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THE DUAL STEINER FORMULA FOR CONVEX COMPACTA

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Abstract. The classical Steiner formula deals with the volume of $K + rD_n$, where K is a convex compact set in $\mathbf{R}^n$, $r \geq 0$, and D_n the Euclidean disc in $\mathbf{R}^n$. We compute the volume of $(K + rD_n)^*$, where K^* means the dual of K. We also represent some Steiner functionals by integrals and prove some inequalities.

1. Introduction and notations

To specify notations we give here some elementary properties of convex compacta in $\mathbf{R}^n$. For details see [**1**].

Let $K(\mathbf{R}^n)$ be the set of all convex compacta in $\mathbf{R}^n$. If $K \in K(\mathbf{R}^n)$ we introduce the function $h_K : \mathbf{R}^n \to \mathbf{R}$ defined by

$$h_K(x) = \max_{y \in K}(x|y), \qquad x \in \mathbf{R}^n$$

where $(x|y) = x_1y_1 + \cdots + x_ny_n$ is the standard scalar product in $\mathbf{R}^n$. The function h_K has the following properties:

1) $h_K(tx) = th_K(x)$, $t \geq 0$, $x \in \mathbf{R}^n$.

2) $h_K(x + y) \leq h_K(x) + h_K(y)$, $x, y \in \mathbf{R}^n$.

3) If $0 \in K$ then $h_K \geq 0$.

4) If $0 \in K^\circ = \operatorname{int}(K)$ then: $h_K = 0$ iff $x = 0$.

If $0 \in K^\circ$, we define the dual K^* of K by $K^* = \{x \in \mathbf{R}^n; h_K(x) \leq 1\}$; then $K^* \in K(\mathbf{R}^n)$, $0 \in K^*$, $(K^*)^* = K$ and $K = \{x \in \mathbf{R}^n; h_{K^*}(x) \leq 1\}$. If $K_1, \ldots, K_m \in K(\mathbf{R}^n)$ and $\lambda_1, \ldots, \lambda_m \in \mathbf{R}_+$ then $K = \lambda_1 K_1 + \cdots + \lambda_m K_m$ is called the Minkowski sum of $K_1, \ldots, K_m$ with coefficients $\lambda_1, \ldots, \lambda_m$ and for K we have $h_K = \lambda_1 h_{K_1} + \cdots + \lambda_m h_{K_m}$.

$K(\mathbf{R}^n)$ is a complete metric space with Hausdorff metric d defined by

$$d(K_1, K_2) = \max_{x \in S_n} |h_{K_1}(x) - h_{K_2}(x)|$$

AMS Subject Classification (1985): Primary 52 A 20 where $S_n = \{x \in \mathbf{R}^n; \|x\| = 1\}$, $\|x\|^2 = (x|x)$. Let $D_n = \{x \in \mathbf{R}^n; \|x\| \leq 1\}$ be the Euclidean disc in $\mathbf{R}^n$. The classical Steiner formula says that

$$V(K + rD_n) = \sum_{m=0}^{n} \binom{n}{m} W_m(K) r^m, \qquad r \geq 0,\ K \in K(\mathbf{R}^n)$$

where $V(K)$ is the volume of K, i.e. the Lebesgue measure of K.

The functions $K \to W_m(K)$, $m = 0, \dots, n$, defined by the Steiner formula, are called the Steiner functionals. They are continuous, invariant under isometries, monotonic, additive and they have many other nice properties. For the proof of these assertions see [**1**].

2. Some auxiliaries

2.1. *Definition.* Let $H(n)$ be the set of all continuous functions $\varphi : \mathbf{R}^n \to [0, \infty)$ such that

1) $\varphi(tx) = t\varphi(x)$, $t \geq 0$, $x \in \mathbf{R}^n$.

2) φ is differentiable a.e. and $\varphi'(x) \neq 0$ a.e.

