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CONVOLUTION IN COLOMBEAU'S SPACES OF GENERALIZED FUNCTIONS PART II. THE CONVOLUTION IN $\mathcal{G}_{\mathbf{a}}$

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Abstract. We investigate various definitions of convolution and the Fourier transform in spaces $\mathcal{G}_{\boldsymbol{a}}$ which are studied in the first part of the paper.

0. Introduction

Colombeau's theory of generalized functions [**3**] made on the problem of multiplication of distributions has a lot of applications in the theory of linear and nonlinear partial differential equations; see the recent monograph [**2**] and the references there. In this paper we are concerned with the convolution in spaces $\mathcal{G}_{\boldsymbol{a}}$ of Colombeau's generalized functions and the relations between the convolution of Schwartz distributions and the generalized convolution of corresponding generalized functions. For this reason some problems on the convolution of Schwartz distributions are examined.

For the notion and the properties of the spaces $\mathcal{G}_{\boldsymbol{a}}$ and the $\boldsymbol{a}$-integral we refer to Part I. In Section 1, we give several new definitions of convolution in the space $\mathcal{G}_{\boldsymbol{a}}$. In Section 2, the relations between different definitions of convolution and the convolution of generalized functions which are determined by convolvable Schwartz distributions are treated. In Section 3, we introduce the $\boldsymbol{a},\mu$-Fourier transform of elements from $\mathcal{G}_{\boldsymbol{a}}$ and give the well known exchange formulae for $\boldsymbol{a} = \boldsymbol{t}$.

1. Definitions of convolution

Colombeau has given two definitions of convolution [**3**]: the convolution when one generalized function has a compact support, and the tempered convolution. Let $\boldsymbol{F}$, $\boldsymbol{G}$ be in $\mathcal{G}$, and let one of them, suppose $\boldsymbol{G}$, have compact support. Then the c-convolution is defined by

$$\boldsymbol{F} \overset{c}{*} \boldsymbol{G}(x) = \int_K \boldsymbol{F}(x-y)\, \boldsymbol{G}(y)\, dy, \qquad x \in \mathbb{R}^n,$$

AMS Subject Classification (1985): Primary 46 F 10 where K is a compact set which contains $\operatorname{supp}(\boldsymbol{G})$ in his interior. It is proved in [**3**] that this convolution exists and is a generalized function.

Let $\boldsymbol{F}$, $\boldsymbol{G}$ be in $\mathcal{G}_\tau$. The τ-convolution is defined by

$$\boldsymbol{F} \overset{\tau}{*} \boldsymbol{G}(x) \int^{\tau} \boldsymbol{F}(x-y)\,\boldsymbol{G}(y)\,dy, \qquad x \in \mathbb{R}^n.$$

It is proved in [**3**] that this convolution exists and is a generalized tempered function.

Now, we shall introduce several new definitions of convolution. Let $\boldsymbol{G}_1$, $\boldsymbol{G}_2$ be in $\mathcal{G}$ and let $\boldsymbol{K}_1$, $\boldsymbol{K}_2$ be their supports. We say that they have compatible supports if for every bounded set I there exists an open bounded set J such that $y \in I$ implies $(y-K_1)\cap K_2 \subset J$. For such $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ we define $\boldsymbol{G}_1 * \boldsymbol{G}_2 = \int_{\overline{J}} \boldsymbol{G}_1(x-y)\,\boldsymbol{G}_2(y)\,dy$, $x \in I$. One can prove that the definition is correct in the same way as Colombeau has proved the correctness of the integration on a compact set [**3**]. In that case we have $\boldsymbol{G}_1 * \boldsymbol{G}_2 = \boldsymbol{G}_2 * \boldsymbol{G}_1$ and $P(D)(\boldsymbol{G}_1 * \boldsymbol{G}_2) = (P(D)\boldsymbol{G}_1) * \boldsymbol{G}_2$. If $g_1, g_2 \in \mathcal{D}'$ have compatible supports, then $G_1 = \mathrm{Cd}(g_1)$, $G_2 = \mathrm{Cd}(g_2)$ also have compatible supports. By using [**1**] one can prove that for such g_1 and g_2, $\boldsymbol{G}_1 * \boldsymbol{G}_2 \approx g_1 * g_2$. Let $\boldsymbol{G}_1$, $\boldsymbol{G}_2$ be in $\mathcal{G}_{\boldsymbol{a}}$. We define

$$\boldsymbol{G}_1 \overset{\boldsymbol{a},\mu}{*} \boldsymbol{G}_2(x) = \int^{\boldsymbol{a},\mu} \boldsymbol{G}_1(x-y)\,\boldsymbol{G}_2(y)\,dy, \qquad x \in \mathbb{R}^n,$$

where μ_ε, $\varepsilon > 0$, is a unit net which corresponds to $\boldsymbol{a}$ (see Part I).

