PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 51 (65), 1992, 62–70

PROPERTIES OF SOLUTIONS OF SOME LINEAR CLASS OF INTEGRODIFFERENTIAL EQUATIONS OF VOLTERRA TYPE

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Abstract. We present a method for calculating a fundamental matrix of the equation (3). In addition we give a formula for a particular solution of the system (1).

We shall associate the linear neutral Volterra integrodifferential equation

$$\begin{aligned} \frac{d}{dt}\left[x(t)-\int_{-\infty}^{t} C(t-s)x(s)\,ds-g(t)\right] \\ = A(t)x(t)+B(t)\int_{-\infty}^{t} F(t-s)x(s)\,ds+f(t) \end{aligned} \tag{1}$$

with

$$\begin{aligned} \frac{d}{dt}\left[x(t)-\int_{t_0}^{t} C(t-s)x(s)\,ds-g(t)\right] \\ = A(t)x(t)+B(t)\int_{t_0}^{t} F(t-s)x(s)\,ds+f(t) \end{aligned} \tag{2}$$

via the resolvent equation

$$\begin{aligned} \frac{d}{dt}\left[Z(t)-\int_{t_0}^{t} C(t-s)Z(s)\,ds\right] \\ = A(t)Z(t)+B(t)\int_{t_0}^{t} F(t-s)Z(s)\,ds, \qquad Z(t_0)=E_n. \end{aligned} \tag{3}$$

Here and hereafter x is an n-vector, $A(t)$ and $B(t)$ are $n\times n$ matrices continuous on $(-\infty,\infty)$, $g,f:(-\infty,\infty)\to\mathbf{R}^n$ are continuous, E_n the $n\times n$ identity matrix, Z an $n\times n$ matrix, and $C(t)$, $F(t)$ are matrices $n\times n$ which can be represented in the form

$$C(t)=\sum_{j=1}^{k}\psi_j(t)\exp(\alpha_j t), \qquad F(t)=\sum_{j=1}^{k}\varphi_j(t)\exp(\beta_j t)$$

AMS Subject Classification (1985): Primary 34 C 27; 34 A 05; 34 C 20 where

$$\psi_j(t) = \sum_{s=0}^{n_j} \psi_{sj} t^s, \qquad \varphi_j(t) = \sum_{s=0}^{n_j} \varphi_{sj} t^s$$

and ψ_{sj}, φ_{sj} are constant $n \times n$ matrices, $\alpha_j, \beta_j = \text{const}$.

In the case where $C(t) = g(t) = 0$ and $A(t) = A$, $B(t) = E_n$ T. Burton [**1**] proved that for any bounded solution $x(t)$ of (2) there exists an integer sequence $n_j \to \infty$ as $j \to \infty$ such that $x(t + n_j T)$ $(T > 0)$ converges to a solution $x^*(t)$ of (1). A similar result can be found in [**2**] for the case $g(t) = 0$ under the assumptions $Z \in L^1\langle 0, \infty)$ and $\lim_{t\to\infty} Z(t) = 0$. In [**3**], the discussion of the T-periodic solution of (1) (in the case $g(t) = 0$) depended heavily on the behaviour of solutions of the integral equation

$$h(t) = \int_0^t C(t-s) h(s)\, ds + f(t).$$

In [**4**] Jianhong Wu proved by using the variation of constants formula for equation (2) that if $Z, Z' \in L^1\langle 0, \infty))$, then there exists a unique globally stable T-periodic solution

$$g(t) + \int_{-\infty}^t Z'(t-s) g(s)\, ds + \int_{-\infty}^t Z(t-s) f(s)\, ds$$

of equation (1), where $Z(t)$ is the solution of equation (3).

The present paper is an extension of [**1**–**4**]. We shall present some facts relative to the existence of periodic and almost periodic solutions of the systems (1)–(2).

