PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 52 (66), 1992, 47–60

CONTINUOUS DEPENDENCE RESULTS FOR SUBDIFFERENTIAL INCLUSIONS

Nikolaos S. Papageorgiou

Abstract. We examine the dependence on a parameter of the solution set of a class of nonlinear evolution inclusions driven by subdifferential operators. We prove that under mild hypotheses on the data, the solution set depends continuously on the parameter for both the Vietoris and Hausdorff topologies. Then we use these results to study the variational stability of class of semilinear parabolic optimal control problems and we also indicate how our work also incorporates the stability analysis of differential variational inequalities.

1. Introduction. Let $T = [0, b]$ and H a separable Hilbert space. We consider the following parametrized family of evolution inclusions of the subdifferential type:

$$-\dot{x} \in \partial\varphi(x(t), \lambda) + F(t, x(t), \lambda) \text{ a.e.} \qquad x(0) = x_0(\lambda) \tag{1}$$

Denote the set of strong solutions (see Section 2) of (1) by $S(\lambda) \subseteq C(T, H)$. The purpose of this note is to study the continuity properties of the multifunction $\lambda \to S(\lambda)$. Analogous continuous dependence results were obtained earlier by Vasilev [**21**] and Lim [**9**] for differential inclusions in $\mathbb{R}^n$ and by Tolstonogov [**19**] and Papageorgiou [**12**], who considered differential inclusions in Banach spaces, but without subdifferential operators present. In fact, their hypotheses are such that preclude the application of their work to multivalued partial differential equations and to distributed parameter optimal control problems. More recently, Kravvaritis and Papageorgiou [**8**], considered evolution inclusions of the subdifferential type and under more restrictive hypotheses on the data, established that the solution multifunction $S(\,\cdot\,)$ has a closed graph (see theorem 4.1 in [**8**]).

In this paper, under general hypotheses on the data (weaker than those in theorem 4.1 of Kravvaritis and Papageorgiou [**8**]), we prove that $S(\,\cdot\,)$ is continuous for both the Vietoris and Hausdorff metric topologies (see theorems 3.2 and 3.3). Then we use these results to establish a sensitivity result for a class of semilinear parabolic distributed parameter optimal control problems.

AMS Subject Classification (1980): Primary 34 G 20, 49 A 20

2. Preliminaries. In what follows, $T=[0,r]$, equipped with the Lebesgue measure dt and H is a separable Hilbert space. Throughout this paper we will be using the following notations:

$$P_{f(c)}(H)=\{A\subseteq H: \text{nonempty, closed, (convex)}\}$$
$$P_{(w)k(c)}(H)=\{A\subseteq H: \text{nonempty, (weakly-) compact, (convex)}\}.$$

A multifunction $F\colon T\to P_f(H)$ is said to be measurable, if for all $x\in H$ $t\to d(x,F(t))=\inf\{\|x-v\|: v\in F(t)\}$ is a measurable $\mathbb{R}_+$-valued function. By S_F^1 we will denote the set of selectors of $F(\,\cdot\,)$ that belong in the Lebesgue-Bochner space $L^1(H)$; i.e. $S_F^1=\{f\in L^1(H): f(t)\in F(t), \text{ a.e.}\}$. This set may be empty. For a measurable $F(\,\cdot\,)$, it is nonempty if and only if $t\to\inf\{\|v\|: v\in F(t)\}\in L_+^1$.

Let $\varphi\colon H\to\bar{\mathbb{R}}=\mathbb{R}\cup\{+\infty\}$. We will say that $\varphi(\,\cdot\,)$ is proper, if it is not identically $+\infty$. Assume that $\varphi(\,\cdot\,)$ is proper, convex and lower semicontinuous (l.s.c.). It is customary to denote this family of $\bar{\mathbb{R}}$-valued functions by $\Gamma_0(H)$. By $\operatorname{dom}\varphi$, we denote the effective domain of $\varphi(\,\cdot\,)$; i.e., $\operatorname{dom}\varphi=\{x\in H:\varphi(x)<\infty\}$. The subdifferential of $\varphi(\,\cdot\,)$ at x is the set $\partial\varphi(x)=\{x^*\in H:(x^*,y-x)\leq\varphi(y)-\varphi(x) \text{ for all } y\in\operatorname{dom}\varphi\}$, where $(\,\cdot\,,\,\cdot\,)$ denotes the inner product of H. If $\varphi(\,\cdot\,)$ is Gateaux differentiable at x, then $\partial\varphi(x)=\{\varphi'(x)\}$. We say that $\varphi(\,\cdot\,)$ is of compact type, if for every $\lambda\in\mathbb{R}$, the level set $\{x\in H:\|x\|^2+\varphi(x)\leq\lambda\}$ is compact. Also for $\mu>0$, we define $J_\mu=(I+\mu\partial\varphi)^{-1}$ (the resolvent of $\partial\varphi(\,\cdot\,)$). It is well known (see for example the book of Brezis [**3**]), that for all $\mu>0$, $D(J_\mu)=H$ and furthermore $J_\mu(\,\cdot\,)$ is nonexpansive.

Let X be a Banach space and $\{A_n,A\}_{n\geq 1}\subseteq 2^X\setminus\{\emptyset\}$. Let s- denote the strong topology on X and w- the weak topology on X. We define:

$$\begin{aligned}
\text{s-}\underline{\lim}\,A_n&=\{x\in X:\lim d(x,A_n)=0\}=\{x\in X: x=\text{s-}\lim x_n, x_n\in A_n, n\geq 1\},\\
\text{s-}\overline{\lim}\,A_n&=\{x\in X:\underline{\lim}\,d(x,A_n)=0\}\\
&=\{x\in X: x=\text{s-}\lim x_{n_k}, x_{n_k}\in A_{n_k}, n_1<n_2<\cdots<n_k<\dots\}.
\end{aligned}$$

It is clear from the above definitions, that we always have: $\text{s-}\underline{\lim}\,A_n\subseteq\text{s-}\overline{\lim}\,A_n\subseteq\text{w-}\overline{\lim}\,A_n$. If $\text{s-}\underline{\lim}\,A_n=\text{s-}\overline{\lim}\,A_n=A$, then we say that the A_n's converge to A in the Kuratowski sense and denote it by $A_n\xrightarrow{K}A$ as $n\to\infty$. If $\text{s-}\underline{\lim}\,A_n=\text{w-}\overline{\lim}\,A_n=A$, then we say that the A_n's converge to A in the Kuratowski–Mosco sense, denoted by $A_n\xrightarrow{K-M}A$.