For $\varphi \in H(n)$ we introduce the sets

$$S_\varphi = \{x \in \mathbf{R}^n; \varphi(x) = 1\}, \qquad D_\varphi = \{x \in \mathbf{R}^n; \varphi(x) \leq\}.$$

By differentiating 1) with respect to t and x we obtain

a) $\varphi'(x)x = \varphi(x)$, a.e., $x \in \mathbf{R}^n$;

b) $\varphi'(tx) = \varphi'(x)$, a.e., $t \geq 0$, $x \in \mathbf{R}^n$.

Here we need some integral formulae involving $\varphi \in H(n)$. We collect all such formulae in the following theorem. For the proof of the theorem see [**2**].

2.2. Theorem. *Let $\varphi, \psi \in H(n)$. Then*

1) $\displaystyle\int_{\mathbf{R}^n} f(x)\, dx = \int_0^\infty \int_{S_\varphi} f(tx) t^{n-1} \frac{dt\, dh(x)}{\|\varphi'(x)\|}$

where h is the Hausdorff $(n-1)$-measure in $\mathbf{R}^n$ and $f \in L_1(\mathbf{R}^n)$;

2) $\displaystyle\int_{\mathbf{R}^n} f(x) \|\varphi'(x)\|\, dx = \int_0^\infty \int_{S_\varphi} f(tx) t^{n-1}\, dt\, dh(x);$

3) $\displaystyle\int_{S_\varphi} \frac{dh(x)}{\|\varphi'(x)\|} = nV(D_\varphi);$

4) $\displaystyle\int_{D_\varphi} \|\varphi'(x)\|\, dx = \frac{1}{n} h(S_\varphi);$

5) $\displaystyle\int_{S_\psi} f\, dh = \int_{S_\varphi} f\left(\frac{x}{\psi(x)}\right) \frac{1}{\psi(x)^n} \cdot \frac{\|\psi'(x)\|}{\|\varphi'(x)\|}\, dh(x)$

where $f \in L_1(S_\psi, h)$;

6) $\displaystyle\int_{D_\psi} f(x)\, dx = \int_{D_\varphi} f\left(\frac{\varphi(x)}{\psi(x)} x\right) \frac{\varphi(x)^n}{\psi(x)^n}\, dx$

where $f \in L_1(D_\psi)$.

2.3. *Note*. Let $K \in K(\mathbf{R}^n)$ and $0 \in K^\circ$. Then by Rademacher's theorem (see [**3**, Theorem 3.1.6, p. 234]) we have $h_K \in H(n)$, $h_{K^*} \in H(n)$ and $K = D_{h_{K^*}}$ and $K^* = D_{h_K}$ and we can apply the theorem above to K and K^*.

3. The dual Steiner formula

3.1. Theorem. *Let $K \in K(\mathbf{R}^n)$ and $0 \in K^\circ$. Then*

1) $V(K^*) = \dfrac{1}{n}\displaystyle\int_{S_n} \frac{dh(x)}{h_K(x)^n}$;

2) $V((K + rD_n)^*) = \dfrac{1}{n}\displaystyle\int_{S_n} \frac{dh(x)}{(h_K(x)+r)^n}$, $r \geq 0$;

3) $W_1(K^*) = \dfrac{1}{n}\displaystyle\int_{S_n} \frac{\|h'_K(x)\|}{h_K(x)^n}\, dh(x)$;

4) $W_1((K + rD_n)^*) = \dfrac{1}{n}\displaystyle\int_{S_n} \frac{\|h'_K(x) + rx\|}{(h_K(x)+r)^n}\, dh(x)$, $r \geq 0$.

Proof: 1) We apply theorem 2.2. 6) to $\psi = h_K$, $\varphi(x) = \|x\|$ and $f = 1$. Hence

$$V(D_\psi) = V(D_{h_K}) = V(K^*) = \int_{D_n} \frac{\|x\|^n\, dx}{h_K(x)^n}$$

and now by theorem 2.2. 1) we have

$$V(K^*) = \int_0^1 \int_{S_n} \frac{t^{n-1}}{h_K(x)^n}\, dt\, dh(x) = \frac{1}{n}\int_{S_n} \frac{dh(x)}{h_K(x)^n}.$$

2) Because of

$$h_{K+rD_n}(x) = h_K(x) + r\|x\|$$

the formula follows from 1).