Putting

$$\int_{t_0}^t \exp(\alpha_j (t - z))(t-z)^s x(z)\, dz = u_{sj}(t),$$
$$\int_{t_0}^t \exp(\beta_j (t - z))(t-z)^s x(z)\, dz = y_{sj}(t), \quad (s = 0, \dots, n_j,\ j = 1, \dots, k)$$

the system (2) becomes equivalent to the system

$$\begin{aligned} \frac{dx}{dt} &= \Big(A(t) + \sum_{j=1}^k \psi_{0j}\Big) x + B(t) \sum_{j=1}^k \sum_{s=0}^{n_j} \varphi_{sj} y_{sj} + \sum_{j=1}^k \sum_{s=1}^{n_j} \psi_{sj} (\alpha_j u_{sj} + s u_{s-1\,j}) \\ &\quad + \sum_{j=1}^k \psi_{0j} \alpha_j u_{0j} + f(t) + g'(t) \end{aligned} \tag{4}$$

$$\frac{du_{0j}}{dt} = x + \alpha_j u_{0j}, \quad \frac{dy_{0j}}{dt} = x + \beta_j y_{0j}, \qquad j = 1, \dots, k$$
$$\frac{du_{sj}}{dt} = \alpha_j u_{sj} + s u_{s-1\,j}, \quad \frac{dy_{sj}}{dt} = s y_{s-1\,j} + \beta_j y_{sj}, \qquad \begin{array}{l} s = 1, \dots, n_j, \\ j = 1, \dots, k, \end{array}$$

with initial conditions

$$y_{sj}(t_0) = 0, \quad u_{sj}(t_0) = 0, \qquad s = 0, \dots, n_j,\ j = 1, \dots, k \tag{5}$$

as only $g'(t)$ exists and is continuous for $t \geq t_0$.

Of course the derivative $x'(t)$ and $\int_{t_0}^{t} C(t-s)x(s)\,ds$ must exist for the representation (4).

THEOREM 1. *Let $W(t,t_0)$, $(W(t_0,t_0)=E_p)$ be the $p\times p$ fundamental matrix of the system* (4) *with $f(t)=g(t)=0$; then the fundamental $n\times n$ matrix $X(t,t_0)$, $(X(t,t_0)=E_n)$ of system* (3) *is the upper left minor of degree n of $W(t,t_0)$, i.e.*

$$W(t,t_0)=\begin{bmatrix} X & P \\ Q & R \end{bmatrix}.$$

Proof. Let $W(t,t_0)$ denote the fundamental $p\times p$ matrix of system (4) with $f(t)=g(t)=0$, where $p=n\bigl(2k+1+2\sum_{j=1}^{k} n_j\bigr)$. Since the general solution of system (4) with $f(t)=g(t)=0$ has the form $W(t,t_0)C$, where C is a p-dimensional vector, then in order to obtain a general solution of problem (4)–(5) it is necessary to equate all components of $W(t,t_0)C$ except of the first n to zero and express some of the $p-n$ arbitrary constants of the vector C in terms of any constants of the vector C. Since $W(t_0,t_0)=E_p$, then $c_{n+1}=\ldots=c_p=0$, where c_i $(i=1,\ldots,p)$ are components of the vector C. Hence the fundamental $n\times n$ matrix $X(t,t_0)$, $(X(t_0,t_0)=E_n)$ of system (3) is equal to the upper left minor of degree n of $W(t,t_0)$.

Let $W(t)$ $(W(t_0)=E_p)$ denote the fundamental matrix of (4) with $f(t)=g(t)=0$ and let $A(t)$, $B(t)$ be periodic of period ω, then by the Floqueta Theorem

$$W(t)=Q(t)\exp(\Lambda(t-t_0)) \tag{6}$$

where $Q(t)$ is periodic of period ω, $Q(t_0)=E_p$, and Λ is a constant matrix.