Let Λ be a complete metric space. A multifunction $G\colon\Lambda\to P_f(X)$ is said upper semicontinuous (u.s.c) (resp. lower semicontinuous (l.s.c.)) if for all $U\subseteq X$ nonempty and open, the set $G^+(U)=\{\lambda\in\Lambda: G(\lambda)\subseteq U\}$ (resp. the set $G^-(U)=\{\lambda\in\Lambda: G(\lambda)\cap U\neq\emptyset\}$) is open in Λ. A multifunction $G(\,\cdot\,)$ which is both u.s.c. and l.s.c., is said to be continuous or Vietoris continuous, to emphasize that it is continuous into the hyperspace $P_f(X)$ equipped with the Vietoris topology (see Klein-Thompson [**7**]). If $\overline{G(\Lambda)}=\overline{\bigcup_{\lambda\in\Lambda}G(\lambda)}$ is compact in X, then the $G(\,\cdot\,)$ is Vietoris continuous if and only if for $\lambda_n\to\lambda$ in Λ, we have $G(\lambda_n)\xrightarrow{K}G(\lambda)$. This follows from remarks 1.6 and 1.8 of DeBlasi and Myjak [**4**].

On $P_f(X)$ we can define a generalized metric, known in the literature as Hausdorff metric, by

$$h(A,B) = \max[\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)].$$

Recall that $(P_f(X),h)$ is a complete metric space. A multifunction $G:\Lambda \to P_f(X)$ is said to be Hausdorff continuous (h-continuous), if it is continuous from Λ into the metric space $(P_f(X),h)$. On $P_k(X)$ the Vietoris and Hausdorff metric topologies coincide (see Klein and Thompson [**7**, Corollary 4.2.3, p. 41]). So a multifunction $G:\Lambda \to P_k(X)$ is Vietoris continuous if and only if it is h-continuous (see DeBlasi and Myjak [**4**, remark 1.9]. Finally a multifunction $G:\Lambda \to P_f(X)$ is said to be d-continuous, if for all $x \in X$, $\lambda \to d(x, G(\lambda))$ is continuous. Clearly if $G(\cdot)$ is h-continuous, then it is d-continuous too.

The following theorem was first proved by the author (see [**12**, theorem 3.1]) and recently improved by Rybinski (see [**18**, theorem 1 and the remark on p. 33]). Here we state improved version obtained by Rybinski [**18**].

Theorem 2.1 **If** *X is a Banach space, $K \in P_{\mathrm{wk}}(X)$, F, $F_n: K \to P_{\mathrm{wkc}}(K)$ are h-Lipschitz multifunctions with the same Lipschitz constant $k \in (0,1)$ such that if $x_n \xrightarrow{s} x$, then $F_n(x_n) \xrightarrow{K-M} F(x)$,* **then** *if $L_n = \{x \in X : x \in F_n(x)\}$ and $L = \{x \in X : x \in F(x)\}$, then $L_n \xrightarrow{K} L$ as $n \to \infty$.*

Remark. The fixed point sets L_n, L are nonempty by Nadler's fixed point theorem [**11**].

Let L be a complete metric space (the parameter space), $T = [0,b]$ and H a separable Hilbert space. The following hypothesis concerning $\varphi(x,\lambda)$ will be in effect throughout this work.

$(H(\varphi))$ $\varphi: X \times \Lambda \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is a function such that:

(1) for every $\lambda \in \Lambda$, $\varphi(\cdot,\lambda)$ is proper, convex, l.s.c (i.e. $\varphi(\cdot,\lambda) \in \Gamma_0(H)$) and of compact type,

(2) if $\lambda_n \to \lambda$ in Λ, then for every $\mu > 0$, we have $(I + \mu\partial\varphi(\cdot,\lambda_n))_\chi^{-1} \to (I + \mu\partial\varphi(\cdot,\lambda))^{-1}x$ for every $x \in H$.

Also we will make the following hypothesis concerning the initial condition x_0 of (1):

(H_0) $\lambda \to x_0(\lambda)$ is continuous from Λ into H and for all $\lambda \in \Lambda$, $x_0(\lambda) \in \operatorname{dom}\varphi(\cdot,\lambda)$.

Given $g \in L^2(H)$, consider the following evolution inclusion:

$$-\dot{x}(t) \in \partial\varphi(x(t),\lambda) + g(t) \text{ a.e.}, \qquad x(0) = x_0(\lambda). \tag{2}$$

From Brezis [**3**, theorem 3.6, p. 72], we know that (2) has a unique strong solution $p(g,\lambda)(\cdot) \in C(T,H)$. So we can define the solution map $p: L^2(H) \times \Lambda \to C(T,H)$ by $(g,\lambda) \to p(g,\lambda)(\cdot)$. The following continuity result concerning $p(\cdot,\cdot)$ can be found in Attouch [**1**, theorem 3.74, p. 388].

THEOREM 2.2 *If the hypotheses $H(\varphi)$ and H_0 hold, then the solution map $p: L^2(H) \times \Lambda \to C(T, H)$, is continuous.*

By a strong solution of evolution inclusion (1), we mean a function $x \in C(T, H)$ such that $x(\cdot)$ is absolutely continuous on any compact subinterval of $(0, b)$, $x(t) \in \operatorname{dom} \varphi(\cdot, \lambda)$ a.e. and $-\dot{x}(t) \in \partial\varphi(x(t), \lambda) + f(t)$ a.e., $f(\cdot) \in L^2(H)$, $f(t) \in F(t, x(t), \lambda)$, a.e., $x(0) = x_0(\lambda)$. We will denote by $S(\lambda) \subseteq C(T, H)$ the set of all strong solutions of the multivalued Cauchy problem (1).

3. Continuous dependence results. In this section, we study the continuity properties of the solution multifunction $S(\cdot)$. For this, we will need the following hypothesis on the orientor field $F(t, x, \lambda)$:

$(H(F))$ $\quad F: T \times H \times \Lambda \to P_{wkc}(H)$ is a multifunction such that:

(1) $t \to F(t, x, \lambda)$ is measurable,

(2) $h(F(t, x, \lambda), F(t, y, \lambda)) \leq k_B(t)\|x - y\|$ a.e. for all $\lambda \in B \subseteq \Lambda$, B compact and with $k_B(\cdot) \in L^1_+$,

(3) $\lambda \to F(t, x, \lambda)$ is d-continuous,

(4) $|F(t, x, \lambda)| = \sup\{\|v\| : v \in F(t, x, \lambda)\} \leq \alpha_B(t) + \beta_B(t)\|x\|$ a.e. for all $\lambda \in B \subseteq \Lambda$, B compact and with $\alpha_B(\cdot), \beta_B(\cdot) \in L^2_+$.

Because of the hypothesis $H(F)$ above, we know that for every $\lambda \in \Lambda$, $S(\lambda)$ is nonempty and compact in $C(T, H)$ (see Kravvaritis and Papageorgiou [**8**, theorem 3.1] and Papageorgiou [**15**, theorem 4.1]).