3) We apply theorem 2.2. 5) to $\psi = h_K$ and $f = 1$. Hence

$$h(S_\psi) = nW_1(D_\psi) = nW_1(K^*) = \int_{S_n} \frac{\|h'_K(x)\|}{h_K(x)^n}\, dh(x).$$

4) Follows from 3) as above.

3.2. Corollary. *Let $K \in K(\mathbf{R}^n)$ and $0 \in K^\circ$. Then*

1) $V((K + rD_n)^*) \leq \dfrac{\alpha(n)}{(\beta + r)^n}$, $r \geq 0$,

where $\alpha(n) = V(D_n)$, $\beta = \min_{x \in S_n} h_K(x)$.

2) $V(K) \leq \dfrac{1}{n}\displaystyle\int_{S_n} h_K(x)^n\, dh(x)$.

3) $W_1((K+rD_n)^*) \leq \dfrac{\alpha(n)}{(\beta+r)^n}(\gamma^2+r^2+2r\|h_K\|)^{1/2}$, $r \geq 0$,

where $\beta = \min_{x\in S_n} h_K(x)$, $\|h_K\| = \max_{x\in S_n} h_K(x)$, $\gamma = \max_{x\in S_n} \|h'_K(x)\|$.

Proof: 1) follows from theorem 3.1 2).

2) Because of $h_K(x)h_{K^*}(y) \geq (x|y)$ we have

$$1 \leq h_K(x)h_{K^*}(x), \qquad x \in S_n$$

and the inequality follows from theorem 3.1. 1).

3) $\|h'_K(x)+rx\|^2 = \|h'_K(x)\|^2 + r^2 + 2rh_K(x)$. Apply now theorem 3.1 4).

3.3. Corollary (Dual Steiner formula). *Let* $K \in K(\mathbf{R}^n)$ *and* $0 \in K^\circ$. *Then*

$$V((K+rD_n)^*) = \sum_{m=0}^{\infty} \frac{1}{n}\binom{n+m-1}{m}(-1)^m r^m \int_{S_n} \frac{dh(x)}{h_K(x)^{n+m}}$$

and the series converges for $0 \leq r < \min_{x\in S_n} h_K(x)$.

Proof. We apply theorem 3.1. 2) by expanding the integrand in a power series by using the formula

$$(1-t)^{-n} = \sum_{m=0}^{\infty} \binom{n+m-1}{m} t^m, \qquad |t| < 1.$$

Hence, we have

$$\frac{1}{(h_K(x)+r)^n} = \sum_{m=0}^{\infty} \binom{n+m-1}{m}(-1)^m r^m \frac{1}{h_K(x)^{n+m}}, \qquad 0 \leq r < \min_{x\in S_n} h_K(x)$$

and the formula follows.

3.4. Corollary. *Let* $K \in K(\mathbf{R}^n)$ *and* $0 \in K^\circ$. *Then*

$$V((D_n+rK)^*) = \sum_{m=0}^{\infty} \frac{1}{n}\binom{n+m-1}{m}(-1)^m r^m \int_{S_n} h_K^m \, dh, \qquad 0 \leq r < 1/\|h_K\|.$$

Proof. Put $1/r$ in place of r in 3.1 2). Then

$$V((D_n+rK)^*) = \frac{1}{n}\int_{S_n} \frac{dh(x)}{(1+rh_K(x))^n}, \qquad r \geq 0.$$

The formula now follows as above.