THEOREM 2. *Let* (6) *be the fundamental matrix solution of* (4) *such that $W(t_0)=E_p$, then*

$$X(t)=Q^*(t)\exp(\Lambda(t-t_0))M$$

will be a fundamental matrix solution of (3) *such that $X(t_0)=E_n$, where $Q^*(t)$ is an $n\times p$ periodic matrix of period ω obtained by deleting the last $p-n$ rows in the matrix $Q(t)$, $M=(m_{ij})$ is a $p\times n$ matrix with the property that $m_{ii}=1$ for $i=1,\ldots,n$ and $m_{ij}=0$ for $i\neq j$.*

Proof. From (6) the general solution of system (4) with $f(t)=g(t)=0$ has the form $Q(t)\exp(\Lambda(t-t_0))C$, where C is a p-dimensional vector. Let $x(t_0)=c^*$, where c^* is an n-dimensional vector. By Theorem 1, in order to obtain a general solution of (2) with $f(t)=g(t)=0$ it is necessary to equate the components $c_{n+1},\ldots,c_p$ to zero.

Hence the general solution of (2) with $f(t)=g(t)=0$ and the fundamental matrix of (3) can be represented by

$$x(t)=Q^*(t)\exp(\Lambda(t-t_0))Mc^* \quad \text{and} \quad X(t)=Q^*(t)\exp(\Lambda(t-t_0))M.$$

Let $W(t)$ ($W(t_0) = E_p$) denote the fundamental matrix of (4). Equation (4) with initial values $x(t_0) = x_0$, $y_{sj}(t_0) = 0$, $u_{sj}(t_0) = 0$, $s = 0, \dots, n_j$, $j = 1, \dots, k$, is equivalent to the equation

$$z(t) = W(t)z_0 + \int_{t_0}^{t} W(t)W^{-1}(r)[\Phi_1(r) + \Phi_2(r)]\,dr \tag{7}$$

where

$$\Phi_1(t) = \operatorname{col}(f(t), 0, 0, 0, 0), \qquad \Phi_2(t) = \operatorname{col}(g'(t), 0, 0, 0, 0),$$
$$z(t) = \operatorname{col}(x(t), y_{0j}(t), y_{sj}(t), u_{0j}(t), u_{sj}(t))$$
$$z_0 = \operatorname{col}(x_0, 0, 0, 0, 0), \qquad (s = 0, \dots, n_j,\ j = 1, \dots, k).$$

The first n components of (7) give the solution of (2) with initial conditions $x(t_0) = x_0$, i.e.

$$x(t) = X(t, t_0)(x_0 - g_0) + g(t) + \int_{t_0}^{t} X(t, r)f(r)\,dr + \int_{t_0}^{t} X'_r(t, r)g(r)\,dr, \tag{8}$$

where $g_0 = g(t_0)$ and $X(t, s)$ is the fundamental $n \times n$ matrix of (3) defined in Theorem 1.

For further consideration we assume that $t_0 = 0$.

Lemma. *If* 1° $A(t), B(t), f(t), g(t)$ *are periodic of period* ω; 2° $x(t)$ *is the solution of* (2), *then* $x(t + \omega)$ *is the solution of* (2) *if and only if*

$$\int_0^{\omega} [C_t(t + \omega - r) + B(t)F(t + \omega - r)]x(r)\,dr = 0. \tag{9}$$

Proof. From the identity

$$\frac{d}{dt}\left[x(t + \omega) - \int_0^{t+\omega} C(t + \omega - r)x(r)\,dr - g(t)\right]$$
$$= A(t)x(t + \omega) + B(t)\int_0^{t+\omega} F(t + \omega - r)x(r)\,dr + f(t)$$

and

$$\frac{d}{dt}\left[x(t + \omega) - \int_0^{t} C(t - r)x(r + \omega)\,dr - g(t)\right]$$
$$= A(t)x(t + \omega) + B(t)\int_0^{t} F(t - r)x(r + \omega)\,dr + f(t)$$

we see that

$$\frac{d}{dt}\int_0^{\omega} C(t + \omega - r)x(r)\,dr + B(t)\int_0^{\omega} F(t + \omega - r)x(r)\,dr = 0.$$

Hence

$$\int_0^{\omega} \big(C_t(t + \omega - r) + B(t)F(t + \omega - r)\big)x(r)\,dr = 0.$$

The sufficiency is obvious.