THEOREM 3.1 *If hypotheses $H(\varphi)$, $H(F)$, H_0 hold and $\lambda_n \to \lambda$ in Λ then $S(\lambda_n) \xrightarrow{K} S(\lambda)$ in $C(T, H)$ as $n \to \infty$.*

Proof. Let $B \subseteq \Lambda$ be a nonempty, compact subset. First we will derive an a priori bound for the elements in $\bigcup_{\lambda \in B} S(\lambda)$. To this end, let $\lambda \in B$, $x(\cdot) \in S(\lambda)$ and let $u_\lambda(\cdot) \in C(T, H)$ be the unique solution of the Cauchy problem:

$$-\dot{u}_\lambda(t) \in \partial\varphi(u_\lambda(t), \lambda) \text{ a.e.,} \qquad u(0) = x_0(\lambda).$$

Exploiting the monotonicity of the subdifferential operator, we have:

$$(-\dot{x}(t) + u_\lambda(t), u_\lambda(t) - x(t)) \leq (f(t), u_\lambda(t) - x(t)) \text{ a.e.}$$

with $f \in L^2(H)$, $f(t) \in F(t, x(t), \lambda)$ a.e. and $-\dot{x}(t) \in \partial\varphi(x(t), \lambda) + f(t)$ a.e. Then we have:

$$\frac{1}{2}\frac{d}{dt}\|x(t) - u_\lambda(t)\|^2 \leq \|f(t)\| \cdot \|x(t) - u_\lambda(t)\| \text{ a.e.}$$

$$\Rightarrow \frac{1}{2}\|x(t) - u_\lambda(t)\|^2 \leq \int_0^1 \|f(s)\| \cdot \|x(s) - u_\lambda(s)\| ds.$$

Apply lemma A.5, p. 157 of Brezis [**3**], to get

$$\|x(t) - u_\lambda(t)\| \leq \int_0^t \|f(s)\| ds \leq \int_0^t (\alpha_B(s) + \beta_B(s)\|x(s)\|) ds$$

$$\Rightarrow \|x(t)\| \leq \|u_\lambda\|_\infty + \int_0^1 (\alpha_B(s) + \beta_B(s)\|x(s)\|) ds$$

From Theorem 2.2, we know that we can find $\theta_B > 0$ such that $\|u_\lambda\|_\infty \leq \theta_B$ for all $\lambda \in B$. Hence, we get

$$\|x(t)\| \leq \theta_B + \int_0^t (\alpha_B(s) + \beta_B(s)\|x(s)\|)ds, \quad t \in T$$

Invoking Gronwall's inequality, we deduce that there exists an $M_B > 0$ such that for all $x \in \bigcup_{\lambda\in B} S(\lambda)$, we have $\|x\|_{C(T,H)} \leq M_B$.

Hence without any loss of generality, we may assume that

$$\begin{aligned}|F(t,x,\lambda)| &= \sup\{\|v\| : v \in F(t,x,\lambda)\} \leq \psi_B(t)\\ &= \alpha_B(t) + \beta_B(t)M_B \text{ a.e.}, \quad \psi_B(\,\cdot\,) \in L^2_+ \qquad \text{for all } \lambda \in B\end{aligned}$$

Then let $K_B = \{h \in L^1(H) : \|h(t)\| \leq \psi_B(t) \text{ a.e.}\}$ (viewed as a subset of $L^1(H)$) and consider the multifunction $R\colon K_B \times B \to P_{fc}(K_B)$ defined by $R(f,\lambda) = S^1_{F(\,\cdot\,,p(f,\lambda)(\,\cdot\,),\lambda)}$.

On $L^1(H)$, consider the norm $\|g\|_B = \int_0^r \exp\left[-L\int_0^t k_B(s)ds\right]\|g(t)\|dt$, $L > 0$, which is clearly equivalent to the usual one. Our claim is that for $L > 1$, the family $\{R(\,\cdot\,,\lambda)_{\lambda\in B}\}$ is h-Lipschitz for this norm with the same Lipschitz constant $\eta_B \in (0,1)$. To this end let $f, g \in K_B$ and let $v \in R(g,\lambda)$. Let also

$$\Gamma(t) = \{u \in F(t,p(f,\lambda)(t),\lambda) : \|v(t) - u\| = d(v(t), F(t,p(f,\lambda)(t),\lambda))\}.$$

Note that for every $t \in T$, $\Gamma(t) \neq 0$ since by the hypothesis $H(F)$, F is $P_{wkc}(H)$-valued. Then observe that

$$\operatorname{Gr}\Gamma = \{(t,u) \in \operatorname{Gr} F(\,\cdot\,,p(f,\lambda)(\,\cdot\,),\lambda) : \|v(t) - u\| - d(v(t), F(t,p(f,\lambda)(t),\lambda)) = 0\}.$$

Because of the hypotheses $H(F)$, (1) and (2) and theorem 3.3 of Papageorgiou [**13**], $\operatorname{Gr} F(\,\cdot\,,p(f,\lambda)(\,\cdot\,),\lambda) \in B(T) \times B(H)$, where $B(T)$ (resp. $B(H)$) is the Borel σ-field of T (resp. of H). Furthermore,

$$(t,u) \to \|v(t) - u\| - d(v(t), F(t,p(f,\lambda)(t),\lambda))$$

is clearly measurable in $t \in T$ and continuous in $u \in H$ (i.e. a Caratheodory function), thus jointly measurable. Therefore $\operatorname{Gr}\Gamma \in B(T) \times B(H)$.

Apply Aumann's selection theorem (see Wagner [**22**, theorem 5.10]) to get $u\colon T \to H$ measurable such that $u(t) \in \Gamma(t)$ a.e. Then we have

$$\begin{aligned}
d_B(v(f,\lambda)) &\leq \|v - u\|_B\\
&= \int_0^r \|v(t) - u(t)\| \exp\left[-L\int_0^t k_B(s)ds\right]dt\\
&= \int_0^r d(v(t), F(t,p(f,\lambda)(t),\lambda)) \exp\left[-L\int_0^t k_B(s)ds\right]dt\\
&\leq \int_0^r h(F(t,p(g,\lambda)(t),\lambda), F(t,p(f,\lambda)(t),\lambda)) \exp\left[-L\int_0^t k_B(s)ds\right]dt\\
&\leq \int_0^r k_B(t)\|p(g,\lambda)(t) - p(f,\lambda)(t)\| \exp\left[-L\int_0^t k_B(s)ds\right]dt.
\end{aligned}$$

As in the beginning of the proof, by exploiting the monotonicity of the subdifferential operator and by using lemma A.5, p. 157 od Brezis [**3**], we get

$$\|p(g,\lambda)(t)-p(f,\lambda)(t)\| \leq \int_0^t \|g(s)-f(s)\|ds \qquad \text{for all } (t,\lambda)\in T\times B.$$

So we have:

$$\begin{aligned}
d_B(v,R(f,\lambda)) &\leq \int_0^r k_B(t)\exp\left[-L\int_0^t k_B(s)ds\right]\int_0^t \|g(s)-f(s)\|ds\,dt \\
&= -\frac{1}{L}\int_0^r \left(\int_0^t \|g(s)-f(s)\|ds\right) d\left(\exp\left[-L\int_0^t k_B(s)ds\right]\right) \\
&= \frac{1}{L}\int_0^r \exp\left[-L\int_0^t k_B(s)ds\right]\|g(s)-f(s)\|ds \\
&\hspace{16em}\text{(by integration by parts)} \\
&\leq \frac{1}{L}\|d-g\|_B.
\end{aligned}$$

Similarly for $w\in R(f,\lambda)$, we can get that $d_B(w,R(g,\lambda))\leq \frac{1}{L}\|g-f\|_N$ implies $\{R(\,\cdot\,,\lambda)\}_{\lambda\in B}$ is h-Lipshitz with constant $1/L$, for the $\|\cdot\|_B$-norm.