3.5. Theorem. *Let* $K \in K(\mathbf{R}^n)$, $0 \in K^\circ$ *and*

$$z(K) = \int_K x \, dx.$$

Then we have

1) $z((K+rD_n)^*) = \dfrac{1}{n+1}\displaystyle\int_{S_n} \frac{x\,dh(x)}{(h_K(x)+r)^{n+1}}$, $r \geq 0$;

2) $z((K+rD_n)^*) = \sum_{m=0}^{\infty} \dfrac{1}{n+1}\dbinom{n+m}{m}(-1)^m r^m \displaystyle\int_{S_n} \frac{x\,dh(x)}{h_K(x)^{n+m+1}}$

and the series converges for $0 \leq r < \min_{x\in S_n} h_K(x)$.

Proof. Let us apply theorem 2.2. 6) to $\psi = h_K$, $\varphi(x) = \|x\|$ and $f(x) = x$. Then we have

$$\begin{aligned} z(K^*) &= \int_{K^*} x\,dx = \int_{D_\psi} x\,dx = \int_{D_n} \frac{\|x\|^{n+1}}{h_K(x)^{n+1}} x\,dx \\ &= \int_0^1 \int_{S_n} \frac{t^n x}{h_K(x)^{n+1}}\,dt\,dh(x) = \frac{1}{n+1}\int_{S_n} \frac{x\,dh(x)}{h_K(x)^{n+1}}, \end{aligned}$$

Now 1) and 2) follow by the same argument as in 3.1, 3.2 and 3.3.

3.6. Using Steiner's vector formula (see [**1**]):

$$z(K+rD_n) = \sum_{m=0}^{n} \binom{n}{m} q_m(K) r^m, \qquad r \geq 0$$

to introduce Steiner vectors $q_m(K)$, $m = 0, \ldots, n$, one can apply the theory above to study the Steiner vectors of K.

The formulae in theorem 3.5 we call dual Steiner vector formulae. Steiner functionals can be used to evaluate some complicated integrals in $\mathbf{R}^n$. Let us give an example.

3.7. Proposition. *Let* $K \in K(\mathbf{R}^n)$. *Then*

$$\int_{\mathbf{R}^n} f(d(x,K))\,dx = f(0)V(K) + \sum_{s=0}^{n-1} n\binom{n-1}{s} W_{s+1}(K)\int_0^\infty f(t)t^s\,dt$$

where $d(x,K) = \min_{y\in K}\|x-y\|$, $f(d(\cdot\,,K)) \in L_1(\mathbf{R}^n)$.

Proof. Let us use the distribution function of $d(\cdot\,,K)$ i.e.

$$V(\{x \in \mathbf{R}^n; d(x,K) \leq r\}) = V(K+rD_n)$$

Hence, we have

$$\begin{aligned} \int_{\mathbf{R}^n} f(d(x,K))\,dx &= f(0)V(K) + \int_0^\infty f(t)\,dV(K+tD_n) \\ &= f(0)V(K) + \int_0^\infty f(t)nW_1(K+tD_n)\,dt \end{aligned}$$

and the formula follows.

3.8. Corollary (Wills formula, 1973).

$$\int_{\mathbf{R}^n} \exp(-\pi d(x,K)^2)\,dx = \sum_{m=0}^{n} \binom{n}{m} \frac{1}{\alpha(m)} W_m(K), \qquad K \in K(\mathbf{R}^n).$$

Proof. Put $f(t) = \exp(-\pi t^2)$ in 3.7.

3.9. Corollary.

$$\int_{K+rD_n} d(x,K)^\alpha\,dx = \sum_{s=0}^{n-1} n\binom{n-1}{s} W_{s+1}(K) \frac{r^{\alpha+s+1}}{\alpha+s+1}, \quad \alpha > 0,\ K \in K(\mathbf{R}^n).$$

Proof. Put $f(t) = t^\alpha \chi_{[0,r]}(t)$ in 3.7.

The technique above can be applied to prove many inequalities. Let us give some examples.