Theorem 3. *If* $A(t), B(t), f(t)$ *and* $g(t)$ *are periodic of period* ω, *then the solution* $x(t)$ *of* (2) *is periodic if and only if* (9) *and* $x(0) = x(\omega)$ *hold.*

Proof. If (9) and $x(0) = x(\omega)$ hold, then the solutions $x(t+\omega)$ and $x(t)$ coincide for $t = 0$. Hence, according to the uniqueness theorem, they coincide for any t and thus $x(t)$ is periodic of period ω. If the solution $x(t)$ of (2) is periodic of period ω, then the above conditions are obviously verified.

Example. Consider the scalar equation

$$\frac{d}{dt}\left[x(t) - \int_0^t \exp(-4(t-s))x(s)\,ds - \frac{3}{2}\cos t\right]$$
$$= -x(t) + 4\int_0^t \exp(-4(t-s))x(s)\,ds + 2\cos t + \sin t.$$

It is not difficult to show that this equation has a periodic solution

$$x^*(t) = 2\sin t + (1/2)\cos t$$

Theorem 4. *If* 1° $A(t), B(t), f(t), g(t)$ *are periodic of period* ω;

2° $\det(E_n - X(\omega,0)) \neq 0$,

then the system (2) *admits a periodic solution of period* ω *if and only if*

$$\begin{aligned} x(t) = X(t,0)[E_n - X(\omega,0)]^{-1}&\left\{\int_0^\omega X(\omega,r)f(r)\,dr + \int_0^\omega X'_r(\omega,r)g(r)\,dr\right\} \\ &+ g(t) + \int_0^t X(t,r)f(r)\,dr + \int_0^t X'_r(t,r)g(r)\,dr \end{aligned} \tag{10}$$

satisfies (9). *This solution is* (10).

Proof. Let $x(t)$ be a periodic solution of period ω of the system (2), then from (8)

$$x(0) = X(\omega,0)(x_0 - g_0) + g(\omega) + \int_0^\omega X(\omega,r)f(r)\,dr + \int_0^\omega X'_r(\omega,r)g(r)\,dr.$$

Hence, since $\det(E_n - X(\omega,0)) \neq 0$ we have that

$$x_0 = g_0 + (E_n - X(\omega,0))^{-1}\left\{\int_0^\omega X(\omega,r)f(r)\,dr + \int_0^\omega X'_r(\omega,r)g(r)\,dr\right\}.$$

The solution $x(t)$ can therefore be written as (10).

The rest of this proof is very similar and therefore is omitted.

Following an argument similar to those of [**4**, Theorem 2; **3**, Theorem 2] we get

Theorem 5. *Let* $C, F \in L^1\langle 0,\infty)$ *and let* $A(t), B(t), f(t), g(t)$ *be periodic of period* ω. *If* $x(t) = x(t,0,x_0)$ *is a bounded solution of* (2) *on* $\langle 0,\infty)$, *then there is a sequence of positive integers* $\{n_j\}$, $n_j \to \infty$ *as* $j \to \infty$, *such that* $\{x(t+n_j\omega)\}$ *converges uniformly on compact subsets of* $(-\infty,\infty)$ *to a function* $x^*(t)$ *which is a solution of* (1).

Proof. Let $C, F \in L^1\langle 0,\infty)$ and let $x(t)$ be a bounded solution of (2) on $\langle 0,\infty)$. We want to show that $\{x(t+n\omega) : n = 1,\dots\}$ is equicontinuous and uniformly bounded on any fixed interval $\langle -k, k\rangle$.