Next, let $[f_n,\lambda_n]\to[f,\lambda]$ in $(K_B,\|\cdot\|_B)\times B \Rightarrow [f_n,\lambda_n]\to[f,\lambda]$ in $L^1(H)\times B$. We will show that $R(f_n,\lambda_n)\xrightarrow{K-M}R(f,\lambda)$. To this end, let $u\in R(f,\lambda)$ and set

$$\begin{aligned}
\gamma_n(t) &= d(u(t)),F(t,p(f_n,\lambda_n)(t),\lambda_n)) \\
&\leq d(u(t),F(t,p(f,\lambda)(t),\lambda_n)) + h(F(t,p(f,\lambda)(t),\lambda_n),F(t,p(f_n,\lambda_n)(t),\lambda_n)) \\
&\leq d(u(t),F(t,p(f,\lambda)(t),\lambda_n)) + k_B(t)\|p(f,\lambda)(t)-p(f_n,\lambda)(t)\| \text{ a.e.}
\end{aligned}$$

Because of hypothesis $H(F)(3)$, we have $d(u(t),F(t,p(f,\lambda)(t),\lambda_n))\to 0$ as $n\to\infty$. Also because of Theorem 2.2, we have $\|p(f,\lambda)(t)-p(f_n,\lambda_n)(t)\|\to 0$ as $n\to\infty$, uniformly on T.

Therefore, we get that $\gamma_n(t)\to 0$ a.e. as $n\to\infty$. As before via Aumann's selection theorem, we can find $u_n(\,\cdot\,)\in K_B$ such that

$$\begin{gathered}
u_n(t)\in F(t,p(f_n,\lambda_n)(t),\lambda_n) \text{ a.e.} \quad\text{and}\quad \|u(t)-u_n(t)\|\leq\gamma_n(t)+1/n \text{ a.e.} \\
\Rightarrow u_n(t)\xrightarrow{s}u(t) \text{ a.e. in } H \text{ as } n\to\infty \\
\Rightarrow u_n\xrightarrow{s}u \text{ in } (L^1(H),\|\cdot\|_B).
\end{gathered}$$

Since $u_n\in R(f_n.\lambda_n)$, $n\geq 1$ we have established that

$$R(f,\lambda)\subseteq \text{s-}\underline{\lim}\,R(f_n,\lambda_n). \tag{3}$$

Next, let $v\in\text{w-}\overline{\lim}\,R(f_n,\lambda_n)$. Denoting subsequences with the same index as original sequences, we know that we can find $v_n\in R(f_n,\lambda_n)$ such that $v_n\xrightarrow{w}v$ in $L^1(H)$. Apply theorem 3.1 of [**14**], to get

$$v(t)\in\overline{\text{conv}}\,\text{w-}\overline{\lim}\,\{f_n(t)\}_{n\geq 1}\subseteq\overline{\text{conv}}\,\text{w-}\overline{\lim}\,F(t,p(f_n,\lambda_n)(t),\lambda_n) \text{ a.e.}$$

Note that for any $v \in H$, we have

$$\begin{aligned} d(v, F(t, p(f,\lambda)(t), \lambda_n)) &\le d(v, F(t, p(f_n,\lambda_n)(t), \lambda_n)) \\ &\qquad + h(F(t, p(f,\lambda)(t), \lambda_n), F(t, p(f_n,\lambda_n)(t), \lambda_n)) \\ &\le d(v, F(t, p(f_n,\lambda_n)(t), \lambda_n)) \\ &\qquad + k_B(t)\|p(f,\lambda)(t) - p(f_n,\lambda_n)(t)\| \text{ a.e.} \end{aligned}$$

Then by passing to the limit as $n \to \infty$ and using Theorem 2.2 together with hypothesis $H(F)(3)$, we get

$$d(v, F(t, p(f,\lambda)(t), \lambda_n)) \le \underline{\lim}\, d(v, F(t, p(f_n,\lambda_n)(t), \lambda)) \text{ a.e.}$$

Invoking theorem 2.2 (iv) of Tsukada [**20**], we get

$$\begin{aligned} \text{w-}\overline{\lim}\, F(t, p(f_n,\lambda_n)(t), \lambda_n) &\subseteq F(t, p(f,\lambda)(t), \lambda) \text{ a.e.} \\ \Rightarrow v(t) &\in F(t, p(f,\lambda)(t), \lambda) \text{ a.e.} \\ \Rightarrow v &\in R(f,\lambda). \end{aligned}$$

Thus we have established that

$$\text{w-}\overline{\lim}\, R(f_n,\lambda_n) \subseteq R(f,\lambda). \tag{4}$$

From (3) and (4) above, we have that if $[f_n,\lambda_n] \to [f,\lambda]$ in $(L^1(H), \|\cdot\|_B) \times B$, then

$$R(f_n,\lambda_n) \xrightarrow{K-M} R(f,\lambda)$$

Let $\Phi(\lambda_n) = \{f \in K_B : f \in R(f,\lambda_n)\}$ and $\Phi(\lambda) = \{f \in K_B : f \in R(f,\lambda)\}$. From Theorem 2.1, we have $\Phi(\lambda_n) \xrightarrow{K} \Phi(\lambda)$ in $L^1(H)$ as $n \to \infty$.

But since $\Psi_B(\cdot) \in L^2_+$ (see the definition of K_B), we can easily see that $\Phi(\lambda_n) \xrightarrow{K} \Phi(\lambda)$ in $L^2(H)$ as $n \to \infty$.

Since the solution map $p(\cdot,\cdot)\colon L^2(H) \times \Lambda \to C(T,H)$ is continuous, we get

$$p(\Phi(\lambda_n),\lambda_n) \xrightarrow{K} p(\Phi(\lambda),\lambda) \text{ in } C(T,H) \text{ as } n \to \infty.$$

But note that $S(\lambda_n) = p(\Phi(\lambda_n),\lambda_n)$ and $S(\lambda) = p(\Phi(\lambda),\lambda)$. So we have $S(\lambda_n) \xrightarrow{K} S(\lambda)$ in $C(T,H)$ as $n \to \infty$. $\diamondsuit$

If we strengthen the hypothesis $H(\varphi)$, using Theorem 3.1 above, we can have the Vietoris continuity of the multifunction $S\colon \Lambda \to P_k(C(T,H))$. The strengthened version of $H(\varphi)$, that we will need, is the following:

$(H(\varphi)')$ $\quad \varphi\colon H \times \Lambda \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is a function such that

(1) for every $\lambda \in \Lambda$, $\varphi(\cdot,\lambda)$ is proper, convex, l.s.c. (i.e. $\phi(\cdot,\lambda) \in \Gamma_0(H)$)

(2) if $\lambda_n \to \lambda$ in Λ, then for every $\mu > 0$ we have $(I + \mu\partial\varphi(\cdot,\lambda_n))^{-1}x \to (I + \mu\partial\varphi(\cdot,\lambda))^{-1}x$ for every $x \in H$,

(3) if $B \subseteq \Lambda$ is compact, then $\overline{\bigcup_{\lambda \in B}\{x \in H : \|x\|^2 + \varphi(x,\lambda) \le \theta\}}$ is compact for every $\theta > 0$ and $\{\varphi(x_0(\lambda),\lambda) : \lambda \in B\}$ is bounded.

