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ON THE ESTIMATES OF THE CONVERGENCE RATE OF THE FINITE DIFFERENCE SCHEMES FOR THE APPROXIMATION OF SOLUTIONS OF HYPERBOLIC PROBLEMS

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Abstract. Some new estimates of the convergence rate for hyperbolic initial-boundary value problems are obtained. For a special case a convergence rate estimate compatible with the smoothness of data is obtained.

1. Introduction

For a broad class of finite difference schemes for elliptic boundary value problems, of major interest are the estimates of the convergence rates compatible with the smoothness of data [**3, 7, 9**], i.e.

$$\|u-v\|_{W_{2,h}^k} \le C\, h^{s-k}\, \|u\|_{W_2^s}\,, \qquad s>k\ .$$

Here u denotes the solution of the original boundary value problem, v denotes the solution of the corresponding finite difference scheme, h is the discretization parameter, W_2^s denotes the Sobolev space, $W_{2,h}^k$ denotes the discrete Sobolev space, and C is a positive generic constant, independent of h and u.

Analogous estimates hold in the parabolic case [**4**]:

$$\|u-v\|_{W_{2,h}^{k,k/2}} \le C\, h^{s-k}\, \|u\|_{W_2^{s,s/2}}\,, \qquad s>k\,.$$

To the contrary, in a hyperbolic case, we only have weak estimates, not compatible with the smoothness of data [**5, 6**]:

$$\|u-v\|_{C_\tau(W_{2,h}^k)} \le C\, h^{s-k-1}\, \|u\|_{W_2^s}\,, \qquad s>k+1\,.$$

Recently, for the hyperbolic projection difference scheme, Zlotnik [**12**] has obtained a convergence rate estimate of the order of $2(s-k)/3$. In this paper we show that, in the same cases, it is possible to obtain better estimates.

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2. State of the problem, preliminaries and denotations

As an example let us consider the initial boundary value problem (IBVP) for the equation of the vibrating string in the domain $Q = (0,1) \times (0,T]$:

$$
(1) \qquad \begin{aligned}
&\partial^2 u/\partial t^2 = \partial^2 u/\partial x^2, && (x,t) \in Q,\\
&u(0,t) = u(1,t) = 0, && t \in [0,t],\\
&u(x,0) = u_0(x), \qquad \partial u(x,0)/\partial t = 0, && x \in (0,1).
\end{aligned}
$$

Let L_q, $q \geq 1$, be Lebesgue spaces of integrable functions, and $W_2^s = W_2^s(0,1)$ be standard Sobolev spaces [**11**]. Let us also introduce spaces $C(W_2^s)$ and $L_q(W_2^s)$ of functions defined on $[0,T]$ with values in W_2^s, and norms

$$
\|u\|_{C(W_2^s)} = \max_{t\in[0,T]} \|u(t)\|_{W_2^s} \qquad \text{and} \qquad \|u\|_{L_q(W_2^s)} = \|\,\|u(t)\|_{W_2^s}\|_{L_q}\,.
$$

In the following, we shall assume that $u_0(x) \in W_2^s(0,1)$, $s \geq 1$, and can be oddly extended preserving the class, for $x < 0$ and $x > 1$. In other words, u_0 satisfies the following conditions

$$
u_0^{(2j)}(0) = u_0^{(2j)}(1) = 0, \qquad j = 0,1,\ldots,\big[(s-1)/2\big].
$$

The solution of the IBVP (1) satisfies an a priori estimate [**8**]

$$
(2) \qquad \max_{t\in[0,1]} \left(\left\| \frac{\partial u}{\partial t} \right\|_{L_2}^2 + \left\| \frac{\partial u}{\partial x} \right\|_{L_2}^2 \right) = \left\| \frac{\partial u(x,0)}{\partial t} \right\|_{L_2}^2 + \left\| \frac{\partial u(x,0)}{\partial x} \right\|_{L_2}^2 = \|u'{}_0\|_{L_2}^2\,.
$$

From (2), we obtain

$$
\|u\|_{C(W_2^1)} \leq C\,\|u_0\|_{W_2^1}\,, \qquad C = \text{const} = \sqrt{1+\pi^{-2}}\,.
$$

Differentiating equation (1), using estimate (2), we obtain the following estimate

$$
(3) \qquad \max_{t\in[0,T]} \left\| \frac{\partial^k u}{\partial x^j \partial t^{k-j}} \right\|_{L_2} \leq \|u_0^{(k)}\|_{L_2}\,, \qquad 1 \leq k \leq [s]\,, \quad 0 \leq j \leq k\,.
$$

Hence, all partial derivatives of the solution $u(x,t)$ of order $\leq [s]$ belong to the space $C(L_2)$. The solution can be oddly extended in x, for $x < 0$ and $x > 1$, and evenly extended in t, for $t < 0$, thus preserving its class.