For $t_2 \ge t_1 \ge -n\omega$, we integrate (2) from $t_1 + n\omega$ to $t_2 + n\omega$ and get

$$\begin{aligned}
&x(t_2+n\omega) - x(t_1+n\omega)\\
&\quad = \int_0^{t_2+n\omega} C(t_2+n\omega-s)x(s)\,ds - \int_0^{t_1+n\omega} C(t_1+n\omega-s)x(s)\,ds\\
&\qquad + \int_{t_1+n\omega}^{t_2+n\omega}\Big(A(t)x(t) + B(t)\int_0^t F(t-s)x(s)\,ds + f(t)\Big)\,dt\\
&\qquad\qquad + g(t_2+n\omega) - g(t_1+n\omega).
\end{aligned}$$

The functions $x(t), f(t), A(t), B(t)$ are bounded and $F \in L^1\langle 0,\infty)$, hence there exist M and N such that $|x(t)| \le M$, $|f(t)| \le M$, $|A(t)| \le M$, $|B(t)| \le M$ for $t \in \langle 0,\infty)$ and $\int_0^\infty |F(t)|\,dt = N < \infty$. Thus

$$\int_{t_1+n\omega}^{t_2+n\omega}\Big|A(t)x(t) + B(t)\int_0^t F(t-s)x(s)\,ds + f(t)\Big|\,dt \le M_1|t_2-t_1|$$

where $M_1 = M(1+MN+M)$. Since $C \in L^1\langle 0,\infty)$, then for any $\varepsilon > 0$, there exists a $T > 0$ such that

$$\int_t^\infty |C(s)|\,ds < \frac{\varepsilon}{8M} \quad \text{for } t \ge T \quad \text{and so} \quad \int_T^\infty |C(t_2-t_1+v) - C(v)|\,dv < \frac{\varepsilon}{4M}.$$

By the continuity of C, there exists a $\delta_1 > 0$ such that for $v \in \langle 0, T\rangle$ and $0 \le t_2 - t_1 \le \delta_1$ we have

$$|C(t_2-t_1+v) - C(v)| < \frac{\varepsilon}{4TM} \qquad \text{and} \qquad \int_0^{t_2-t_1} |C(v)|\,dv < \frac{\varepsilon}{4M}.$$

Thus

$$\begin{aligned}
&\Big|\int_0^{t_2+n\omega} C(t_2+n\omega-s)x(s)\,ds - \int_0^{t_1+n\omega} C(t_1+n\omega-s)x(s)\,ds\Big|\\
&\le M\int_0^{t_1+n\omega} |C(t_2-t_1+v) - C(v)|\,dv + M\int_0^{t_2-t_1} |C(v)|\,dv\\
&\le M\int_0^T |C(t_2-t_1+v) - C(v)|\,dv + M\int_T^\infty |C(t_2-t_1+v) - C(v)|\,dv\\
&\quad + M\int_0^{t_2-t_1} |C(v)|\,dv \le \frac{3}{4}
\end{aligned}$$

if $0 \le t_2 - t_1 \le \delta_1$.

By the continuity of g, there exists a $\delta_2 > 0$ such that

$$0 \le (t_2+n\omega) - (t_1+n\omega) \le \delta_2 \quad \text{imply} \quad |g(t_2+n\omega) - g(t_1+n\omega)| \le \varepsilon/8.$$

Let $\delta = \min(\delta_1, \varepsilon/8M_1, \delta_2)$. Then we have

$$|x(t_2 + n\omega) - x(t_1 + n\omega)| \le \varepsilon \qquad \text{if} \qquad 0 \le t_2 - t_1 \le \delta.$$

This implies that $\{x(t+n\omega)\}$ is equicontinuous and uniformly bounded on any fixed interval $\langle -k, k\rangle$, $k = 1, 2, \ldots$. By Ascoli's theorem there is a subsequence $\{x(t + n_1\omega)\}$ of the $x(t+n\omega)$'s converging uniformly on $\langle -1, 1\rangle$, which contains a subsequence $\{x(t + n_2\omega)\}$ on $\langle -2, 2\rangle$. Proceeding inductively we obtain a subsequence, say $\{x(t + n_j\omega)\}$, converging uniformly on any fixed interval $\langle -k, k\rangle$ to a continuous function $x^*(t)$.

