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EMBEDDING DERIVATIVES OF $\mathcal{M}$-HARMONIC TENT SPACES INTO LEBESGUE SPACES

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Abstract. A characterization is given of those measures μ on B, the open unit ball in $\mathbb{C}^n$, such that differentiation of order m maps the $\mathcal{M}$-harmonic tent space $\mathcal{H}^p$ boundedly into $L^q(\mu)$, $0 < p < q < \infty$.

1. Introduction. Let B be the open unit ball in $\mathbb{C}^n$ with (normalized) volume measure ν and let S denote its boundary. For the most part we will follow the notation and terminology of Rudin [**5**]. If $\alpha > 1$ and $\xi \in S$ the Koranyi approach regions are defined by

$$D_\alpha(\xi) = \left\{ z \in B : |1 - \langle z, \xi \rangle| < \tfrac{1}{2}\alpha(1 - |z|^2) \right\}.$$

For any function f on B, we define a scale of maximal functions by

$$M_\alpha f(\xi) = \sup\{|f(z)| : z \in D_\alpha(\xi)\}.$$

For simplicity of notation, we write simply $D(\xi)$ for $D_2(\xi)$ and Mf for $M_2 f$. For $0 < p < \infty$, the tent space $T^p = T^p(B)$ is defined to be the space of all continuous functions f on B such that $Mf \in L^p(\sigma)$. Here σ denotes the rotation invariant probability measure on S. We note that the use of approach regions of "aperture" 2 in the definition of T^p is merely a convenience: approach regions of any other aperture would yield the same class of functions with an equivalent norm.

Let $\widetilde{\Delta}$ be the invariant Laplacian on B. That is, $(\widetilde{\Delta} f)(z) = \Delta(f \circ \varphi_z)(0)$, $f \in C^2(B)$, where Δ is the ordinary Laplacian and φ_z the standard automorphism of B ($\varphi_z \in \mathrm{Aut}(B)$) taking 0 to z (see [**5**]). A function f defined on B is $\mathcal{M}$-harmonic, $f \in \mathcal{M}$, if $\widetilde{\Delta} f = 0$.

We shall call $\mathcal{H}^p = \mathcal{M} \cap T^p$ $\mathcal{M}$-harmonic tent space.

For $f \in \mathcal{M}$ let

$$\partial f(z) = \left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n}, \frac{\partial f}{\partial \bar{z}_1}, \dots, \frac{\partial f}{\partial \bar{z}_n} \right)$$

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and for any positive integer m we write

$$\partial^m f(z) = (\partial^\alpha \overline{\partial}^\beta f(z))_{|\alpha|+|\beta|=m} \quad \text{and} \quad |\partial^m f(z)|^2 = \sum_{|\alpha|+|\beta|=m} |\partial^\alpha \overline{\partial}^\beta f(z)|^2,$$

where $\partial^\alpha \overline{\partial}^\beta f(z) = \dfrac{\partial^{|\alpha|+|\beta|} f(z)}{\partial z_1^{\alpha_1} \dots \partial z_n^{\alpha_n} \partial \bar{z}_1^{\beta_1} \dots \partial \bar{z}_n^{\beta_n}}$, α and β are multi-indices.

Let μ be a positive measure on B and consider the problem of determining what conditions on μ imply $|\partial^\beta f| \in L^q(\mu)$, whenever $f \in \mathcal{H}^p$. A standard application of the closed graph theorem leads to the following equivalent problem: Characterize such μ for which there exists a constant C satisfying

$$\left(\int_B |\partial^\beta f|^q \, d\mu\right)^{1/q} \le C \left(\int_S |Mf|^p \, d\sigma\right)^{1/p} = C\|f\|_{\mathcal{H}^p}.$$

The purpose of this paper is to present a solution of this problem in the case $0 < p < q < \infty$. To state it we need some more notations.

For $\xi \in S$ and $\delta > 0$ the following nonisotropic balls are defined

$$B(\xi, \delta) = \{z \in B : |1 - \langle z, \xi\rangle| < \delta\}, \qquad Q(\xi, \delta) = \{\eta \in S : |1 - \langle \eta, \xi\rangle| < \delta\}.$$

For each $E \subset S$ we define the α-tent over E by $T_\alpha(E) = \left(\bigcup_{\xi \notin E} D_\alpha(\xi)\right)^c$, the complement being taken in B. We write $T(E)$ for $T_2(E)$. For $z \in B$ and r, $0 < r < 1$, $E_r(z) = \{w \in B : |\varphi_z(w)| < r\}$, we will let $|E_r(z)| = \nu(E_r(z))$.

Throughout the paper r will be fixed and we will occasionally write $E(z)$ instead of $E_r(z)$. Constants will be denoted by C which may indicate a different constant from one occurrence to the next.