Theorem 3.2 *If the hypotheses $H(\varphi)'$, $H(F)$ and H_0 hold, then $S\colon\Lambda\to P_k(C(T,H))$ is Vietoris continuous.*

Proof. First, note that for any $\lambda\in\Lambda$ and any compact set C containing $x_0(\lambda)$, we have that $\inf\{\varphi(x,\lambda): x\in C\}=\varphi(\hat{x},\lambda)$ for some $\hat{x}\in C$ (Weierstrass theorem). Since $\partial(\varphi(x,\lambda)-\varphi(\hat{x},\lambda))=\partial\varphi(x,\lambda)$, we deduce that we may assume without any loss of generality that for every $\lambda\in\Lambda$, $\varphi(\,\cdot\,,\lambda)\geq 0$.

Let $B\subseteq\Lambda$ be compact and let $V_B=\{h\in L^2(H):\|h(t)\|\leq\psi_B(t)\text{ a.e.}\}$, where $\psi_B(\,\cdot\,)\in L^2_+$ is as in the proof of Theorem 3.1. Let $W=p(V_B,B)$, where $p(\,\cdot\,,\,\cdot\,)$ is the solution map. Our claim is that W is relatively compact in $C(T,H)$. So let $x\in W$ and $0\leq t\leq t'\leq r$. We have:

$$\begin{aligned}\|x(t')-x(t)\| &= \left\|\int_t^{t'}\dot{x}(s)ds\right\|\leq\int_t^{t'}\|\dot{x}(s)\|ds\\ &\leq\left[\int_0^r\chi_{[t,t']}(s)^2ds\right]^{1/2}\left[\int_0^r\|\dot{x}(s)\|^2ds\right]^{1/2}.\end{aligned}$$

But from theorem 3.6, p. 72 of Brezis [**3**], we have

$$\left[\int_0^r\|\dot{x}(s)\|^2ds\right]^{1/2}\leq\|\psi_B\|_2+\sup_{\lambda\in B}\varphi(x_0,\lambda)=M<\infty$$

(see (3) of the hypothesis $H(\varphi)'$). So we get that $\|x(t')-x(t)\|\leq M(t'-t)^{1/2}$ implies W is equicontinuous. Furthermore, using once more theorem 3.6 of Brezis [**3**], we obtain

$$\begin{aligned}&\|\dot{x}(t)\|^2+\frac{d}{dt}\varphi(x(t),\lambda)=(h(t),\dot{x}(t))\\ &\Rightarrow\frac{d}{dt}\varphi(x(t),\lambda)\leq(h(t),\dot{x}(t))\text{ a.e.}\\ &\Rightarrow\varphi(x(t),\lambda)\leq\varphi(x_0,\lambda)+\int_0^t\|h(s)\|\cdot\|\dot{x}(s)\|ds\\ &\qquad\qquad\leq\varphi(x_0,\lambda)+\|h\|_2\,\|\dot{x}\|_2\\ &\qquad\qquad\leq\varphi(x_0,\lambda)+\|\psi_B\|_2M\\ &\qquad\qquad\leq M_1\text{ for all }\lambda\in B,\end{aligned}$$

(see (3) of the hypothesis $H(\varphi)'$). Thus

$$\overline{W(T)}=\overline{\{x(t):x(\,\cdot\,)\in W\}}\subseteq\overline{\bigcup_{\lambda\in B}\{v\in H:\|v\|^2+\varphi(v,\lambda)\leq M_1\}}\in P_k(H)$$

(see (3) of the hypothesis $H(\varphi)'$). Therefore by the Arzela-Ascoli theorem, we deduce that W is compact in $C(T,H)$ and $S(\lambda)\subseteq\bar{W}$ for all $\lambda\in B$. Combining this fact with Theorem 3.1 above, we get that $S|_B$ is Vietoris continuous. Since $B\subseteq\Lambda$ was an arbitrary compact subset, from lemma δ p. 71 of [**16**] and remark 1.7 of DeBlasi and Myjak [**4**], we conclude that $S(\,\cdot\,)$ is Vietoris continuous. $\Diamond$

Finally, recalling that the Vietoris and Hausdorff metric topologies coincide on $P_k(C(T,H))$ (see Section 2), we also have

THEOREM 3.3 *If hypotheses $H(\varphi)'$, $H(F)$ and H_0 hold, then $S\colon\Lambda \to P_k(C(T,H))$ is h-continuous.*

4. Sensitivity analysis in optimal control. In this section, we use the previous theorems to study the variational stability of a class of nonlinear distributed parameter optimal control problems.

So let $T=[0,r]$ and $Z=[0,b]$. Let Λ be a complete metric space (the parameter space). We consider the following parametrized parabolic optimal control problem:

$$\begin{gathered}\int_0^b \eta(z,x(r,z),\lambda)dz \to \inf = m(\lambda) \text{ such that}\\ \frac{\partial x(t,z)}{\partial t} - \frac{\partial}{\partial z}\left(\alpha(z,\lambda)\frac{\partial x}{\partial z}\right) = f(t,z,x(t,z),\lambda)u(t,z) \text{ a.e.} \qquad (5)\\ x(0,z)=x_0(z,\lambda), x(t,0)=x(t,b)=0 \quad \text{and} \quad |u(t,z)|\le v(t,z,\lambda) \text{ a.e.}\\ u(\,\cdot\,,\,\cdot\,) \text{ measurable.}\end{gathered}$$

We will need the following hypotheses on the data of (5):

$(H(a))$ $0<m_1\le a(t,z)\le m_2$ a.e.

$(H(f))$ $f\colon T\times Z\times\mathbb{R}\times\Lambda\to\mathbb{R}$ is a function such that

(1) $(t,z)\to f(t,z,x,\lambda)$ is measurable,

(2) $|f(t,z,x,\lambda)-f(t,z,x',\lambda)|\le k_B(t,z)|x-x'|$ a.e. with $k_B\in L^1(T\times Z)$, $\lambda\in B\subseteq\Lambda$, B =compact,

(3) $\lambda\to f(t,z,x,\lambda)$ is continuous,

(4) $|f(t,z,x,\lambda)|\le a_B(t,z)+c_B(t,z)|x|$ a.e. with $a_B\in L^2(T\times Z)$, $c_B\in L^\infty(T\times Z)$, $\lambda\in B\subseteq\Lambda$, B =compact.