Now, we show that $x^*(t)$ is a solution of (1). Integrating (2) from $n_j\omega$ to $t + n_j\omega$, we have

$$\begin{aligned}
x(t+n_j\omega) - x(n_j\omega) &= \int_0^{t+n_j\omega} C(t + n_j\omega - s)x(s)\,ds - \int_0^{n_j\omega} C(n_j\omega - s)x(s)\,ds \\
&\quad + \int_{n_j\omega}^{t+n_j\omega}\left(A(s)x(s) + B(s)\int_0^s F(s-v)x(v)\,dv + f(s)\right) ds \\
&\quad + g(t+n_j\omega) - g(n_j\omega) \\
&= \int_{-n_j\omega}^{t} C(t-v)x(v+n_j\omega)\,dv - \int_{-n_j\omega}^{0} C(-v)x(v+n_j\omega)\,dv \\
&\quad + \int_0^t \Bigg(A(v+n_j\omega)x(v+n_j\omega) \\
&\quad + B(v+n_j\omega)\int_{-n_j\omega}^{v} F(v-u)x(u+n_j\omega)\,du + f(v)\Bigg) dv + g(t) - g(n_j\omega) \\
&= \int_{-n_j\omega}^{t} C(t-v)x(v+n_j\omega)\,dv - \int_{-n_j\omega}^{0} C(-v)x(v+n_j\omega)\,dv \\
&\quad + \int_0^t \left(A(v)x(v+n_j\omega) + B(v)\int_{-n_j\omega}^{v} F(v-u)x(u+n_j\omega)\,du + f(v)\right) dv \\
&\quad + g(t) - g(n_j\omega).
\end{aligned}$$

Since $C, F \in L^1\langle 0, \infty)$, by Lebesgue's dominated convergence theorem by letting $j \to \infty$, we have

$$\begin{aligned}
x^*(t) - x^*(0) &= \int_{-\infty}^{t} C(t-v)x^*(v)\,dv - \int_{-\infty}^{0} C(-v)x^*(v)\,dv \\
&\quad + \int_0^t \left(A(v)x^*(v) + B(v)\int_{-\infty}^{v} F(v-u)x^*(u)\,du + f(v)\right) dv + g(t) - g(\infty).
\end{aligned}$$

Hence, by differentiation, we have

$$\begin{aligned}
&\frac{d}{dt}\left[x^*(t) - \int_{-\infty}^{t} C(t-v)x^*(v)\,dv - g(t)\right] \\
&\qquad = A(t)x^*(t) + B(t)\int_{-\infty}^{t} F(t-v)x^*(v)\,dv + f(t)
\end{aligned}$$

and so the limit function $x^*(t)$ is a solution of (1).

Let $A(t)$ and $B(t)$ be periodic and f, g — almost periodic and $\operatorname{Re}\alpha_j < 0$, $\operatorname{Re}\beta_j < 0$. Let $x(t)$ be an almost periodic solution of (2). Then

$$\begin{aligned}\int_0^t & e^{\beta_j(t-z)}(t-z)^s x(z)\,dz \\ &= -\int_{-\infty}^0 e^{\beta_j(t-z)}(t-z)^s x(z)\,dz + \int_{-\infty}^t e^{\beta_j(t-z)}(t-z)^s x(z)\,dz,\end{aligned}$$

where $\int_{-\infty}^t e^{\beta_j(t-z)}(t-z)^s x(z)\,dz$ is an almost periodic function. Since $x(t)$ is an almost periodic solution of (2), then

$$\sum_{s=0}^{n_j} B(t)\varphi_{sj}\int_{-\infty}^0 e^{\beta_j(t-z)}(t-z)^s x(z)\,dz \equiv 0 \qquad (j=1,\dots,k). \tag{11}$$