Proposition 1. *Assume $\boldsymbol{G}_1, \boldsymbol{G}_2 \in \mathcal{G}_{\boldsymbol{a}}$. Then*:

a) $\boldsymbol{G}_1 \overset{\boldsymbol{a},\mu}{*} \boldsymbol{G}_2 \in \mathcal{G}$;

b) $\partial^\alpha(\boldsymbol{G}_1 \overset{\boldsymbol{a},\mu}{*} \boldsymbol{G}_2) = (\partial^\alpha \boldsymbol{G}_1 \overset{\boldsymbol{a},\mu}{*} \boldsymbol{G}_2)$, *where* $\alpha \in \mathbb{N}_0^n$;

c) $\overline{\partial}_j^h(\boldsymbol{G}_1 \overset{\boldsymbol{a},\mu}{*} \boldsymbol{G}_2) = (\overline{\partial}_j^h \boldsymbol{G}_1) \overset{\boldsymbol{a},\mu}{*} \boldsymbol{G}_2$, *where* $h \in \mathcal{H}$, $j \in \{1,\dots,n\}$ *(see Part I)*;

d) *Let $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ be in $\overset{\odot}{\mathcal{G}}_{\boldsymbol{a}}$. Then $\boldsymbol{G}_1 \overset{\boldsymbol{a},\mu}{*} \boldsymbol{G}_2 \in \mathcal{G}_{\boldsymbol{a}}$.*

The assertion in a) means the following: for $G_1, G_2 \in \mathcal{E}_{\boldsymbol{a}}$, $N_1, N_2 \in \mathcal{N}_{\boldsymbol{a}}$

$$(G_1+N_1) \overset{\boldsymbol{a},\mu}{*} (G_2+N_2) = G_1 \overset{\boldsymbol{a},\mu}{*} G_2 + (G_1 \overset{\boldsymbol{a},\mu}{*} N_2 + N_1 \overset{\boldsymbol{a},\mu}{*} G_2 + N_1 \overset{\boldsymbol{a},\mu}{*} N_2) \in \mathcal{G},$$

because $G_1 \overset{\boldsymbol{a},\mu}{*} G_2 \in \mathcal{E}_M$ and $(G_1 \overset{\boldsymbol{a},\mu}{*} N_2 + N_1 \overset{\boldsymbol{a},\mu}{*} G_2 + N_1 \overset{\boldsymbol{a},\mu}{*} N_2) \in \mathcal{N}$. Assertion in d) is similar.

Proof. We shall prove only a). Let G_1 and G_2 be in $\mathcal{E}_{\boldsymbol{a}}$. We adopt the notation for G_1 and ∂^β in (12) from part I by using symbols with subindex $_1$, and for G_2 with subindex $_2$. Then, for given compact set K and every $\beta \in \mathbb{N}_0^n$ let $N = [\gamma_1 + \gamma_2 + N_1 + N_2 + n] + 1$. Because of Lebesgue's theorem for the differentiation under the integral sign, we have

$$\begin{aligned}
&\left|\partial^\beta \int G_1(\phi_\varepsilon, x-y)\,G_2(\phi_\varepsilon, y)\mu_\varepsilon(y)\,dy\right| = \left|\int \partial^\beta G_1(\phi_\varepsilon, x-y)\,G_2(\phi_\varepsilon, y)\mu_\varepsilon(y)\,dy\right| \\
&\le \sup_{\substack{|y| \le \boldsymbol{a}(b/\varepsilon)+r \\ x \in K}} \bigl\{c_1\theta_1(|x-y|)\bigr\}\varepsilon^{-N_1} \cdot \sup_{|y| \le \boldsymbol{a}(b/\varepsilon)+r} \bigl\{c_2\theta_2(|y|)\bigr\}\varepsilon^{-N_2} c_5 \varepsilon^{-n} b^n \\
&\le c\varepsilon^{-N}, \qquad \phi_\varepsilon \in \mathcal{A}_N, \quad x \in K, \quad \varepsilon \in (0,\infty),
\end{aligned}$$

where $c = c_1c_2c_3c_4c_5b^{n+\gamma_1+\gamma_2}$, $\eta = \min\{\eta_1, \eta_2\}$; c_3, c_4, γ_1, γ_2 are given by $\theta(\bar{c} + \boldsymbol{a}(x)) \le c_3x^{\gamma_1}$, $\theta(\boldsymbol{a}(x)) \le c_4x^{\gamma_2}$, $\bar{c} = \sup\{|x| : x \in K\} + r$, $c_5 = \pi^{n/2}\Gamma\big((n+2)/2\big)$. Thus we have proved that $G_1 \overset{\boldsymbol{a},\mu}{*} G_2 \in \mathcal{E}_M$. Similarly, one can prove that if G_1 or G_2 belongs to $\mathcal{N}_{\boldsymbol{a}}$, then $G_1 \overset{\boldsymbol{a},\mu}{*} G_2 \in \mathcal{N}$. □

COROLLARY. *If* $\overset{\odot}{\Theta}_{\boldsymbol{a}} = \Theta_{\boldsymbol{a}}$ *then* $\boldsymbol{G}_1 \overset{\boldsymbol{a},\mu}{*} \boldsymbol{G}_2 \in \mathcal{G}_{\boldsymbol{a}}$. *Particularly, let* $\boldsymbol{G}_1$ *and* $\boldsymbol{G}_2$ *be in* $\mathcal{G}_{\boldsymbol{t}}$. *Then* $\boldsymbol{G}_1 \overset{\boldsymbol{t},\mu}{*} \boldsymbol{G}_2 \in \mathcal{G}_{\boldsymbol{t}}$.