Let $\overline{\omega}_h$ be a uniform mesh with the stepsize $h = 1/n$ on $[0,1]$, $\omega_h = \overline{\omega}_h \cap (0,1)$ and $\omega_h^- = \omega_h \cup \{0\}$. Let v_x and $v_{\overline{x}}$ denote the upward and backward finite differences:

$$
v_x = \big(v(x+h) - v(x)\big)/h\,, \qquad v_{\bar{x}} = \big(v(x) - v(x-h)\big)/h\,.
$$

We define the following discrete norms

$$
\|v\|_h = \|v\|_{L_{2,h}} = \Big\{ h \sum_{x\in\omega_h} v^2(x) \Big\}^{1/2}, \qquad [\![v]\!]_h = [\![v]\!]_{L_{2,h}} = \Big\{ h \sum_{x\in\omega_h^-} v^2(x) \Big\}^{1/2},
$$

$$
\text{and} \qquad \|v\|_{W_{2,h}^1} = \big(\|v\|_h^2 + [\![v]\!]_h^2\big)^{1/2}.
$$

Let $\overline{\omega}_\tau$ be a uniform mesh with the stepsize $\tau = T/(m-1/2)$ on $[-\tau/2, T]$, $\omega_\tau = \overline{\omega}_\tau \cap (0,T)$ and $\omega_\tau^- = \omega_\tau \cup \{-\tau/2\}$. We shall introduce the following notations

$$v = v(t), \quad \hat{v} = v(t+\tau), \quad \check{v} = v(t-\tau), \quad v^j = v\big((j-1/2)\tau\big),$$
$$\overline{v} = (v+\hat{v})/2, \quad v_t = (\hat{v}-v)/\tau, \quad v_{\bar{t}} = (v-\check{v})/\tau.$$

For functions defined on the mesh $\overline{\omega}_h \times \overline{\omega}_\tau$ we define the following norms

$$\|v\|_{C_\tau(W_{2,h}^1)} = \max_{t\in\omega_\tau^-} \|v(\cdot,t)\|_{W_{2,h}^1}$$

and

$$\|v\|_{L_{q,\tau}(L_{2,h})} = \left\{\tau \sum_{t\in\omega_\tau} \|v(\cdot,t)\|_{L_{2,h}}^q\right\}^{1/q}.$$

Let S_x a S_t denote the Steklov smoothing operators in x and t

$$S_x f(x,t) = \frac{1}{h}\int_{x-h/2}^{x+h/2} f(\xi,t)\,d\xi, \qquad S_t f(x,t) = \frac{1}{\tau}\int_{t-\tau/2}^{t+\tau/2} f(x,\eta)\,d\eta.$$

Finally, let C denote the positive generic constant, independent of h and τ.

3. Second order finite difference schemes

We approximate the IBVP (1) by the following weighted finite difference scheme (FDS) [**10**]

$$v_{t\bar{t}} = [\sigma\,\hat{v} + (1-2\sigma)\,v + \sigma\,\check{v}]_{x\bar{x}}, \qquad x\in\omega_h, \quad t\in\omega_\tau, \tag{4}$$

$$v(0,t) = v(1,t) = 0, \qquad t\in\overline{\omega}_\tau, \tag{5}$$

$$v^0 = v^1 = u_0(x), \qquad x\in\omega_h. \tag{6}$$

The solution of the FDS (4–6) satisfies the relation

$$N^2(v) \equiv \|v_t\|_h^2 + \tau^2\,(\sigma-0.25)\,[\![v_{tx}\|_h^2 + [\![\overline{v}_x\|_h^2 = [\![v_x^0\|_h^2\,.$$

From here, for $\sigma \ge 1/4$, we obtain

$$\max_{t\in\omega_\tau^-} [\![\overline{v}_x\|_h \le [\![v_x^0\|_h. \tag{7}$$

The inequality (7) holds also for $\sigma < 1/4$, if

$$\tau \le h\sqrt{\frac{1-c_0}{1-4\,\sigma}}, \qquad c_0 = \text{const} \in (0,1) \qquad \text{(conditional stability)}.$$