It is easy to see that

$$\sum_{s=0}^{n_j} B(t)\varphi_{sj}\int_{-\infty}^0 e^{\beta_j(t-z)}(t-z)^s x(z)\,dz \qquad (j=1,\dots,k)$$

is an almost periodic function. In this case, if

$$\sum_{s=0}^{n_j} \varphi_{sj}\int_{-\infty}^0 e^{\alpha_j(t-z)}(t-z)^s x(z)\,dz \equiv 0 \qquad (j=1,\dots,k). \tag{11'}$$

then

$$\sum_{s=0}^{n_j} \varphi_{sj}\int_0^t e^{\alpha_j(t-z)}(t-z)^s x(z)\,dz$$

is an almost periodic function.

On the ground of conditions (11), (11′), the solution $x(t)$ of the system (2) is the solution of the system (1).

Note that (1) is equivalent to the system

$$\begin{aligned}\frac{dx}{dt} &= \Big[A(t) + \sum_{j=1}^{k}\psi_{0j}\Big] + B(t)\sum_{j=1}^{k}\sum_{s=0}^{n_j}\varphi_{sj}w_{sj} \\ &\quad + \sum_{j=1}^{k}\sum_{s=1}^{n_j}\psi_{sj}(\alpha_j v_{sj} + s v_{s-1\,j}) + \sum_{j=1}^{k}\psi_{0j}\alpha_j v_{0j} + f(t) + g'(t)\end{aligned} \tag{12}$$

$$\begin{aligned}&dv_{0j}/dt = x + \alpha_j v_{0j}, && dw_{0j}/dt = x + \beta_j w_{0j} && s = 1,\dots,n_j,\\ &dv_{sj}/dt = \alpha_j v_{sj} + s v_{s-1\,j}, && dw_{sj}/dt = s w_{s-1\,j} + \beta_j w_{sj} && j = 1,\dots,k\end{aligned}$$

where

$$w_{sj}(t) = \int_{-\infty}^t e^{\beta_j(t-z)}(t-z)^s x(z)\,dz, \qquad v_{sj}(t) = \int_{-\infty}^t e^{\alpha_j(t-z)}(t-z)^s x(z)\,dz.$$

According to the assumption, $x(t)$ is an almost periodic function, so $w_{sj}(t)$ and $v_{sj}(t)$ are almost periodic functions.

Theorem 6. *If $A(t)$ and $B(t)$ are periodic, f, g are almost periodic and* $\operatorname{Re}\alpha_j < 0$, $\operatorname{Re}\beta_j < 0$ $(j = 1, \dots, k)$, *then $x(t)$ is an almost periodic solution of the system* (2) *if and only if $(x(t), w_{sj}(t), v_{sj}(t))$ is an almost periodic solution of the system* (12) *and conditions* (11)–(11′) *are satisfied.*

Proof. If the system (2) admits an almost periodic solution $x(t)$, then the conditions (11)–(11′) are obviously verified and (12) has an almost periodic solution.

Conversely, if (12) has an almost periodic solution and conditions (11)–(11′) are satisfied, then the solution $x(t)$ of (2) is almost periodic.

Remark. If $A(t) = A = \text{const}$, $B(t) = B = \text{const}$, $t_0 = 0$, then the system (2) has a general solution of the form

$$x(t) = X(t)(x_0 - g_0) + g(t) + \int_0^t X(t-s)f(s)\,ds + \int_0^t X'(t-s)g(s)\,ds.$$

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(Received 20 07 1990)