If for every pair of unit nets $\mu_{1\varepsilon}$, $\mu_{2\varepsilon}$, $\boldsymbol{G}_1 \overset{\boldsymbol{a},\mu_1}{*} \boldsymbol{G}_2 \approx \boldsymbol{G}_1 \overset{\boldsymbol{a},\mu_2}{*} \boldsymbol{G}_2$, then we say that there exists the associated $\boldsymbol{a}$-convolution $\boldsymbol{G}_1 \overset{\boldsymbol{a}}{*} \boldsymbol{G}_2 = \boldsymbol{G}_1 \overset{\boldsymbol{a},\mu_1}{*} \boldsymbol{G}_2$. If for every pair of unit nets $\mu_{1\varepsilon}$, $\mu_{2\varepsilon}$, $\boldsymbol{G}_1 \overset{\boldsymbol{a},\mu_1}{*} \boldsymbol{G}_2 - \boldsymbol{G}_1 \overset{\boldsymbol{a},\mu_2}{*} \boldsymbol{G}_2 \in \mathcal{N}$ ($\in \mathcal{N}_{\boldsymbol{a}}$), then there exist the $\boldsymbol{a}$-convolution in $\mathcal{G}$ (in $\mathcal{G}_{\boldsymbol{a}}$) $\boldsymbol{G}_1 \overset{\boldsymbol{a}}{*} \boldsymbol{G}_2 = \boldsymbol{G}_1 \overset{\boldsymbol{a},\mu_1}{*} \boldsymbol{G}_2 = \boldsymbol{G}_1 \overset{\boldsymbol{a},\mu_2}{*} \boldsymbol{G}_2$.

If the equality holds in g.d. (g.t.d.) sense, then there exist g.d. (g.t.d.) $\boldsymbol{a}$-convolution $\boldsymbol{G}_1 \overset{\boldsymbol{a}}{*} \boldsymbol{G}_2$.

2. Relations between different convolutions

Colombeau has proved [**3**] that if $\boldsymbol{G}_1$, $\boldsymbol{G}_2$ are from $\mathcal{G}_\tau$ and one of them has a compact support, then $\boldsymbol{G}_1 \overset{c}{*} \boldsymbol{G}_2 = \boldsymbol{G}_1 \overset{\tau}{*} \boldsymbol{G}_2$. Let $\boldsymbol{G}_1, \boldsymbol{G}_2 \in \mathcal{G}_{\boldsymbol{a}}$ and one of them has a compact support. Then there exists $\boldsymbol{G}_1 \overset{\boldsymbol{a},\mu}{*} \boldsymbol{G}_2$ and $\boldsymbol{G}_1 \overset{c}{*} \boldsymbol{G}_2 = \boldsymbol{G}_1 \overset{\boldsymbol{a},\mu}{*} \boldsymbol{G}_2$ for every μ_ε, $\varepsilon > 0$; thus $\boldsymbol{G}_1 \overset{\boldsymbol{a}}{*} \boldsymbol{G}_2 = \boldsymbol{G}_1 \overset{c}{*} \boldsymbol{G}_2$.

Example. Let $G(\phi_\varepsilon, x) = 1$, $x \in \mathbb{R}^n$, $\varepsilon > 0$. Clearly, $\boldsymbol{G} \in \mathcal{G}_\tau(\mathbb{R})$. Then

$$\boldsymbol{G} \overset{\tau}{*} \boldsymbol{G}(\phi_\varepsilon, x) = \int F(\phi)(\varepsilon x)\, dx = 2\pi\phi(0)/\varepsilon, \qquad \varepsilon > 0,$$

$$\boldsymbol{G} \overset{\tau,\mu}{*} \boldsymbol{G}(\phi_\varepsilon, x) = \int_{-1/\varepsilon a}^{1/\varepsilon a} dx = 2/\varepsilon a, \qquad \varepsilon > 0,$$

where μ_ε, $\varepsilon > 0$, is a unit net. So $\boldsymbol{G} \overset{\boldsymbol{t},\mu}{*} \boldsymbol{G}$ is not associated with $\boldsymbol{G} \overset{\tau}{*} \boldsymbol{G}$. □

Since we shall compare the Schwartz's convolution of distribution and the $\boldsymbol{a}, \mu$-convolutions of corresponding generalized functions, we need several assertions concerning Schwartz's distributions.

If $\phi \in \mathcal{A}_1$, then we put $\delta_\nu(x) = \phi_{1/\nu}(x) = \nu^n\phi(\nu x)$, $x \in \mathbb{R}^n$, $\nu \in \mathbb{N}$. This is a δ-sequence (for the general definitions we refer to [**1**]). For a unit net μ_ε we put $\varepsilon = 1/\nu$, $\nu \in \mathbb{N}$ and the corresponding sequence will be called a unit sequence and denoted by μ_ν (instead of $\mu_{1/\nu}$). Such sequences belong to the set of special approximate unit sequences introduced in [**9**] (see [**4**]): a sequence from $\mathcal{D}$, $\nu \in \mathbb{N}$, is a special approximate unit if

(1)
- (i) For every compact set $K \subset \mathbb{R}^n$ there is $\nu_K > 0$ such that $\mu_\nu(x) = 1$, $x \in K$, $\nu > \nu_K$;
- (ii) For every $m \in \mathbb{N}_0$, $p_m(\mu_\nu) \le c_m$, $\nu \in \mathbb{N}$, where $p_m(\varphi) = \sup\{|\partial^\alpha\varphi(x)| : |\alpha| \le m,\, x \in \mathbb{R}^n\}$, $\varphi \in \mathcal{D}$.