3.10. Theorem. *Let $K_1, K_2 \in K(\mathbf{R}^n)$, $0 \in K_1^\circ$, $0 \in K_2^\circ$, and $\lambda \in [0,1]$. Then*

$$V((\lambda K_1 + (1-\lambda)K_2)^*) \le V(K_1^*)^\lambda V(K_2^*)^{1-\lambda}.$$

Proof. By theorem 2.2. 1) we have

$$n!V(K_1^*) = \int_{\mathbf{R}^n} \exp(-h_{K_1}(x))\,dx$$

and also

$$n!V((\lambda K_1 + (1-\lambda)K_2)^*) = \int_{\mathbf{R}^n} \exp(-\lambda h_{K_1}(x) - (1-\lambda)h_{K_2}(x))\,dx.$$

If $\lambda = 0$ or $\lambda = 1$ the inequality is trivial. Hence, we can assume that $0 < \lambda < 1$. Let us apply Hölder's inequality for

$$p = \frac{1}{\lambda}, \quad q = \frac{1}{1-\lambda}, \quad \frac{1}{p} + \frac{1}{q} = \lambda + 1 - \lambda = 1.$$

Then we have

$$\begin{aligned}
&n!V((\lambda K_1 + (1-\lambda)K_2)^*)\\
&\le \left(\int_{\mathbf{R}^n} \exp(-h_{K_1}(x))\,dx\right)^\lambda \cdot \left(\int_{\mathbf{R}^n} \exp(-h_{K_2}(x))\,dx\right)^{1-\lambda}\\
&= (n!V(K_1^*))^\lambda \cdot (n!V(K_2^*))^{1-\lambda} = n!V(K_1^*)^\lambda V(K_2^*)^{1-\lambda}.
\end{aligned}$$

3.11 Corollary.

1) $V((K_1 + K_2)^*) \le 2^{-n} V(K_1^*)^{1/2} V(K_2^*)^{1/2}$, $0 \in K_1^\circ$, $0 \in K_2^\circ$.

2) $V((K+rD_n)^*)^2 \leq \dfrac{\alpha(n)}{4^n \cdot r^n} V(K^*)$, $0 \in K^\circ$, $r > 0$.

Proof. 1) Put $\lambda = 1/2$ in 3.10.

2) follows from 1).

3.12. THEOREM. *Let* $K_1, K_2 \in K(\mathbf{R}^n)$, $0 \in K_1^\circ$, $0 \in K_2^\circ$. *Then*

$$W_1((K_1+K_2)^*) \leq W_1(K_1^*) + W_1(K_2^*).$$

Proof. By theorem 2.2. 2) applied to $f(t) = e^{-t}$, $\varphi = h_{K_1}$ we have

$$n! W_1(K_1^*) = \int_{\mathbf{R}^n} \exp(-h_{K_1}(x)) \|h'_{K_1}(x)\| \, dx.$$

Hence, we have

$$\begin{aligned} n! W_1((K_1+K_2)^*) &= \int_{\mathbf{R}^n} \exp(-h_{K_1}(x) - h_{K_2}(x)) \|h'_{K_1}(x) + h'_{K_2}(x)\| \, dx \\ &\leq \int_{\mathbf{R}^n} \exp(-h_{K_1}(x) - h_{K_2}(x)) \|h'_{K_1}(x)\| \, dx \\ &\quad + \int_{\mathbf{R}^n} \exp(-h_{K_1}(x) - h_{K_2}(x)) \|h'_{K_2}(x)\| \, dx \\ &\leq \int_{\mathbf{R}^n} \exp(-h_{K_1}(x)) \|h'_{K_1}(x)\| \, dx + \int_{\mathbf{R}^n} \exp(-h_{K_2}(x)) \|h'_{K_2}(x)\| \, dx \\ &= n! W_1(K_1^*) + n! W_1(K_2^*). \end{aligned}$$

REFERENCES

[**1**] K. Leichtweiss, *Konvexe Mengen*, Nauka, Moscow, 1985 (in Russian).

[**2**] L. Dedić, *On Haar measure on* $SL(n,R)$, Publ. Inst. Math. Belgrade **47** (**61**) (1990), 56–60.

[**3**] H. Federer, *Geometric measure theory*, Nauka, Moscow, 1987 (in Russian).

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