Theorem. *Let $0 < p < q < \infty$. For a positive measure μ on B and a positive integer m, necessary and sufficient condition for*

$$\left(\int_B |\partial^m f|^q \, d\mu\right)^{1/q} \le C\|f\|_{\mathcal{H}^p} \tag{1.1}$$

is that there exists a constant C for which

$$\mu(E(z)) \le C(1 - |z|)^{nq/p + mq}, \qquad z \in B. \tag{1.2}$$

For holomorphic functions Theorem was proved by Shirokov and Luecking (see [**3**, **4**, **6**, **7**]).

2. Proof of Theorem.

The following two preliminary lemmas will be needed in the proof of Theorem.

LEMMA 2.1 [**2**]. *Let $k \geq m$ be non-negative integers, $0 < p < \infty$ and $0 < r < 1$. There exists a constant $C = C(k, m, p, r, n)$ such that if $f \in \mathcal{M}$ then*

$$|\partial^k f(w)|^p \leq C(1-|w|)^{(m-k)p} \int_{E_r(w)} |\partial^m f(z)|^p (1-|z|)^{-n-1}\, d\nu(z), \quad \textit{for all } w \in B.$$

LEMMA 2.2. *Let $1 < \alpha < \infty$, $0 < p < \infty$ and let μ be a finite Borel measure in B. In order that there exist a constant C such that*

$$\int_B |f(z)|^{\alpha p}\, d\mu(z) \leq C\|f\|^{\alpha p}_{\mathcal{H}^p}, \qquad \textit{for all } f \textit{ in } \mathcal{H}^p, \tag{2.1}$$

it is necessary and sufficient that there exists a constant C for which

$$\mu(B(\xi, \delta)) \leq C\delta^{n\alpha}, \qquad \xi \in C,\ \delta > 0. \tag{2.2}$$

Proof. The necessity of the condition (2.2) follows upon applying the inequality (2.1) to appropriate f (see [**1**]).

It is easy to see that the condition (2.2) is equivalent to

$$\mu(T(Q(\xi, \delta))) \leq C\delta^{n\alpha}, \tag{2.3}$$

for some constant C and for all $\xi \in S$, $\delta > 0$. The sufficiency can be gotten by the following argument. For $\lambda > 0$, let $E_\lambda = \{\xi \in S : Mf(\xi) > \lambda\}$. By Whitney decomposition theorem there is a family $\mathcal{B}$ of sets $Q = Q(\xi, \delta)$ such that $E_\lambda = \bigcup\{Q : Q \in \mathcal{B}\}$ and a countable disjoint subfamily $\{Q_n = Q_n(\xi_n, \delta_n)\}$ of $\mathcal{B}$ such that each Q in $\mathcal{B}$ is in some $cQ_n = Q_n(\xi_n, c\delta_n)$. Then $\{z \in B : |f(z)| > \lambda\} \subset \bigcup T(cQ_n)$, so

$$\begin{aligned}\mu(\{|f| > \lambda\}) &\leq \sum_n \mu(T(cQ_n)) \leq C\sum_n [\sigma(cQ_n)]^\alpha \\ &\leq C\Big(\sum_n \sigma(Q_n)\Big)^\alpha \leq C(\sigma(E_\lambda))^\alpha.\end{aligned}$$

Integrating this inequality with respect to $\lambda^{p-1}\, d\lambda$ we get

$$\begin{aligned}\int_B |f|^{\alpha p}\, d\mu &= p\int_0^\infty \lambda^{\alpha p-1}\mu(\{|f| > \lambda\})\, d\lambda \\ &\leq C\sum_{n=-\infty}^{\infty} 2^{k\alpha p}\mu(\{|f| > \lambda\}) \leq C\bigg(\sum_{k=-\infty}^{\infty} 2^{kp}\big(\mu(\{|f| > \lambda\})\big)^{1/\alpha}\bigg)^\alpha \\ &\leq C\bigg(\sum_{k=-\infty}^{\infty} 2^{kp}\sigma(\{Mf > \lambda\})\bigg)^\alpha \leq C\|Mf\|^{\alpha p}_{\mathcal{H}^p}.\end{aligned}$$

Proof of Theorem. The necessity of the condition (1.2) follows from the Shirokov-Luecking theorem mentioned above.

The sufficiency is obtained by the same arguments as in [**4**]:

$$(1-|z|)^{mq}|\partial^m f(z)|^q \le C\int_{E(z)} |f(w)|^q (1-|w|)^{-n-1}\,d\nu(w),$$

by Lemma 2.1. Integrating both sides with respect to $(1-|z|)^{-mq}\,d\mu(z)$ and using Fubini's theorem on the right, we obtain

$$\int_B |\partial^m f(z)|^q\,d\mu(z) \le C\int_B |f(w)|^q \mu(E(w))(1-|w|)^{-mq-n-1}\,d\nu(w).$$

By Lemma 2.2 and Theorem 2 of [**1**], we conclude that

$$\int_B |\partial^m f(z)|^q\,d\mu(z) \le C\|f\|^q_{\mathcal{H}^p}.$$

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