Proposition 2. *Let h_k, $k \in \mathbb{N}$, be a sequence of distributions from $\mathcal{D}'$. If*

$$\text{(a)}\qquad \left\{\begin{array}{l} \textit{there exists } m \in \mathbb{N}_0 \textit{ such that for every } \varepsilon > 0 \textit{ there exist a compact set} \\ K \subset \mathbb{R}^n \textit{ and } k_0 \in \mathbb{N} \textit{ with the property: } \varphi \in \mathcal{D},\ \operatorname{supp}(\varphi) \cap K = \varnothing \Rightarrow \\ |\langle h_k, \varphi\rangle| \le \varepsilon p_m(\varphi) \textit{ if } k > k_0, \end{array}\right.$$

then

$$\text{(b)}\qquad \left\{\begin{array}{l} \textit{for every special approximate unit } \mu_\nu \textit{ the sequence } \langle h_k, \mu_\nu\rangle, \textit{ converges} \\ (\textit{when } \nu \to \infty\,) \textit{ uniformly for } k \in \mathbb{N}. \end{array}\right.$$

Proof. The proof is similar to the proof of "(b) $\Rightarrow$ (c)" in [**4**, (1.1) Proposition] with the remark that for θ_r as in this proof and a special approximate unit μ_ν we have $\langle h_k\theta_r, \mu_p - \mu_q\rangle = 0$ for $p, q > \nu_0$. $\square$

It is proved in [**4**] that the definitions of convolutions of Schwartz, Vladimirov, Schiraishi, Chevalley and Mikusiński are equivalent (see also [**5**]). Recall, [**4**, (1.1) Proposition and (1.3) Theorem], $f, g \in \mathcal{D}'$ are convolvable iff one of the following equivalent conditions is satisfied:

(I) For every $\varphi \in \mathcal{D}$, $f(x)g(y)\varphi(x+y) \in \mathcal{D}'_{L^1}$, $x, y \in \mathbb{R}^n$;

(II) For every $\varphi \in \mathcal{D}$ and every special approximate unit μ_ν in $\mathcal{D}(\mathbb{R}^{2n})$, $\langle f(x)g(y), \varphi(x+y)\mu_\nu(x,y)\rangle$ converges when $\nu \to \infty$;

(Special approximate unit means that a μ_ν has a compact support.)

$$\text{(III)}\qquad \left\{\begin{array}{l} \text{For every } \varphi \in \mathcal{D} \text{ there is an } m \in \mathbb{N}_0 \text{ such that for every } \varepsilon > 0 \\ \text{there is a compact set } K \subset \mathbb{R}^{2n} \text{ such that if } \psi \in \mathcal{D}(\mathbb{R}^{2n}), \text{ and} \\ \operatorname{supp}(\psi) \cap K = \varnothing, \text{ then } |\langle f(x)g(y), \varphi(x+y)\psi(x,y)\rangle| \le \varepsilon p_m(\psi). \end{array}\right.$$

Note that there are several other equivalent conditions.

Proposition 3. a) *Let $f, g \in \mathcal{D}'$ be convolvable, and let δ_k be a delta sequence. Then for $h_k(x,y) = (f * \delta_k)(x)\,(g * \delta_k)(y)$, $k \in \mathbb{N}$, $x, y \in \mathbb{R}^n$, the condition* (a) *from Proposition* 2 *holds* (*with the same m, K as in* (III)). *Particularly, for any strong approximate unit μ_ν, $\nu \in \mathbb{N}$, from $\mathcal{D}(\mathbb{R}^{2n})$ we have that $\langle (f * \delta_k)(x)\,(g * \delta_k)(y), \mu_\nu(x,y)\rangle$, $\nu \to \infty$, converges uniformly for $k \in \mathbb{N}$.*

b) $\langle f * g, \varphi\rangle = \lim\limits_{\nu\to\infty} \langle (f * \delta_\nu)(x)\,(g * \delta_\nu)(y)\,\varphi(x+y), \mu_\nu(x,y)\rangle$, $\varphi \in \mathcal{D}(\mathbb{R}^n)$.

Proof. a) By using the condition (III) and the notation from there, we have

$$\langle f(x)g(y)\varphi(x+y), \psi(x,y)\rangle = \langle f(x)g(y), \varphi(x+y)\psi(x,y)\rangle = \langle \partial^l F(x)\partial^s G(y), \theta(x,y)\rangle,$$

where $\theta(x,y) = \varphi(x+y)\psi(x,y) \in \mathcal{D}(\mathbb{R}^{2n})$, $\operatorname{supp}(\theta) \subset I_x \times I_y$; I_x and I_y are bounded intervals in $\mathbb{R}^n$ and $f = \partial^l F$ in I_x and $g = \partial^s G$ in I_y for some $l, s \in \mathbb{N}_0^n$ and some continuous functions F and G. Thus we obtain

$$|\langle f(x)g(y), \varphi(x+y)\psi(x,y)\rangle| = \left| \iint_{I_x \times I_y} F(x)G(y)(-\partial_x^l)(-\partial_y^s)\theta(x,y)\,dx\,dy \right| \le \varepsilon p_m(\psi).$$

Assume now that (a) from Proposition 2, with ε replaced with 2ε, does not hold. Since on I_x, respectively on I_y, we have $f*\delta_k = \partial^l(F*\delta_k)$, $g*\delta_k = \partial^s(G*\delta_k)$, $k \in \mathbb{N}$, after the same procedure we get that there is a subsequence r_k, $k \in \mathbb{N}$, of natural numbers such that

$$\left| \iint_{I_x \times I_y} (F*\delta_{r_k})(x)\,(G*\delta_{r_k})(y)\,(-\partial_x^l)(-\partial_y^s)\theta(x,y)\,dx\,dy \right| > 2\varepsilon p_m(\psi), \quad k \in \mathbb{N}.$$

