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INDEPENDENT VERTEX SETS IN SOME COMPOUND GRAPHS

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Abstract. Let G be an n-vertex graph and $R_1, R_2, \dots, R_n$ distinct rooted graphs. The compound graph $G[R_1, R_2, \dots, R_n]$ is obtained by identifying the root of R_i with the i-th vertex of G, $i = 1, 2, \dots, n$. We determine the number of independent vertex sets and the independence polynomial of $G[R_1, R_2, \dots, R_n]$. Several special cases of these results are pointed out.

1. Introduction

Consider finite graphs without loops. If G is such a graph, then $\boldsymbol{V}(G)$ will denote its vertex set. Any subset of $\boldsymbol{V}(G)$, such that no two elements of it are mutually adjacent, is called an independent vertex set of the graph G. Let $\mathrm{Ind}(G)$ be the set of all independent vertex sets of G.

The number $\sigma(G)$ of independent vertex sets of the graph G, i.e. the cardinality of $\mathrm{Ind}(G)$, has been examined in a number of recent papers [**1**–**11**]. In particular, Prodinger and Tichy [**7**, **11**] called the quantity $\sigma(G)$ "the Fibonacci number of the graph G". The motivation for this was the fact that if P_n is the path-graph with n vertices, then $\sigma(P_n)$ is equal to the $(n+1)$-th Fibonacci number.

In the present paper we determine the number of independent vertex sets of the compound graph $G[R_1, R_2, \dots, R_n]$ constructed in the following manner.

Let G be a graph with the vertex set $\boldsymbol{V}(G) = \{v_1, v_2, \dots, v_n\}$. Let further $R_1, R_2, \dots, R_n$ be distinct rooted graphs; the root of R_i is denoted by r_i, $i = 1, 2, \dots, n$. Then $G[R_1, R_2, \dots, R_n]$ is the graph obtained by identifying the vertex v_i of G with the root r_i of R_i, simultaneously for $i = 1, 2, \dots, n$ (see Fig. 1).

Denote by R_i° the graph obtained by deleting from R_i the root-vertex r_i and the edges incident to it. Denote by $R_i^\bullet$ the graph obtained by deleting from R_i the root-vertex r_i, the vertices adjacent to r_i and all the incident edges. Then the main result of our work can be formulated as follows.

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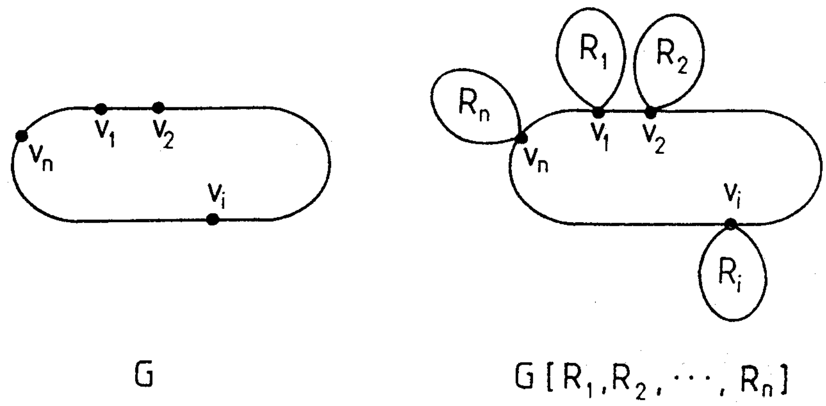

Fig. 1

THEOREM 1. *Let I be an independent vertex set of the graph G. Define*

$$\sigma_i(I) = \begin{cases} \sigma(R_i^\circ) & \text{if } v_i \notin I, \\ \sigma(R_i^\bullet) & \text{if } v_i \in I. \end{cases}$$

Then

$$\sigma(G[R_1, R_2, \ldots, R_n]) = \sum_{I \in \mathrm{Ind}(G)} \prod_{i=1}^{n} \sigma_i(I).$$

Instead of Theorem 1 we prove a somewhat stronger result, namely Theorem 2. In order to do this we need some preparations.

2. The independence polynomial

Denote by $n(G,k)$ the number of distinct k-element independent vertex sets of the graph G. Then the polynomial

$$\omega(G) = \omega(G,x) = \sum_{k \geq 0} n(G,k)\, x^k \tag{1}$$

is called the independence polynomial of the graph G [**3**, **5**, **6**]. Evidently, $\omega(G,1) = \sigma(G)$.

The basic properties of the independence polynomial have been determined by Gutman and Harary [**5**] and recently by Hoede and Li [**6**].

