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ON ISOMORPHISMS OF L^1 SPACES OF ANALYTIC FUNCTIONS ONTO l^1

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Abstract. It is proved that an L^1_φ space of analytic functions in the unit disc, with the weight $\varphi'(1-|z|)$, is isomorphic to the Lebesgue sequence space l^1 only if φ is "normal". The converse is known from the papers of Shields and Williams [**13**] and Lindenstrauss and Pelczynski [**4**]. The key of our proof are three classical results: Paley's theorem on lacunary series, Pelczynski's theorem on complemented subspaces of l^1 and Lindenstrauss-Pelczynski's theorem on the equivalence of unconditional bases in l^1.

Throughout the paper we assume that φ is a quasi-normal function [**7**]. This means that φ is defined, increasing and continuously differentiable on the interval $(0,1]$, $\varphi(0+)=0$, and that

(Q) there is a constant β such that $\varphi(t)/t^\beta$ is almost decreasing for $0<t<1$.

(A function ψ is almost decreasing if $\psi(t_1)\le C\psi(t_1)$ for $t_2>t_1$, where C is a constant, see [**1, 15**]. If, in addition,

(N) there is a constant $\alpha>0$ such that $\varphi(t)/t^\alpha$ is almost increasing for $0<t<1$,

then φ is said to be *normal* [**13, 15**].

For a (complex-valued) harmonic function f on the unit disc U, we define the quantities

$$\|f\|_{1,\varphi}=\frac{1}{\pi}\int_U |f(z)|\varphi'(1-|z|)\,dm(z)$$

(dm denotes the Lebesgue measure, and φ' the derivative of φ) and

$$\|f\|_{\infty,\varphi}=\sup\{|f(z)|\varphi(1-|z|):z\in U\}.$$

In this note we are concerned with the following spaces:

$$A^p(\varphi)=\{f:\|f\|_{p,\varphi}<\infty,\ f \text{ analytic in } U\}\quad (p=1,\infty),$$

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$$A_0(\varphi) = \{f \in A^\infty(\varphi) : |f(z)|\varphi(1-|z|) = o(1),\ z \to 1^-\}.$$

Analogous spaces of harmonic functions are denoted by $h^p(\varphi)$ and $h_0(\varphi)$.

It is known that each of the harmonic spaces is isomorphic, via a multiplier transform, to a space of Lipschitz functions on the unit circle [**8**, **9**]. On the other hand, it was shown by Shields and Williams [**14**] that $h^p(\varphi)$ is isomorphic to l^p, and $h_0(\varphi)$ is isomorphic to c_0, the space of null-sequences (see Remarks at the end). And it follows from the results of Lindenstrauss and Pelczynski [**4**] and Shields and Williams [**13**] that h can be replaced by A if φ is normal. Our aim here is to prove the converse. More precisely, we have the following result.

Theorem. *If one of the following assertions* (i), (ii) *or* (iii) *holds, then the function φ is normal*:

(i) *$A^1(\varphi)$ is isomorphic to l^1*;

(ii) *$A^\infty(\varphi)$ is isomorphic to l^∞*;

(iii) *$A_0(\varphi)$ is isomorphic to c_0.*

Before discussing the case of $A^1(\varphi)$ we note that the other two cases are contained implicitly in [**14**, Theorem 7] and [**15**]. Namely, if $A^\infty(\varphi)$ is isomorphic to l^∞, then $A^\infty(\varphi)$ is complemented in $h^\infty(\varphi)$, because l^∞ is complemented in every space containing it (see [**5**, p. 105]). Then, by using a method of Rudin [**13**], we conclude that $A^\infty(\varphi)$ is complemented by the "analytic" projection, i.e. that $h^\infty(\varphi)$ is self-conjugate and hence, by [**15**], φ is normal.

If $A_0(\varphi)$ is isomorphic to c_0, we can use a theorem of Sobczyk that asserts that c_0 is complemented in every separable space containing it. We can also reduce the problem to the case of A^∞ by using the fact, proved by Rubel and Shields [**11**], that the second dual of $A_0(\varphi)$ is isometrically isomorphic to $A^\infty(\varphi)$.

In order to discuss the case of $A^1(\varphi)$ we use three famous theorems.

Paley Theorem [**6**]. *If $\{m_n\}_0^\infty$ is a lacunary sequence of positive integers, then there is a constant $C < \infty$ such that*

$$\sum |\hat{f}(m_n)|^2 \le C\|f\|_1^2, \qquad f \in H^1,$$

where $\|\ \|_1$ stands for the norm in the Hardy class H^1.