This is in a contradiction with the fact that $F*\delta_{r_k} \to F$ uniformly on I_x and $G*\delta_{r_k} \to G$ uniformly on I_y, when $k \to \infty$. The assertion a) is proved.

b) The previous part implies that for every $k \in \mathbb{N}$, $f*\delta_k$ and $g*\delta_k$ are convolvable. Put

$$a_{k,\nu} = \langle (f*\delta_k)(x)\,(g*\delta_k)(y)\,\varphi(x+y), \mu_\nu(x,y)\rangle, \qquad k,\nu \in \mathbb{N}.$$

We have

$$\begin{aligned} a_{k,\nu} &\overset{\nu\to\infty}{\longrightarrow} a_k = \langle (f*\delta_k)*(g*\delta_k), \varphi\rangle \overset{k\to\infty}{\longrightarrow} \langle f*g, \varphi\rangle, \\ a_{k,\nu} &\overset{k\to\infty}{\longrightarrow} a_\nu = \langle f(x)g(y)\varphi(x+y), \mu_\nu(x,y)\rangle \overset{\nu\to\infty}{\longrightarrow} \langle f*g, \varphi\rangle. \end{aligned}$$

Since Proposition 3 implies $a_{k,\nu} \overset{\nu\to\infty}{\longrightarrow} a_k$ uniformly for $k \in \mathbb{N}$, from the well known properties of a double sequence we have

$$\lim_{\substack{k\to\infty \\ \nu\to\infty}} a_{k,\nu} = \lim_{k\to\infty}\lim_{\nu\to\infty} a_{k,\nu} = \lim_{\nu\to\infty}\lim_{k\to\infty} a_{k,\nu} = \lim_{\nu\to\infty} a_{\nu,\nu}.$$

All above implies the assertion b). □

By [**8**] we have (for $\varphi \in \mathcal{D}(\mathbb{R}^n)$)

$$\langle f*g, \varphi\rangle = \lim_{k\to\infty} \langle f(x)\,(g*\check{\varphi})(x), \mu_k(x)\rangle = \lim_{k\to\infty} \langle f(x)g(x)\varphi(x+y), \mu_k(x)\rangle,$$

where μ_k, $k \in \mathbb{N}$, is a strong unit sequence from $\mathcal{D}(\mathbb{R}^n)$.

In the same way as Proposition 3 one can prove

PROPOSITION 4. $\lim_{m\to\infty} \langle f_m(x)g_m(y), \mu_m(x)\varphi(x+y)\rangle = \langle f*g, \varphi\rangle$, $\varphi \in \mathcal{D}(\mathbb{R}^n)$, *where* $f_n = f*\delta_m$, $g_m = g*\delta_m$, $m \in \mathbb{N}$ *and* μ_m, $m \in \mathbb{N}$, *is a strong unit sequence from* $\mathcal{D}(\mathbb{R}^n)$.

PROPOSITION 5. *Let* f, g *be in* $\mathcal{D}'$, $\boldsymbol{G}_1 = \mathrm{Cd}(f)$, $\boldsymbol{G}_2 = \mathrm{Cd}(g)$. *If there exists* $f*g$ *in the distributional sense and for some* $\boldsymbol{a} \in A$, $\boldsymbol{G}_1$ *and* $\boldsymbol{G}_2$ *are in* $\mathcal{G}_{\boldsymbol{a}}$, *then there exists* $\boldsymbol{G}_1 \overset{\boldsymbol{a},\mu}{*} \boldsymbol{G}_2$, *and* $\boldsymbol{G}_1 \overset{\boldsymbol{a},\mu}{*} \boldsymbol{G}_2 \approx f*g$, *for all unit nets* μ_ε, $\varepsilon > 0$, *and thus* $\boldsymbol{G}_1 \overset{\boldsymbol{a}}{*} \boldsymbol{G}_2 \approx f*g$.

Proof. Proposition 1 a) implies $\boldsymbol{G}_1 \overset{\boldsymbol{a},\mu}{*} \boldsymbol{G}_2 \in \mathcal{G}$ for every unit net μ_ε, $\varepsilon > 0$, and Proposition 4 implies that

$$\lim_{\varepsilon\to 0} \iint G_1(\phi_\varepsilon, x-y)\,G_2(\phi_\varepsilon, y)\,\mu_\varepsilon(y)\,\varphi(x)\,dx\,dy = \langle f*g, \varphi\rangle.$$

This implies the assertion. □

COROLLARY. *With the same notation as in Proposition 5 we have* $\boldsymbol{G}_1 \overset{\boldsymbol{a}}{*} \boldsymbol{G}_2 = \boldsymbol{G}_2 \overset{\boldsymbol{a}}{*} \boldsymbol{G}_1$.

Proof. It follows from $f * g = g * f$ in $\mathcal{D}'$. □

COROLLARY. *If* g_1, g_2 *are in* $\mathcal{D}'_F$ *and* $g_1 * g_2$ *exists in* $\mathcal{D}$; *then there exists an* $\boldsymbol{a} \in A$ *such that for any* μ, $\boldsymbol{G}_1 \overset{\boldsymbol{a},\mu}{*} \boldsymbol{G}_2$ *exists. Thus* $\boldsymbol{G}_1 \overset{\boldsymbol{a}}{*} \boldsymbol{G}_2 \approx g_1 * g_2$, *where* $\boldsymbol{G}_1 = \mathrm{Cd}(g_1)$, $\boldsymbol{G}_2 = \mathrm{Cd}(g_2)$.