Two of these properties will be needed in the subsequent considerations:

(a) If v is a vertex of the graph G and N_v is the set containing v and its first neighbors, then
$$\omega(G) = \omega(G-v) + x\,\omega(G-N_v). \tag{2}$$

(b) If $G_1 \cup G_2$ is the graph composed of components G_1 and G_2, then
$$\omega(G_1 \cup G_2) = \omega(G_1)\,\omega(G_2). \tag{3}$$

THEOREM 2. *Let I be an independent vertex set of the graph G. Define*

$$\omega_i(I) = \begin{cases} \omega(R_i^\circ) & \text{if } v_i \notin I, \\ \omega(R_i^\bullet) & \text{if } v_i \in I. \end{cases} \tag{4}$$

Then

$$\omega(G[R_1, R_2, \dots, R_n], x) = \sum_{I \in \mathrm{Ind}(G)} x^{|I|} \prod_{i=1}^{n} \omega_i(I) \tag{5}$$

where $|I|$ stands for the number of elements of I.

Evidently, Theorem 1 is a special case of Theorem 2, obtained by setting $x = 1$ in formula (5).

3. Proof of Theorem 2

We demonstrate the validity of Theorem 2 by induction on the number n of vertices of the graph G. If $n = 1$, then $\boldsymbol{V}(G) = \{v_1\}$ and therefore $G[R_1, R_2, \dots, R_n]$ coincides with R_1. On the other hand, for $n = 1$ the set $\mathrm{Ind}(G)$ consists of only two elements, namely $I_1 = \varnothing$ and $I_2 = \{v_1\}$. Bearing in mind (4) we have $\omega_1(I_1) = \omega(R_1^\circ)$ and $\omega_1(I_2) = \omega(R_1^\bullet)$. Consequently, the right-hand side of (5) is equal to $\omega(R_1^\circ) + x\,\omega(R_1^\bullet)$. Because of (2) this latter expression is equal to $\omega(G)$.

Thus the statement of Theorem 2 is true for $n = 1$. In a similar manner one can check that Theorem 2 is satisfied for $n = 2$ and $n = 3$.

Assume now that Theorem 2 holds for all graphs G with less than n vertices. In order to accomplish the inductive proof we have to show that this assumption implies the validity of Theorem 2 for the graphs G having n vertices.

Suppose that $n \geq 3$ and apply formula (2) to the vertex v_n of the graph G. Without loss of generality we may label the vertices of G so that v_n is adjacent to $v_{n-1}, \dots, v_{n-d}$. Then by using (3),

$$\begin{aligned} \omega(G[R_1, R_2, \dots, R_n]) &= \omega(R_n^\circ)\,\omega((G - v_n)[R_1, R_2, \dots, R_n]) \\ &\quad + \omega(R_n^\bullet)\,\omega(R_{n-1}^\circ) \cdot \ldots \cdot \omega(R_{n-d}^\circ)\,\omega((G - N_{v_n})[R_1, R_2, \dots, R_n]). \end{aligned} \tag{6}$$

The subgraphs $G - v_n$ and $G - N_{v_n}$ have $n-1$ and $n-1-d$ vertices, respectively. Therefore according to the induction hypothesis:

$$\omega((G - v_n)[R_1, R_2, \dots, R_n]) = \sum_{I \in \mathrm{Ind}(G - v_n)} x^{|I|} \prod_{i=1}^{n-1} \omega_i(I) \tag{7}$$

$$\omega((G - N_{v_n})[R_1, R_2, \dots, R_n]) = \sum_{I \in \mathrm{Ind}(G - N_{v_n})} x^{|I|} \prod_{i=1}^{n-d-1} \omega_i(I). \tag{8}$$

The set $\mathrm{Ind}(G)$ can be partitioned into two disjoint subsets $\mathrm{Ind}^\circ(G)$ and $\mathrm{Ind}^\bullet(G)$, such that $\mathrm{Ind}^\circ(G)$ is the set of independent vertex sets of G which do not contain the vertex v_n whereas $\mathrm{Ind}^\bullet(G)$ is the set of those independent vertex sets of G which do contain v_n. It is easy to see that

$$\mathrm{Ind}^\circ(G) = \mathrm{Ind}(G - v_n) \tag{9}$$

$$\mathrm{Ind}^\bullet(G) = \big\{I \cup \{v_n\} \mid I \in \mathrm{Ind}(G - N_{v_n})\big\}. \tag{10}$$

Bearing in mind (7)–(10), the relation (6) is transformed into

$$\begin{aligned}\omega(G[R_1, R_2, \dots, R_n]) = \omega(R_n^\circ) \sum_{I \in \mathrm{Ind}^\circ(G)} x^{|I|} \prod_{i=1}^{n-1} \omega_i(I) \\ + x\,\omega(R_n^\bullet)\,\omega(R_{n-1}^\circ) \cdot \dots \cdot \omega(R_{n-d}^\circ) \sum_{I \in \mathrm{Ind}^\bullet(G)} x^{|I|-1} \prod_{i=1}^{n-d-1} \omega_i(I).\end{aligned} \tag{11}$$