$(H(r))$ $(t,z)\to v(t,z,\lambda)$ is measurable, $\lambda\to v(t,z,\lambda)$ is continuous and $|v(t,z,\lambda)|\le\theta_B(t,z)$ a.e. with $\theta_B(\,\cdot\,,\,\cdot\,)\in L^\infty(T\times Z)$, $\lambda\in B\subseteq\Lambda$, B =compact.

$(H(\eta))$ $\eta\colon Z\times\mathbb{R}\times\Lambda\to\mathbb{R}$ is an integrand such that

(1) $z\to\eta(z,x,\lambda)$ is measurable,

(2) $(x,\lambda)\to\eta(z,x,\lambda)$ is continuous,

(3) $|\eta(z,x,\lambda)|\le\psi_{1B}(z)+\psi_{2B}(z)|x|^2$ a.e. with $\psi_{1B}(\cdot)\in L^2_+$, $\psi_{2B}\in L^\infty_+$, $\lambda\in B\subseteq\Lambda$, B-compact.

(H_0) $x_0(\cdot,\lambda)\in H^1_0(Z)$ and $\lambda\to x_0(\cdot,\lambda)$ is continuous from Λ into $L^2(Z)$.

(H_c) If $\lambda_n\to\lambda$ in Λ, then $\frac{1}{a(\cdot,\lambda_n)}\xrightarrow{w}\frac{1}{a(\cdot,\lambda)}$ in $L^2(Z)$.

Let $Q(\lambda)\subseteq C(T,L^2(Z))$ be the set of optimal trajectories of (5).

THEOREM 4.1 *If hypotheses $H(a)$, $H(f)$, $H(r)$, $H(\eta)$, H_0 and H_c hold, then for every $\lambda \in \Lambda$, $Q(\lambda) \neq \emptyset$, $Q: \Lambda \to P_k(C(T, L^2(Z)))$ is u.s.c. and $m: \Lambda \to \mathbb{R}$ is continuous.*

Proof. Let $H = L^2(Z)$ and $A_H(x,\lambda) = \frac{\partial}{\partial x}\left(a(z,\lambda)\frac{\partial x}{\partial z}\right)$ with $D(A_H((\cdot,\lambda)) = \left\{x \in H_0^1)(Z) : \frac{\partial}{\partial z}\left(a(z,\lambda)\frac{\partial x}{\partial z}\right) \in L^2(Z)\right\}$. Then from Attouch [**1**, p. 379], we know that $A_H(\cdot,\lambda)$ is maximal monotone and linear on $L^2(Z)$ and furthermore, $A_H(\cdot,\lambda) = \partial\varphi(\cdot,\lambda)$, where

$$\varphi(x,\lambda) = \begin{cases} \dfrac{1}{2}\displaystyle\int_Z a(z,\lambda)\left(\frac{\partial x}{\partial z}\right)^2 dz, & \text{if } x \in H_0^1(Z) \\ +\infty, & \text{otherwise} \end{cases}$$

Because of the hypothesis H_c and using theorem 29 of Zhikov, Kozlov and Oleinik [**23**], we have that if $\lambda_n \to \lambda$ in Λ, then $A_H(\cdot,\lambda_n) \xrightarrow{G} A_H(\cdot,\lambda)$ as $n \to \infty$, and this by theorem 3.62, p.365 of Attouch [**1**], tells us that $(I + \mu\partial\varphi(\cdot,\lambda_n))^{-1} \to (I + \mu\partial\varphi(\cdot,\lambda))^{-1}$ as $n \to \infty$, for all $\mu \in \mathbb{R}$. Let $\hat{f}: T \times H \times \Lambda \to H$ is defined by $\hat{f}(t,x,\lambda)(\cdot) = f(t,\cdot,x(\cdot),\lambda)$ and $\hat{U}(t,\lambda) = \{u \in L^2(Z) : |u(z)| \leq v(t,z,\lambda) \text{ a.e.}\}$. Set

$$F(t,x,\lambda) = \hat{f}(t,x,\lambda)\hat{U}(t,\lambda) \in P_{wkc}(L^2(Z)).$$

We will now check that $F(\cdot,\cdot)$ satisfies the hypothesis $H(F)$. To this end, let $w \in H = L^2(Z)$ be given. Then we have

$$\begin{aligned} d(w, F(t,x,\lambda)) &= \inf\left\{\|w - \hat{f}(t,x,\lambda)u\|_{L^2(Z)} : v \in \hat{U}(t,\lambda)\right\} \\ &= \inf\left[\int_Z |w(z) - f(t,z,x(z),\lambda)u(z)|^2 dz : u \in \hat{U}(t,\lambda)\right]^{1/2} \\ &= \left[\inf\int_Z |w(z) - f(t,z,x(z),\lambda)u(z)|^2 dz : u \in \hat{U}(t,\lambda)\right]^{1/2} \\ &= \left(\int_Z \inf\{|w(z) - f(t,z,x(z),\lambda)u|^2 : u \in U(t,z,\lambda)\} dz\right)^{1/2} \\ &\qquad\qquad [\mathbf{6}, \text{Theorem 2.2}] \\ &= \left(\int_Z d(w(z), G(t,z,\lambda))^2 dz\right)^{1/2} \end{aligned}$$

with $G(t,z,\lambda) = f(t,z,x(z),\lambda)U(t,z,\lambda)$ and $U(t,z,\lambda) = [-v(t,z,\lambda), v(t,z,\lambda)]$. But note that because of the hypotheses $H(f)$, $H(r)$, it is clear that $(t,z) \to G(t,z,\lambda)$ is measurable and so

$$\begin{aligned} & t \to \left(\int_Z d(w(z), G(t,z,\lambda))^2 dz\right)^{1/2} \quad \text{is measurable} \\ \Rightarrow\ & t \to d(w, F(t,x,\lambda)) \quad \text{is measurable} \\ \Rightarrow\ & t \to F(t,x,\lambda) \quad \text{is measurable} \end{aligned}$$

Also note that because of (2) of the hypothesis $H(f)$, if $x, y \in L^2(Z)$, then we have:

$$h(F(t,x,\lambda), F(t,y,\lambda)) \leq \|\hat{f}(t,x,\lambda) - \hat{f}(t,y,\lambda)\|_2 \, \|v\|_\infty r \leq \hat{k}\|x-y\|_2, \quad \hat{k} > 0.$$