3. Fourier transform

We shall define the $\boldsymbol{a},\mu$-Fourier transform of elements from $\mathcal{G}_{\boldsymbol{a}}$. Let $\boldsymbol{G} \in \mathcal{G}_{\boldsymbol{a}}$. We define $F_{\boldsymbol{a},\mu} : \mathcal{G}_{\boldsymbol{a}} \to \mathcal{G}_{\boldsymbol{t}}$ by $F_{\boldsymbol{a},\mu}(\boldsymbol{G})(x) = \int^{\boldsymbol{a},\mu} \boldsymbol{G}(y) e^{-ixy}\, dy$, $x \in \mathbb{R}^n$.

PROPOSITION 6. $F_{\boldsymbol{a},\mu} : \mathcal{G}_{\boldsymbol{a}} \to \mathcal{G}_{\boldsymbol{t}}$.

Proof. Let $G \in \mathcal{N}_{\boldsymbol{a}}$. For $c > 0$ we have

$$\begin{aligned}\left| \int G(\phi_\varepsilon, y) e^{-ixy} \mu_\varepsilon(y)\, dy \right| &\le \int_{|y| \le \boldsymbol{a}(b/\varepsilon)+r} |G(\phi_\varepsilon, y)|\, dy \\ &\le c\varepsilon^{\alpha(q)-N} \le c(1+|x|)\varepsilon^{\alpha(q)-N}.\end{aligned}$$

This means that $F_{\boldsymbol{a},\mu}(\mathcal{N}_{\boldsymbol{a}}) \subset \mathcal{N}_{\boldsymbol{t}}$. Similarly, we have $F_{\boldsymbol{a},\mu}(\mathcal{E}_{\boldsymbol{a}}) \subset \mathcal{E}_{\boldsymbol{t}}$. □

If for every two unit nets μ^1_ε, μ^2_ε, $\varepsilon > 0$, which correspond to $\boldsymbol{a}$, $F_{\boldsymbol{a},\mu^1}(\boldsymbol{G}) = F_{\boldsymbol{a},\mu^2}(\boldsymbol{G})$ (g.t.d.), then we say that there exists the $\boldsymbol{a}$-Fourier transform in g.t.d. sense $F_{\boldsymbol{a}}(\boldsymbol{G}) = F_{\boldsymbol{a},\mu^1}(\boldsymbol{G})$. In the sequel, we shall consider $\boldsymbol{t}$-Fourier transform.

PROPOSITION 7. *Let* $\boldsymbol{G}_1$, $\boldsymbol{G}_2$ *be in* $\mathcal{G}_{\boldsymbol{t}}$ *and* μ_ε, $\varepsilon > 0$, *be a unit net. Then*:

a) $\langle F_{\boldsymbol{t},\mu}(\boldsymbol{G}), \varphi\rangle = \langle \boldsymbol{G}, F(\varphi)\rangle$. *Particularly, for every* $\boldsymbol{G} \in \mathcal{G}_{\boldsymbol{t}}$ *there exists* $F_{\boldsymbol{t}}(\boldsymbol{G})$ (*in g.t.d. sense*);

b) *If* $F_{\boldsymbol{t}}(\boldsymbol{G}_1) = F_{\boldsymbol{t}}(\boldsymbol{G}_2)$ *(g.t.d.), then* $\boldsymbol{G}_1 = \boldsymbol{G}_2$ *(g.t.d.)*;

c) $F_{\boldsymbol{t}}(\boldsymbol{G}_1 \overset{\boldsymbol{t},\mu}{*} \boldsymbol{G}_2) = F_{\boldsymbol{t}}(\boldsymbol{G}_1) F_{\boldsymbol{t}}(\boldsymbol{G}_2)$ *(g.t.d.)*;

d) $F_{\boldsymbol{t}}(\partial^\alpha \boldsymbol{G}) = (ix)^\alpha F_{\boldsymbol{t}}(\boldsymbol{G})$ *(g.t.d.)*, $\alpha \in \mathbb{N}_0^n$.

Proof. a) We have

$$\begin{aligned}\langle F_{\boldsymbol{t},\mu}(\boldsymbol{G}), \varphi\rangle &= \int \left(\int^{\boldsymbol{t},\mu} \boldsymbol{G}(y) e^{-ixy}\, dy \right) \varphi(x)\, dx = \int^{\boldsymbol{t},\mu} \boldsymbol{G}(y) F(\varphi)(y)\, dy \\ &= \int \boldsymbol{G}(y) F(\varphi)(y)\, dy = \langle \boldsymbol{G}, F(\varphi)\rangle.\end{aligned}$$

b) This assertion follows from Proposition 7 a) and the fact that the Fourier transform is a bijection of $\mathcal{S}$ onto $\mathcal{S}$.