For all $I \in \mathrm{Ind}^\circ(G)$, $v_n \notin I$ and therefore $\omega_n(I) = \omega(R_n^\circ)$. For similar reasons, the relations $\omega_n(I) = \omega(R_n^\bullet)$ and $\omega_j(I) = \omega(R_j^\circ)$, $j = 1, \dots, d$, are satisfied for all $I \in \mathrm{Ind}^\bullet(G)$. Consequently, equation (11) becomes

$$\omega(G[R_1, R_2, \dots, R_n]) = \sum_{I \in \mathrm{Ind}^\circ(G)} x^{|I|} \prod_{i=1}^{n} \omega_i(I) + \sum_{I \in \mathrm{Ind}^\bullet(G)} x^{|I|} \prod_{i=1}^{n} \omega_i(I)$$

and formula (5) follows from the fact that $\mathrm{Ind}^\circ(G) \cup \mathrm{Ind}^\bullet(G) = \mathrm{Ind}(G)$. This completes the proof of Theorem 2.

4. Special cases

4.1. All R_i are isomorphic. The graph $G[R_1, R_2, \dots, R_n]$ in which all R_i, $i = 1, 2, \dots, n$ are isomorphic to the rooted graph R is denoted by $G[R]$. For the compound graphs $G[R]$ formula (5) is much simplified by the fact that the product $\prod_{i=1}^n \omega_i(I)$ depends only on the cardinality k of the independent vertex set I and is equal to $\omega(R - r)^{n-k}\,\omega(R - N_r)^k$ where r stands for the root of R. Since the number of k-element independent vertex sets of the graph G is equal to $n(G, k)$ we further have

$$\omega(G[R]) = \sum_{k \geq 0} x^k\, n(G, k)\, \omega(R - r)^{n-k}\, \omega(R - N_r)^k. \tag{12}$$

This, bearing in mind the definition of $\omega(G)$, immediately leads to Corollary 2.1.

Corollary 2.1. $\omega(G[R], x) = \omega(R - r, x)^n\, \omega(G, \omega(R - N_r)/\omega(R - r))$.

Corollary 2.2. *If k^* is the maximum cardinality of an independent vertex set of the graph G, then the polynomial $\omega(R - r)^{n-k^*}$ divides the polynomial $\omega(G[R])$.*

4.2. The corona. The corona $G \circ Q$ of the graphs G and Q is obtained from G and n copies of Q, so that each vertex of G is joined to all vertices of a copy of Q. Whence, $G \circ Q$ is a special case of $G[R]$ when the root r of R is adjacent to all other vertices of R. In this notation, $Q = R - r$.

COROLLARY 2.3. $\omega(G \circ Q, x) = \omega(Q, x)^n \, \omega(G, 1/\omega(Q))$.

COROLLARY 2.4. *If k^* is the maximum cardinality of an independent vertex set of the graph G, then the polynomial $\omega(Q)^{n-k^*}$ divides the polynomial $\omega(G \circ Q)$.*

4.3. Some more special cases. If G is the complete graph K_n then $\mathrm{Ind}(G)$ consists of $n+1$ elements: the empty set and n one-element sets, each containing one vertex of G. Formula (5) gives then

$$\omega(K_n[R_1, R_2, \ldots, R_n]) = x^0 \prod_{i=1}^{n} \omega_i(\varnothing) + \sum_{j=1}^{n} \prod_{i=1}^{n} \omega_i(\{v_j\}).$$

Bearing in mind (4) we arrive at

COROLLARY 2.5. $\omega(K_n[R_1, R_2, \ldots, R_n]) = \left[1 + x \sum_{j=1}^{n} \frac{\omega(R_j^\bullet)}{\omega(R_j^\circ)}\right] \prod_{i=1}^{n} \omega(R_i^\circ)$.

It is easy to deduce combinatorial formulas for the $n(G, k)$-numbers of the path P_n and the circuit C_n [**2**, **5**]. Then equations (12) and (2) lead to

COROLLARY 2.6.

$$\omega(P_n(R)) = \sum_{k \geq 0} \binom{n-1-k}{k} \omega(R-r)^{n-k} [\omega(R) - \omega(R-r)]^k,$$

$$\omega(C_n(R)) = \sum_{k \geq 0} \frac{n}{n-k} \binom{n-k}{k} \omega(R-r)^{n-k} [\omega(R) - \omega(R-r)]^k.$$

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