We will also show that for every $w \in L^2(Z)$, $\lambda \to d(w, F(t,x,\lambda))$ is continuous. To this end, let $\lambda_n \to \lambda$ and let $u \in \hat{U}(t,\lambda)$. Because of the hypothesis $H(r)$, clearly $\hat{U}(t,\cdot)$ is continuous and so we can find $u_n \in \hat{U}(t,\lambda_n)$, $u_n \xrightarrow{s} u$ in $L^2(Z)$. We have:

$$\begin{aligned} &d(w, F(t,x,\lambda_n)) \leq \|w - \hat{f}(t,x,\lambda_n)u_n\|_2 \Rightarrow \\ &\overline{\lim}\, d(w, F(t,x,\lambda_n)) \leq \|w - \hat{f}(t,x,\lambda)u\|_2 \end{aligned}$$

since $\lambda \to \hat{f}(t,x,\lambda)$ is continuous (part (3) of the hypothesis $H(f)$). Since $u \in \hat{U}(t,\lambda)$ was arbitrary, we get

$$\overline{\lim}\, d(w, F(t,x,\lambda_n)) \leq d(w, F(t,x,\lambda)). \tag{6}$$

On the other hand, let $u_n \in \hat{U}(t,\lambda_n)$ be such that

$$d(w, F(t,x,\lambda_n)) = \|w - \hat{f}(t,x,\lambda_n)u_n\|_2.$$

Its existence follows from the fact that $\hat{U}(t,\lambda_n) \in P_{wkc}(L^2(Z))$. Since $\theta_B(\cdot,\cdot) \in L^\infty(T \times Z)$, $B = \{\lambda_n, \lambda\}_{n \geq 1}$ (see the hypothesis $H(r)$), by passing to a subsequence if necessary, we may assume that $u_n \xrightarrow{w^*} u$ in $L^\infty(Z)$. Then for every $p(\cdot) \in L^2(Z)$, we have

$$\begin{aligned} (\hat{f}(t,x,\lambda_n)u_n, p)_{L^2(Z)} &= \int_Z f(t,z,x(z),\lambda_n)u_n(z)p(z)dz \\ \to (\hat{f}(t,x,\lambda)u, p)_{L^2(Z)} &= \int_Z f(t,z,x(z),\lambda)u(z)p(z)dz \text{ as } n \to \infty. \end{aligned}$$

Hence $\hat{f}(t,x,\lambda_n)u_n \xrightarrow{w} \hat{f}(t,x,\lambda)u$ in $L^2(Z)$ and clearly $u \in \hat{U}(t,\lambda)$. Recalling that the norm is weakly l.s.c., we get

$$\begin{aligned} &\|w - \hat{f}(t,x,\lambda)u\|_2 \leq \underline{\lim}\, \|w - \hat{f}(t,x,\lambda_n)u_n\|_2 \\ &\Rightarrow d(w, F(t,x,\lambda)) \leq \underline{\lim}\, d(w, F(t,x,\lambda_n)). \end{aligned} \tag{7}$$

From (6) and (7) above, we conclude that if $\lambda \to d(w, F(t,x,\lambda))$ is continuous, then $\lambda \to F(t,x,\lambda)$ is d-continuous. Finally, note that

$$|F(t,x,\lambda)| \leq \|a_B(t,\cdot)\|_2 \|r\|_\infty + \|c_B\|_2 \|r\|_\infty \|x\|_2, \lambda \in B \subseteq \Lambda, \quad B = \text{compact}.$$

So we have satisfied the hypothesis $H(F)$.

Next let $\hat{\eta}: H \times \Lambda \to \mathbb{R}$ be defined by $\hat{\eta}(x,\lambda) = \int_Z \eta(z, x(z), \lambda)dz$. Using the hypothesis $H(\eta)$, we can easily check that $\hat{\eta}(\cdot,\cdot)$ is in fact continuous. Now rewrite problem (5) in the following equivalent abstract form:

$$\begin{aligned} &\hat{\eta}(x(b),\lambda) \to \inf = m(\lambda) \quad \text{such that} \\ -&\dot{x}(t) \in \partial\varphi(x(t),\lambda) + F(t,x(t),\lambda) \text{ a.e.} \quad x(0) = x_0(\lambda). \end{aligned} \tag{8}$$

We know (see Theorem 3.1) that for every $\lambda \in \Lambda$, the problem (8) above has a nonempty set $S(\lambda)$ of admissible trajectories, which is compact in $C(T, L^2(Z))$. Since $\hat{\eta}(\cdot,\cdot)$ is continuous, we deduce that $Q(\lambda) \neq \emptyset$ for every $\lambda \in \Lambda$.

Next we will establish the continuity of the value function $m(\cdot)$. So let $\lambda_n \to \lambda$ in Λ. Let $x \in S(\lambda)$ such that $m(\lambda) = \hat{\eta}(x, \lambda)$. From Theorem 3.1, we know that $S(\lambda_n) \xrightarrow{K} S(\lambda)$ in $C(T, L^2(Z))$ and so we can find $x_n \in S(\lambda_n)$, $n \geq 1$ such that $x_n \xrightarrow{s} x$ in $C(T, L^2(Z))$. Then we have:

$$m(\lambda_n) \leq \hat{\eta}(x_n, \lambda_n) \Rightarrow \overline{\lim}\, m(\lambda_n) \leq \lim\, \hat{\eta}(x_n, \lambda_n) = \hat{\eta}(x, \lambda) = m(\lambda). \tag{9}$$

Note that if $B \subseteq \Lambda$ is compact, then for any $\beta > 0$,

$$\bigcup_{\lambda \in B} \{x \in H_0^1(Z) : \|x\|_2^2 + \varphi(x, \lambda) \leq \beta\}$$

is bounded in $L^2(Z)$. Since $H_0^1(Z)$ embeds compactly in $L^2(Z)$ (Sobolev embedding theorem), we have that

$$\overline{\bigcup_{\lambda \in B} \{x \in H_0^1(Z) : \|x\|_2^2 + \varphi(x, \lambda) \leq \beta\}} \quad \text{is compact in } L^2(Z).$$

Then from the proof of Theorem 3.2, we know that $\overline{\bigcup_{\lambda \in B} S(\lambda)} \in P_k(C(T, L^2(Z)))$. So if $\lambda_n \to \lambda$ in Λ, $B = \{\lambda_n, \lambda\}_{n \geq 1}$ and $x_n \in S(\lambda_n)$ is such that $m(\lambda_n) = \hat{\eta}(x_n, \lambda_n)$, by passing to a subsequence if necessary, we may assume that $x_n \xrightarrow{s} x$ in $C(T, L^2(Z))$. Then we have

$$\hat{\eta}(x, \lambda) = \lim\, \hat{\eta}(x_n, \lambda_n) \Rightarrow m(\lambda) \leq \underline{\lim}\, m(\lambda_n). \tag{10}$$

From (9) and (10) above, we get the continuity of $m(\,\cdot\,)$. Using the continuity of $m(\,\cdot\,)$, we can easily check that

$$\text{s-lim}\, Q(\lambda_n) \subseteq Q(\lambda) \Rightarrow Q\restriction_B \quad \text{is u.s.c}$$

and this by lemma δ of [**16**] implies that $Q(\cdot)$ is u.s.c. $\diamondsuit$

Remark. Our result extends the results of Przyluski [**17**], who considers linear systems and the parameter λ appears only on the control constraint set.