From the initial conditions (6) it follows that

$$\begin{aligned}
[\![v_x^0\|_h = [\![u_{0,x}\|_h &= \left\{h\sum_{x\in\omega_h^-}\left[\frac{u_0(x+h)-u_0(x)}{h}\right]^2\right\}^{1/2} \\
&= \left\{h\sum_{x\in\omega_h^-}\left(\frac{1}{h}\int_x^{x+h} u_0'(\xi)\,d\xi\right)^2\right\}^{1/2} \\
&\le \left\{\sum_{x\in\omega_h^-}\int_x^{x+h}[u_0'(\xi)]^2\,d\xi\right\}^{1/2} = \|u_0'\|_{L_2} \le \|u_0\|_{W_2^1}\,.
\end{aligned} \tag{8}$$

Using the inequality [**10**]

$$\|v\|_h \leq [\![v_x\|_h / \big(2\sqrt{2}\big)\,,$$

from (7) and (8) we obtain

$$\|\overline{v}\|_{C_\tau(W^1_{2,h})} \leq C\,\|u_0\|_{W^1_2}\,. \tag{9}$$

Let u be the solution of IBVP (1) and v the solution of FDS (4–6). The error $z = u - v$ satisfies the conditions

$$z_{t\bar t} = [\sigma\,\hat z + (1-2\sigma)\,z + \sigma\check z]_{x\bar x} + \psi, \qquad x \in \omega_h, \quad t \in \omega_\tau\,, \tag{10}$$

$$z(0,t) = z(1,t) = 0, \qquad t \in \overline{\omega}_\tau, \tag{11}$$

$$z^0 = z^1 = u(x,\tau/2) - u_0(x), \qquad x \in \omega_h, \tag{12}$$

where $\psi = u_{t\bar t} - [\sigma\,\hat u + (1-2\sigma)\,u + \sigma\,\check u]_{x\bar x}$.

The a priori estimate

$$\max_{t\in\omega_\tau^-} [\![\overline{z}_x\|_h \leq \max_{t\in\omega_\tau^-} N(z) \leq [\![z^0_x\|_h + \frac{1}{\sqrt{c}}\,\|\psi\|_{L_{1,\tau}(L_{2,h})} \tag{13}$$

where $c = 1$ for $\sigma \geq 1/4$, and $c = c_0$ for $\sigma < 1/4$, holds.

Estimating z^0_x and ψ, using the Bramble-Hilbert lemma [**1, 2**], for $c_1 h \leq \tau \leq c_2 h$, we obtain the estimate [**5**]

$$\max_{t\in\omega_\tau^-} [\![\overline{z}_x\|_h \leq C\,h^{s-2}\,\|u\|_{W^s_2(Q)}\,, \qquad 2 \leq s \leq 4\,,$$

i.e.

$$\|\overline{z}\|_{C_\tau(W^1_{2,h})} \leq C\,h^{s-2}\,\|u\|_{W^s_2(Q)}\,, \qquad 2 \leq s \leq 4\,. \tag{14}$$

On the other hand, using

$$z^0_x = \big[u(x,\tau/2) - u(x,0)\big]_x = \frac{1}{h}\int_x^{x+h}\int_0^{\tau/2}\int_0^t \frac{\partial^3 u(\xi,\eta)}{\partial t^2\partial x}\,d\eta\,dt\,d\xi$$

we easily obtain

$$\begin{aligned} [\![z^0_x\|_h &\leq \left\{h\sum_{x\in\omega_h^-} h^{-2}\,h\,(\tau/2)^3 \int_x^{x+h}\int_0^{\tau/2}\left(\frac{\partial^3 u(\xi,t)}{\partial t^2\partial x}\right)^2 dt\,d\xi\right\}^{1/2} \\ &\leq \frac{\tau^2}{4}\max_{t\in[0,T]}\left\|\frac{\partial^3 u}{\partial t^2\partial x}\right\|_{L_2}\,. \end{aligned} \tag{15}$$

Using relations $S^2_x\big(\partial^2 u/\partial x^2\big) = u_{x\bar x}$ and $S^2_t\big(\partial^2 u/\partial t^2\big) = u_{t\bar t}$, and equation (1), we can represent the function ψ in the following manner

$$\begin{aligned} \psi(x,t) &= \left(S^2_t\frac{\partial^2 u}{\partial t^2} - S^2_x S^2_t\frac{\partial^2 u}{\partial t^2}\right) - \left(S^2_x\frac{\partial^2 u}{\partial x^2} - S^2_x S^2_t\frac{\partial^2 u}{\partial x^2}\right) - \sigma\,\tau^2\,S^2_x S^2_t\frac{\partial^4 u}{\partial x^2\partial t^2} \\ &= -\frac{1}{h\tau}\int_{x-h}^{x+h}\int_x^{\xi}\int_{t-\tau}^{t+\tau}(\xi-\eta)\left(1-\frac{|\xi-x|}{h}\right)\left(1-\frac{|\zeta-t|}{\tau}\right)\frac{\partial^4 u(\eta,\zeta)}{\partial x^2\partial t^2}\,d\zeta\,d\eta\,d\xi \end{aligned}$$

