im}\frac{\bar a_m}{1-\bar a_m r}=\operatorname{Im}\frac{\bar a_m}{|1-\bar a_m r|^2}\quad(<0,\text{ all } m,\text{ or } >0,\text{ all } m)$$

and $|\operatorname{Im}\bar a_m|\cdot|1-\bar a_m|^{-1}>\sin(\alpha/2)$, it follows that the expression

$$\sum_{m=1}^{p}\frac{|1-\bar a_m|(1-|a_m|^2)}{|1-\bar a_m r|^{2n+2}}B_{m-1}(r)^*P_mB_{m-1}(r)$$

also stays bounded as $r\to 1-$. Hence it follows easily that $R_{N+1}(B)<\infty$.

In the case when $\theta(z)\equiv B(z)$ and $a_j\in\alpha$ for all j, it follows by induction hypothesis and Lemma 2 that there exists an r_1, $0<r_1<1$, such that $|a_j-1|>r_1$ for all j. Hence $R_{N+1}(B)\le r_1^{-N-1}R_0(B)$ follows.

Assume now that $\theta(z)\equiv F(z)$, with $0\le y(x)<5\pi/6$ or $7\pi/6<y(x)\le 2\pi$, all x. In the proof of this case we will follow the analogy with the proof of the first of cases considered above. First, by differentiating the equality

$$K(F;w,z)=\int_0^l F^u(w)^*k_u(w,z)\,dE(u)F^u(z)$$

for $w=z=r$, where

$$k_u(w,z)=(1-\overline{w}z)^{-1}\big(\overline{v(u,w)}+v(u,z)\big)=2(1-e^{iy(u)}\overline{w})^{-1}(1-e^{-iy}z)^{-1},$$

we obtain the analogue of (33)

$$K^{n-1,n}(F;r,r)=2\int_0^l\frac{\partial^{n-1}}{\partial\overline{w}^{\,n-1}}\left(\frac{F^u(w)^*}{1-e^{iy(u)}\overline{w}}\right)\bigg|_{w=r}dE(u)\,\frac{\partial^n}{\partial z^n}\left(\frac{F^u(z)}{1-e^{-iy(u)}z}\right)\bigg|_{z=r}.$$

The analogue of (34) is the following expression:

$$\int_0^l F^u(r)^*\frac{\partial^{2n-1}}{\partial\overline{w}^{\,n-1}\partial z^n}k_u(r,r)\,dE(u)F^u(r).$$

Since

$$\frac{\partial^{2n-1}}{\partial\overline{w}^{\,n-1}\partial z^n}k_u(r,r)=2(n-1)!n!\frac{e^{-iy(u)}}{|1-e^{-iy(u)}|^{2n}(1-e^{-iy(u)}r)},$$

since

$$\operatorname{Im}\left(\frac{e^{-iy(u)}}{1-e^{-iy(u)}r}\right)=-\frac{\sin y(u)}{|1-e^{-iy(u)}r|^2}\quad(<0,\text{ all } u,\text{ or } >0,\text{ all } u),$$

and $|\sin y(u)|\cdot|1-e^{-iy(u)}|^{-1}>\sin(\alpha/2)$, it follows that the expression

$$\int_0^l|1-e^{-iy(u)}r|^{-2n-2}|1-e^{-iy(u)}|F^u(r)^*dE(u)F^u(r)$$

is bounded as $r\to 1-$. Hence we conclude easily that $R_{N+1}(F)<\infty$, because of $\operatorname{Tr}(F^u(1)^*dE(u)F^u(1))=du$.

If $\theta(z) \equiv F(z)$ and $5\pi/6 \le y(x) \le 7\pi/6$, all x, then $|1 - e^{-iy(x)}| \ge 3^{1/2}$, so that $R_{N+1}(F) \le 3^{-(N+1)/2} l$.

In the general case the statement follows from the fact that, according to Note 6, θ has divisors of all types considered, such that $\det\theta(z)$ is the product of these divisors, which implies that $R_{N+1}(\theta)$ is the sum of quantities R_{N+1} of these divisors.

$(\mathrm{e}_1) \Rightarrow (\mathrm{a}_2)$. Established in Lemma 3.

The case $N = 2n + 1$, $n \in \mathbf{N}$. It is clear that $(\mathrm{c}_1) \Rightarrow (\mathrm{a}_1)$ is true.

$(\mathrm{b}_1) \Rightarrow (\mathrm{a}_1)$. Since (b_1) for N implies (b_1) for $N-1$, it follows, by the induction hypothesis, that (a_2) for $N-1$ is satisfied. Thus, (a_1) is true for N.

$(\mathrm{a}_1) \Rightarrow (\mathrm{d}_1)$. This statement can be proved in the same way as $(\mathrm{b}_1) \Rightarrow (\mathrm{b}_2)$.