Our formation of the problem also incorporates "differential variational inequalities" (see Aubin-Celina [**2**, p. 264]). These are differential inclusions of the following form:

$$-\dot{x} \in N_{K(\lambda)}(x(t)) + F(t, x(t), \lambda) \text{ a.e.} \quad x(0) = x_0(\lambda). \tag{11}$$

Recall that the normal cone $N_{K(\lambda)}(x)$ to the closed, convex set $K(\lambda) \subseteq \mathbb{R}^k$ at the point x, is defined to be the set $N_{K(\lambda)}(x) = \partial\delta_{K(\lambda)}(x)$, where $\delta_{K(\lambda)}(x) = 0$ if $x \in K(\lambda)$, $+\infty$ otherwise (indicator function of the set $K(\lambda)$.) In fact, problem

(11) is equivalent to the following "projected differential inclusion" (see Aubin and Cellina [**2**])

$$\dot{x}(t) \in \operatorname{proj}(F(t, x(t), \lambda); T_{k(\lambda)}(x(t))) \text{ a.e.} \quad x(0) = x_0(\lambda) \tag{12}$$

which is a natural approximation to a viability problem, in which the well-known Nagumo tangential condition is not satisfied. The problem (12) was first considered by Henry [**5**], for producing planing procedures in mathematical economics, where the use of viable trajectories is essential. Problem (11) also arises in theoretical mechanics, in the study of unilateral problems.

Note that if $K: \Lambda \to P_{fc}(\mathbb{R}^k)$ is continuous, then for $\lambda_n \to \lambda$ in Λ we have $\delta_{K(\lambda_n)}(\cdot) \xrightarrow{\tau} \delta_{K(\lambda)}(\cdot)$, where τ denotes the convergence in the epigraphical sense (see Mosco [**10**]). So by theorem 3.66, p.373 of Attouch [**1**], we have that for all $\mu > 0$,

$$\begin{aligned}(I + \mu\partial\delta_{K(\lambda_n)})^{-1}x = (I + \mu N_{K(\lambda_n)})^{-1}x \to (I + \mu\partial\delta_{K(\lambda)})^{-1} \\ = (I + \mu N_{K(\lambda)})^{-1}x \quad \text{for all } x \in \mathbb{R}^k.\end{aligned}$$

Thus if $S(\lambda_n)$, $S(\lambda)$ are the solution sets for (11), by Theorem 3.1, we have $S(\lambda_n) \xrightarrow{K} S(\lambda)$ in $C(T, \mathbb{R}^k)$. Furthermore, if $K(\cdot)$ is $P_{kc}(\mathbb{R}^k)$-valued, then for $B \subseteq \Lambda$ compact, we have $K(B) \in P_k(\mathbb{R}^k)$ and so the hypothesis $H(\varphi)'$ is satisfied. Thus, via Theorems 3.2 and 3.3, we can get that $S(\cdot)$ is Vietoris and Hausdorff continuous from Λ into $P_k(C(T, \mathbb{R}^k))$.

REFERENCES

[**1**] H. Attouch, *Variational Convergence for Functions and Operators*, Pitman, Boston 1984.

[**2**] J. P. Aubin and A. Cellina, *Differential Inclusions*, Springer–Verlag, Berlin 1983.

[**3**] H. Brezis, *Operateurs Maximaux Monotones*, North Holand, Amsterdam 1973.

[**4**] F. DeBlasi and J. Myjak, *On continuous approximations for multifuctions*, Pacific. J. Math. **123** (1986), 9–31.

[**5**] C. Henry, *Differential equations with discontinuous right-hand side for planning procedures*, J. Econom. Theory **4** (1972), 545–551.

[**6**] F. Hiai and H. Umegaki, *Integrals, conditional expectations and Martingales of multivalued functions*, J. Multivariate Anal. **7** (1977), 149–182.

[**7**] E. Klein and A. Thompson, *Theory of Correspondences*, Wiley, New York 1984.

[**8**] D. Kravvaritis and N. S. Papageorgiou, *Multivalued perturbations of subdifferential type evolution equations in Hilbert space*, J. Differential Equations **76** (1988), 238–255.

[**9**] T.-C. Lim, *On fixed point stability of set-valued contractive mappings with applications to generalized differential equations*, J. Math. Anal. Appl. **110** (1985), 436–441.

[**10**] U. Mosco, *On the continuity of the Young-Fenchel transformation*, J. Math. Anal. Appl. **35** (1971), 518–535.

[**11**] S.B. Nadler, *Multivalued contraction mappings*, Pacific. J. Math. **30** (1969), 475–488.

[**12**] N.S. Papageorgiou, *A stability result for differential inclusions in Banach spaces*, J. Math. Anal. Appl. **118** (1986), 232–246.

[**13**] N.S. Papageorgiou, *On measurable multifunctions with applications to random multivalued equations*, Math. Japon. **32** (1987), 437–464.

[**14**] N.S. Papageorgiou, *Convergence theorems for Banach space valued integrable multifunctions*, Internat J. Math. Math. Sci. **10** (1987), 433–442.

[**15**] N. S. Papageorgiou, *On evolution inclusions associated with time dependent convex subdifferentials*, Comm. Math. Univ. Carolinae **31** (1990), 517–527.

[**16**] N. S. Papageorgiou, *On infinite dimensional control systems with state and control constraints*, Proc. Indian Acad. Sci. **100** (1990), 65–77.

[**17**] K. Przyluski, *Remarks on continuous dependence of an optimal control on parameters*, in: *Game Theory and Math Economics*, (eds. O. Moeschlin, D. Pallashke) North Holland, Amsterdam, 1981, pp. 331–337.

[**18**] L. Rybinski, *A fixed point approach in the study of the solution sets of Lipschitzian functional differential inclusion*, J. Math. Anal. Appl. **160** (1991), 24–46.

[**19**] A. Tolstonogov, *On the dependence on a parameter of a solution of a differential inclusion with a nonconvex second member*, Differential Equations **18** (1982), 1105–1113.

[**20**] M. Tsukada, *Convergence of best approximations in smooth Banach spaces*, J. Approx. Theory **40** (1984), 301–309.

[**21**] A. Vasilev, *Continuous dependence of the solutions of differential inclusions on the parameter*, Ukrainian Math. J. **35** (1983), 520–524.

[**22**] D. Wagner, *Survey of measurable selection theorems*, SIAM J. Control Optim. **15** (1977), 859–903.

[**23**] V. Zhikov, S. Kozlov and O. Oleinik, *G-convergence of parabolic operators*, Russian Math. Surveys **36** (1981), 9–60.

National Technical University
Department of Mathematics
Zografou Campus
Athens 15773, GREECE

and

Florida Institute of Technology
Department of Applied Mathematics
150 West University Blvd.
Melbourne, Florida 32901-6988, USA

Mailing address:

Florida Institute of Technology
Department of Applied Mathematics
150 W. University Blvd.
Melbourne, Florida 32901-6988, USA

(Received 05 02 1992)