$$+\frac{1}{h\tau}\int_{x-h}^{x+h}\int_{t-\tau}^{t+\tau}\int_{t}^{\eta}(\eta-\zeta)\left(1-\frac{|\xi-x|}{h}\right)\left(1-\frac{|\eta-t|}{\tau}\right)\frac{\partial^4 u(\xi,\zeta)}{\partial x^2\partial t^2}\,d\zeta\,d\eta\,d\xi$$
$$-\frac{\sigma\tau^2}{h\tau}\int_{x-h}^{x+h}\int_{t-\tau}^{t+\tau}\left(1-\frac{|\xi-x|}{h}\right)\left(1-\frac{|\eta-t|}{\tau}\right)\frac{\partial^4 u(\xi,\eta)}{\partial x^2\partial t^2}\,d\eta\,d\xi\,.$$

From this we obtain

$$|\psi(x,t)|\le\frac{C\,(h^2+\tau^2)}{\sqrt{h\tau}}\left\|\frac{\partial^4 u}{\partial x^2\partial t^2}\right\|_{L_2(e)},$$

where $e=(x-h,x+h)\times(t-\tau,t+\tau)$, and

$$\|\psi\|_{L_{1,\tau}(L_{2,h})}\le C\,(h+\tau)^2\max_{t\in[0,T]}\left\|\frac{\partial^4 u}{\partial x^2\partial t^2}\right\|_{L_2(0,1)}. \tag{16}$$

From (13), (15), (16) and (3) we obtain the following convergence rate estimates for FDS (4–6)

$$\max_{t\in\omega_\tau^-}[\![\overline{z}_x|\!|_h\le C\,(h+\tau)^2\,\|u_0\|_{W_2^4}\,,\qquad\text{i.e.}$$
$$\|\overline{z}\|_{C_\tau(W_{2,h}^1)}\le C\,(h+\tau)^2\,\|u_0\|_{W_2^4}\,. \tag{17}$$

On the other hand, from the self-evident inequalities

$$\max_{t\in\omega_\tau^-}[\![\overline{z}_x|\!|_h\le\max_{t\in\omega_\tau^-}[\![\overline{u}_x|\!|_h+\max_{t\in\omega_\tau^-}[\![\overline{v}_x|\!|_h\le\max_{t\in[0,T]}\left\|\frac{\partial u}{\partial x}\right\|_{L_2}+\|u_0'\|_{L_2}\le 2\,\|u_0'\|_{L_2}$$

we obtain

$$\|\overline{z}\|_{C_\tau(W_{2,h}^1)}\le C\,\|u_0\|_{W_2^1}\,. \tag{18}$$

By the K-method for the real interpolation [**11**] we introduce the function spaces $(W_2^k,W_2^{k+1})_{\theta,2}$ $(0<\theta<1,\ k=0,1,2,\dots)$. Let R denote the linear operator defined by $Ru_0=\overline{z}$. From (17) and (18) it follows that R is a bounded operator from W_2^4 into $D\equiv C_\tau(W_{2,h}^1)$ and also from W_2^1 into D. Therefore, R is a bounded operator from $(W_2^1,W_2^4)_{\theta,2}$ into D, and the interpolation inequality

$$\|R\|_{(W_2^1,W_2^4)_{\theta,2}\to D}\le\|R\|_{W_2^1\to D}^{1-\theta}\,\|R\|_{W_2^4\to D}^{\theta} \tag{19}$$

holds. Here

$$\|R\|_{A\to B}=\sup_{u\ne 0}\frac{\|Ru\|_B}{\|u\|_A}$$

is the standard operator norm of $R:A\to B$.

From (17–19) we get

$$\|\overline{z}\|_{C_\tau(W_{2,h}^1)}\le C\,(h+\tau)^{2\theta}\,\|u_0\|_{(W_2^1,W_2^4)_{\theta,2}}\,.$$

Further from [**11**], we have

$$(W_2^1, W_2^4)_{\theta,2} = W_2^{1-\theta+4\theta} = W_2^{3\theta+1}\,, \qquad 0 < \theta < 1\,.$$

Setting $3\theta + 1 = s$, we finally obtain the required convergence rate estimate

$$\|\overline{z}\|_{C_\tau(W_{2,h}^1)} \le C\,(h+\tau)^{\frac{2}{3}(s-1)}\,\|u_0\|_{W_2^s}\,, \qquad 1 \le s \le 4\,. \tag{20}$$

The estimate of the form (20) is obtained in [**12**].