c) The corollary of Proposition 1 implies $\boldsymbol{G}_1 \overset{\boldsymbol{t},\mu}{*} \boldsymbol{G}_2 \in \mathcal{G}_{\boldsymbol{t}}$. For any $\varphi \in \mathcal{S}$ we have

$$\begin{aligned}
\langle F_{\boldsymbol{t}}(\boldsymbol{G}_1 \overset{\boldsymbol{t},\mu}{*} \boldsymbol{G}_2), \varphi\rangle &= \langle \boldsymbol{G}_1 \overset{\boldsymbol{t},\mu}{*} \boldsymbol{G}_2, F(\varphi)\rangle \\
&= \iint \left(\int^{\boldsymbol{t}} \boldsymbol{G}_1(x_1 - y)\, \boldsymbol{G}_2(y)\varphi(z)\, dy \right) e^{-ix_1 z}\, dz\, dx \\
&= \iint \left(\int^{\boldsymbol{t}} \boldsymbol{G}_1(x_1 - y)\, \boldsymbol{G}_2(y)\varphi(z)\, e^{-i(x_1-y)z} e^{-iyz}\, dy \right) dz\, dx_1 .
\end{aligned}$$

If we put $x = x_1 - y$, $y = y$, $z = z$, we obtain

$$\begin{aligned}
\langle F_{\boldsymbol{t}}(\boldsymbol{G}_1 \overset{\boldsymbol{t},\mu}{*} \boldsymbol{G}_2), \varphi\rangle &= \iint \left(\int^{\boldsymbol{t}} \boldsymbol{G}_1(x)\, \boldsymbol{G}_2(y)\, \varphi(z) e^{-ixz} e^{-iyz}\, dy \right) dz\, dx \\
&= \int \left(\int F_{\boldsymbol{t}}(\boldsymbol{G}_2)(z)\, \boldsymbol{G}_1(x) \varphi(z) e^{-ixz}\, dz \right) dx = \int F_{\boldsymbol{t}}(F_{\boldsymbol{t}}(\boldsymbol{G}_2\varphi))(x)\, \boldsymbol{G}_1(x)\, dx \\
&= \int^{\boldsymbol{t}} F_{\boldsymbol{t}}(F_{\boldsymbol{t}}(\boldsymbol{G}_2\varphi))(x)\, \boldsymbol{G}_1(x)\, dx = \int^{\boldsymbol{t}} \left(\int F_{\boldsymbol{t}}(\boldsymbol{G}_2)(z)\, \boldsymbol{G}_1(x) \varphi(z) e^{-ixz}\, dz \right) dx \\
&= \int F_{\boldsymbol{t}}(\boldsymbol{G}_1)(z)\, F_{\boldsymbol{t}}(\boldsymbol{G}_2)(z) \varphi(z)\, dz.
\end{aligned}$$

This follow from Proposition 7 a), since $F_{\boldsymbol{t}}(\boldsymbol{G})\varphi$ is a rapidly decreasing function for fixed $\varepsilon > 0$.

d) We have

$$\begin{aligned}
\langle F_{\boldsymbol{t}}(\partial^\alpha \boldsymbol{G}), \varphi\rangle &= \langle \partial^\alpha \boldsymbol{G}, F(\varphi)\rangle = \langle \boldsymbol{G}, \partial^\alpha F(\varphi)\rangle\, (-1)^{|\alpha|} \\
&= \langle \boldsymbol{G}, F((ix)^\alpha \varphi)\rangle\, (-1)^{|\alpha|} = (-1)^{|\alpha|}\, \langle (ix)^\alpha F_{\boldsymbol{t}}(\boldsymbol{G}), \varphi\rangle. \square
\end{aligned}$$

Proposition 7 implies the following one.

PROPOSITION 8. *If* $\boldsymbol{G}_1, \boldsymbol{G}_2, \boldsymbol{G}_3 \in \mathcal{G}_{\boldsymbol{t}}$ *and* μ_ε, $\varepsilon > 0$, *is a unit net, then*:

(i) $\boldsymbol{G}_1 \overset{\boldsymbol{t},\mu}{*} \boldsymbol{G}_2 = \boldsymbol{G}_2 \overset{\boldsymbol{t},\mu}{*} \boldsymbol{G}_1$ *(g.t.d.)*;

(ii) $(\boldsymbol{G}_1 \overset{\boldsymbol{t},\mu}{*} \boldsymbol{G}_2) \overset{\boldsymbol{t},\mu}{*} \boldsymbol{G}_3 = \boldsymbol{G}_1 \overset{\boldsymbol{t},\mu}{*} (\boldsymbol{G}_2 \overset{\boldsymbol{t},\mu}{*} \boldsymbol{G}_3)$ *(g.t.d.)*;

(iii) $\partial^\alpha(\boldsymbol{G}_1 \overset{\boldsymbol{t},\mu}{*} \boldsymbol{G}_2) = \partial^\alpha \boldsymbol{G}_1 \overset{\boldsymbol{t},\mu}{*} \boldsymbol{G}_2$ *(g.t.d.)*, $\alpha \in \mathbb{N}_0^n$.