Pelczynski Theorem [**10**]. *Every infinite dimensional complemented subspace of l^1 is isomorphic to l^1.*

Lindenstrauss-Pelczynski Theorem [**3**]. *Every normalized unconditional basis of l^1 is equivalent to the canonical basis of l^1.*

Proof of the Theorem. Assume that $A^1(\varphi)$ is isomorphic to l^1. Let Y denote the subspace spanned by the elements $h_n(z) = z^{2^n}$, and $Pf(z) = \sum \hat{f}(2^n) z^{2^n}$, $f \in A^1(\varphi)$. By the Paley theorem,

$$M_1(r, Pf) \le M_2(r, Pf) \le C M_1(r, f), \qquad 0 < r < 1, \tag{1}$$

where C is independent of r, f. (Here we use the usual notation for the integral means of f [**2**]). Multiplying inequality (1) by $\varphi'(1-r)\,dr$ and then integrating from 0 to 1, we conclude that P is a bounded projection of $A^1(\varphi)$ onto Y. Hence, by Pelczynski's theorem, Y is isomorphic to l^1. Using (1) again we see that

$$M_1(r,f) \asymp M_2(r,f), \qquad f \in Y,\ 0<r<1,$$

which implies that $\{h_n\}$ is an unconditional basis of Y. Hence, by Lindenstrauss-Pelczynski's theorem, there is a constant $c>0$ such that

$$\Big\|\sum a_n h_n\Big\|_Y \geq c\sum |a_n|\,\|h_n\|_Y \tag{2}$$

for scalar sequences $\{a_n\}$.

To deduce from (2) that φ is normal, we need some calculation. We have

$$\|h_n\|_Y = \int_0^1 \varphi'(1-r)r^{2^n}\,dr,$$

whence, by Lemma 3.2 of [**7**],

$$\|h_n\|_Y \asymp \varphi(2^{-n}). \tag{3}$$

Extend φ so that

$$\varphi(t)\varphi(1/t) = \varphi(1)^2, \qquad t>1, \tag{4}$$

and let F denote the inverse function. In view of condition (Q), there is a constant $m \geq 2$ such that $F(mt)/F(t) \geq 4$ for $t>0$. It follows that the interval

$$E_t := [\log F(t), \log F(mt)] \qquad (\log = \log_2)$$

contains at least two integers. For a fixed t with $\log F(t) > 0$ define $\{a_n\}$ in the following way:

$$a_n = \begin{cases} \varphi(2^n) & \text{for } n \in E_t, \\ 0, & \text{otherwise,} \end{cases}$$

and let $f_t = \sum a_n h_n$. It follows from (2), by (3) and (4), that

$$\|f_t\|_Y \geq c\big(\log F(mt) - \log F(t)\big), \tag{5}$$

where $c>0$ is independent of t. On the other hand, by Parseval's relation,

$$M_1^2(r,f_t) \leq M_2^2(r,f_t) = \sum |a_n|^2 r^{2^{n+1}} \leq \sum \varphi(2^n)^2 r^{2F(t)} \qquad (n \in E_t).$$

Since $\varphi(2^n) \leq \varphi(F(mt)) = mt$ for $n \in E_t$, we see that

$$M_1(r,f_t) \leq mtr^{F(t)}\big(\log F(mt) - \log F(t) + 1\big)^{1/2}.$$

Multiply thus by $\varphi'(1-r)\,dr$, then integrate and use the estimate

$$\int_0^1 r^x \varphi'(1-r)\,dr \asymp \varphi(1/x), \qquad x>1$$

(see [**7**, Lemma 3.2]). As a result we obtain

$$\|f_t\|_Y \leq C\big(\log F(mt) - \log F(t)\big)^{1/2}.$$

From this and (5) we find that $F(2t) \leq KF(t)$ for $t>0$, where K is a constant. As a consequence, $F(t)/t^\beta$ is almost decreasing for $t>0$, where $\beta = \log_2 K$. Hence, φ satisfies the condition (N) with $\alpha = 1/\beta$, which was to be proved.

Remarks. Shields and Williams [**14**] proved that $h_0(\varphi)$ is isomorphic to c_0 if φ satisfies some regularity conditions. However, it follows from Theorems 1 and 7 of [**14**] and the relation $h_0(\varphi)^* = h^1(\varphi)$ [**7**] that $h_0(\varphi)$ is isomorphic to c_0 whenever φ is quasi-normal. It would be interesting to construct an explicit isomorphism between $h_0(\varphi)$ and c_0, or between $h^1(\varphi)$ and l^1. In the case of $A^1(\varphi)$, where φ is a power function, an explicit isomorphism onto l^1 was constructed by Wojtaszczyk [**17**].

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