$(\mathrm{d}_1) \Rightarrow (\mathrm{e}_1)$. According to Note 5 and to factorization (2), the boundedness of $K^{n,n}(\theta; r, r)$ as $r \to 1-$ implies that for some $M > 0$ and r_0, $0 < r_0 < 1$, the following holds

$$\mathrm{Tr}(K^{n,n}(B; r, r)) \le M, \qquad r \in [r_0, 1), \tag{35}$$

and

$$\mathrm{Tr}(\partial^{2n}/\partial\overline{w}^{\,n}\partial z^n (B(w)^* U^* K(F; w, z) U B(z))|_{w=r,\, z=r}) \le M, \quad r \in [r_0, 1). \tag{36}$$

By the induction hypothesis, by Note 5 and Lemma 3, we may assume that $R_N(f) < \infty$ and that $\|f^{(j)}(r)\|_1 \le M_{N-1}$, $r < 1$, $0 \le j \le N-1$, and also that the limit $\lim_{r\to 1} f(r) := f(1)$ exists and that $f(1)$ is a unitary operator, for $f = \theta$ and every divisor of θ.

By differentiating the equality (24) for $w = z = r$, we obtain the relation

$$K^{n,n}(B; r, r) = \sum_{m=1}^{p} \frac{\partial^{2n} k_m(r, r)}{\partial\overline{w}^{\,n}\partial z^n}, \tag{37}$$

which shows, by taking into account the definition of the kernel k_m, that (35) and the induction hypothesis imply boundedness, as $r \to 1-$, of the expression

$$\sum_{m=1}^{p} B_{m-1}(r)^* K^{n,n}(b_m; r, r) B_{m-1}(r). \tag{38}$$

But, since $K^{n,n}(b_m; r, r) = (n!)^2 |a_m|^{2n} |1 - \bar{a}_m r|^{-2n-2}(1 - |a_m|^2) P_m$, it follows that the expression

$$\sum_{m=1}^{p} |1 - \bar{a}_m r|^{-2n-2}(1 - |a_m|) B_{m-1}(r)^* P_m B_{m-1}(r) \tag{39}$$

also stays bounded as $r \to 1-$. Hence already it follows that

$$R_{N+1}(B) < \infty, \tag{40}$$

for $\mathrm{Tr}(B_{m-1}(1)^* P_m B_{m-1}(1)) = p_m$.

In order to establish such a fact for F, we will follow the analogy with the reasoning just applied. By differentiating the relation (27), and putting $(1-\overline{w}z)^{-1}(\overline{v(u,w)}+v(u,z)) = k_u(w,z)$ there, we obtain the relation

$$\frac{\partial^{2n}}{\partial\overline{w}^{\,n}\partial z^n}(B(w)^* U^* K(F;w,z) U B(z)) = \int_0^l \frac{\partial^{2n}}{\partial\overline{w}^{\,n}\partial z^n}(\theta^u(w)^* k_u(w,z)\, dE(u)\theta^u(z)),$$

which can be considered as the analogue od (37). Now, (36) and induction hypothesis imply that the following expression (the analogue of (38)) is bounded as $r \to 1-$:

$$\int_0^l \theta^u(r)^* \frac{\partial^{2n}}{\partial\overline{w}^{\,n}\partial z^n} k_u(r,r)\, dE(u)\theta^u(r).$$

But, since

$$\partial^{2n} k_u(r,r)/\partial\overline{w}^{\,n}\partial z^n = 2(n!)^2 |1-e^{-iy(u)}r|^{-2n-2},$$

it follows that the expression

$$\int_0^l |1-e^{-iy(u)}r|^{-2n-2}\theta^u(r)^*\, dE(u)\theta^u(r)$$

is also bounded as $r \to 1-$. Hence it follows easily that

$$R_{N+1}(F) < \infty, \tag{41}$$

for $\mathrm{Tr}(\theta^u(1)^*\, dE(u)\theta^u(1)) = du$.

Since $R_{N+1}(\theta) = R_{N+1}(B) + R_{N+1}(F)$, the statement follows from (40) and (41).

(e_1) $\implies$ (c_1) $\wedge$ (b_1). Established in Lemma 3.

This completes the proof of Theorem 2.

REFERENCES

[**1**] P. R. Ahern, D. N. Clark, *Radial N-th derivatives of Blaschke products*, Math. Scand. **28** (1971), 189–201.

[**2**] D. R. Georgijević, *Radial derivatives of some analytic operator functions*, in: *Complex Analysis and Applications '85*, Bulgarian Academy of Sciences, Sofia, 1986, 239–248.

[**3**] J. Rovnyak, *Some Hilbert Spaces of Analytic Functions*, Dissertation, Yale Univ., 1963.

[**4**] B. Sz.-Nagy, *Positive definite kernels generated by operator-valued analytic functions*, Acta Sci. Math. **26** (1965), 191–192.

[**5**] D. R. Georgijević, *Differentiation and integration as linear functionals in Hilbert space with reproducing kernel II*, Saopštenja Maš. Fak. Beograd, No. **1–2** (1982), 50–51 (in Serbocroatian).

[**6**] Ю. П. Гинзбург, *О мультипликативных представленијх ограниченных аналитических оператор-функций*, ДАН СССР **170** (1966), 23–26.

[**7**] И. Ц. Гохберг, М. Г. Крейн, *Введение в теорию линейных несамосопряженных операторов в гильбертовом пространстве*, Наука, Москва, 1965.

[**8**] L. V. Ahlfors, *Conformal Invariants: Topics in Geometric Function Theory*, McGraw-Hill, New York, 1973.

Katedra za matematiku
Mašinski fakultet
11000 Beograd
Jugoslavija

(Received 18 12 1989)