Let us define the inverse $\boldsymbol{a}, \mu$-Fourier transform of elements from $\mathcal{G}_{\boldsymbol{a}}$. Let $\boldsymbol{G} \in \mathcal{G}_{\boldsymbol{a}}$. We define $F^{-1}_{\boldsymbol{a},\mu} : \mathcal{G}_{\boldsymbol{a}} \to \mathcal{G}_{\boldsymbol{t}}$ by

$$F^{-1}_{\boldsymbol{a},\mu}(\boldsymbol{G})(x) = (2\pi)^{-n} \int^{\boldsymbol{a},\mu} \boldsymbol{G}(y) e^{ixy}\, dy, \qquad x \in \mathbb{R}^n .$$

All the facts which hold for $F_{\boldsymbol{a},\mu}$, hold also for $F^{-1}_{\boldsymbol{a},\mu}$. Furthermore, we have

$$\langle F_{\boldsymbol{t}}(F^{-1}_{\boldsymbol{t}}(\boldsymbol{G})), \varphi\rangle = \langle F^{-1}_{\boldsymbol{t}}(\boldsymbol{G}), F(\varphi)\rangle = \langle \boldsymbol{G}, \varphi\rangle, \qquad \boldsymbol{G} \in \mathcal{G}_{\boldsymbol{t}},\ \varphi \in \mathcal{D},$$

i.e. $F^{-1}_{\boldsymbol{t}}$ is the inverse of $F_{\boldsymbol{t}}$ in the g.t.d. sense. For unit nets $\mu_{1,\varepsilon}$, $\mu_{2,\varepsilon}$, $\varepsilon > 0$, we have

$$\begin{aligned}
\langle \boldsymbol{G}_1 \overset{\boldsymbol{t},\mu_1}{*} \boldsymbol{G}_2, \varphi\rangle &= \langle F_{\boldsymbol{t}}(F^{-1}_{\boldsymbol{t}}(\boldsymbol{G}_1 \overset{\boldsymbol{t},\mu_1}{*} \boldsymbol{G}_2)), \varphi\rangle = \langle F_{\boldsymbol{t}}(F^{-1}_{\boldsymbol{t}}(\boldsymbol{G}_1) F^{-1}_{\boldsymbol{t}}(\boldsymbol{G}_2)), \varphi\rangle \\
&= \langle F_{\boldsymbol{t}}(F^{-1}_{\boldsymbol{t}}(\boldsymbol{G}_1 \overset{\boldsymbol{t},\mu_2}{*} \boldsymbol{G}_2)), \varphi\rangle = \langle \boldsymbol{G}_1 \overset{\boldsymbol{t},\mu_2}{*} \boldsymbol{G}_2, \varphi\rangle.
\end{aligned}$$

This implies that there exists the g.t.d. $\boldsymbol{t}$-convolution for every $\boldsymbol{G}_1$, $\boldsymbol{G}_2$ from $\mathcal{G}_{\boldsymbol{t}}$: $\boldsymbol{G}_1 \overset{\boldsymbol{t}}{*} \boldsymbol{G}_2 = \boldsymbol{G}_1 \overset{\boldsymbol{t},\mu_1}{*} \boldsymbol{G}_2$.

PROPOSITION 9. *Let G be in $\mathcal{G}_{\boldsymbol{t}}$ such that $\boldsymbol{G} \approx g$, $g \in \mathcal{S}'$. Then $x_j\boldsymbol{G} \approx x_j g$; $F_{\boldsymbol{t}}(\boldsymbol{G}) \approx F(g)$, and $F_{\boldsymbol{t}}(\overline{\partial}_j^h \boldsymbol{G})(x) \approx ix_j F_{\boldsymbol{t}}(\boldsymbol{G})(x)$, for $h \in \mathcal{H}$, $j \in \{1, \dots, n\}$.*

Proof. One can easily prove the first two assertions by using Proposition 8 of Part I. So we shall prove only the last one. Let $\varphi \in \mathcal{D}$. Then, by Proposition 7 d),

$$\begin{aligned}\langle F_{\boldsymbol{t}}(\overline{\partial}_j^h G), \varphi\rangle(\phi_\varepsilon) &= \int F_{\boldsymbol{t}}(\partial_j G(\phi_\varepsilon, \cdot) * \phi_{h(\varepsilon)})(x)\, \varphi(x)\, dx \\ &= \int_{\operatorname{supp}(\phi)} ix_j F_{\boldsymbol{t}}(G)(\phi_\varepsilon, x)\, F_{\boldsymbol{t}}(\phi_{h(\varepsilon)})(x)\, \varphi(x)\, dx.\end{aligned}$$

We shall use the fact that for any compact set K $|1 - F(\phi_{h(\varepsilon)})(x)| \le ch(\varepsilon)^n$, $x \in K$. Since for $g \in \mathcal{S}'$, $F(\partial_j g)(x) = ix_j F(g)(x)$, $x \in \mathbb{R}^n$ and (i) and (ii) hold, we have that there exists a function $B \in L^1$, which depends on φ, and there exist an $N \in \mathbb{N}$, and an $\eta > 0$ such that

$$|ix_j F_{\boldsymbol{t}}(G)(\phi_\varepsilon, x)\, \varphi(x)| \le B(x), \qquad x \in \mathbb{R}^n,\ 0 < \varepsilon < \eta,\ \phi \in \mathcal{A}_N.$$

Let $A(\phi_\varepsilon) = \big(\langle F_{\boldsymbol{t}}(\overline{\partial}_j^h G), \varphi\rangle - \langle ix_j F_{\boldsymbol{t}}(G), \varphi\rangle\big)(\phi_\varepsilon)$. Then

$$|A(\phi_\varepsilon)| = \int_{\operatorname{supp}(\varphi)} B(x)\, |1 - F(\phi_{h(\varepsilon)})(x)|\, dx,$$

and $\lim_{\varepsilon \to 0,\, \phi \in \mathcal{A}_N} |A(\phi_\varepsilon)| = 0$, because $h(\varepsilon) \to 0$ as $\varepsilon \to 0$. $\square$

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