4. Fourth–order scheme

Let us approximate equation (1) by

$$v_{t\bar t} = v_{x\bar x} + \frac{\tau^2 - h^2}{12}\, v_{t\bar t x\bar x}\,. \tag{21}$$

Here observe that (21) reduces to (4) for $\sigma = 1/2 - h^2/(12\tau^2)$. The scheme is stable for

$$\tau \le h\sqrt{1 - 3\,c_0/2}\,, \qquad c_0 = \mathrm{const} \in (0, 2/3)\,.$$

The initial conditions can be approximated by

$$v^0 = v^1 = u_0 + \frac{\tau^2}{8}\, u_{0,x\bar x}\,, \qquad x \in \omega_h\,. \tag{22}$$

Then,

$$[\![v_x^0 \|_h \le [\![u_{0,x} \|_h + C\,\tau^2\, h^{-2}\, [\![u_{0,x} \|_h \le \|u_0'\|_{L_2}$$

and the a priori estimates (7) and (9) hold.

The error $z = u - v$ satisfies the conditions (10), (11) and

$$z^0 = z^1 = u(x, \tau/2) - u_0(x) - 0.125\,\tau^2\, u_{0,x\bar x}\,, \tag{23}$$

as well as the a priori estimate (13).

The following representations hold:

$$\begin{aligned} z_x^0 = &-\frac{\tau^2}{8h^2}\int_{x-h}^{x+h}\int_x^{\xi}\int_{\eta}^{\eta+h}(\xi-\eta)\left(1-\frac{|\xi-x|}{h}\right)u_0^{(5)}(\zeta)\,d\zeta\,d\eta\,d\xi \\ &+\frac{1}{6h}\int_0^{\tau/2}\int_x^{x+h}\left(\frac{\tau}{2}-s\right)^3\frac{\partial^5 u(\xi,\eta)}{\partial t^4\partial x}\,d\xi\,d\eta \end{aligned}$$

and

$$\begin{aligned} \psi(x,t) = \frac{1}{6h\tau}\int_{x-h}^{x+h}\int_x^{\xi}\int_{t-\tau}^{t+\tau}\left[\frac{h^2}{2}(\xi-\eta)-(\xi-\eta)^3\right]\times \\ \times\left(1-\frac{|\xi-x|}{h}\right)\left(1-\frac{|\zeta-t|}{\tau}\right)\frac{\partial^6 u(\eta,\zeta)}{\partial x^4\partial t^2}\,d\zeta\,d\eta\,d\xi \end{aligned}$$

$$- \frac{1}{6h\tau} \int_{x-h}^{x+h} \int_{t-\tau}^{t+\tau} \int_{t}^{\zeta} \left[\frac{\tau^2}{2}(\zeta - \eta) - (\zeta - \eta)^3 \right] \times$$
$$\times \left(1 - \frac{|\xi - x|}{h}\right) \left(1 - \frac{|\zeta - t|}{\tau}\right) \frac{\partial^6 u(\xi, \eta)}{\partial x^4 \partial t^2} \, d\eta \, d\zeta \, d\xi .$$

Herefrom we obtain

$$\llbracket z_x^0 \|_h \le C\, (\tau^2 \, h^2 + h^4)\, \|u_0^{(5)}\|_{L_2} \le C\, h^4 \, \|u_0\|_{W_2^5} \,, \tag{24}$$
$$\|\psi\|_{L_{1,\tau}(L_{2,h})} \le C\, (h^4 + \tau^4) \left\| \frac{\partial^6 u}{\partial x^4 \partial t^2} \right\|_{C(L_2)} \le C\, h^4 \, \|u_0\|_{W_2^6}$$

and

$$\|\overline{z}\|_{C_\tau(W_{2,h}^1)} \le C\, h^4 \, \|u_0\|_{W_2^6} \,. \tag{25}$$

Further,

$$\max_{\tau \in \omega_\tau^-} \llbracket \overline{z}_x \|_h \le \max_{\tau \in \omega_\tau^-} \llbracket \overline{u}_x \|_h + \max_{\tau \in \omega_\tau^-} \llbracket \overline{v}_x \|_h$$
$$\le \max_{t \in [0,T]} \left\| \frac{\partial u}{\partial x} \right\|_{L_2} + \llbracket v_x^0 \|_h \le C\, \|u_0'\|_{L_2} \le C\, \|u_0\|_{W_2^1} \,.$$

From here follows the inequality (18).

From (25) and (18), by interpolation we obtain the following convergence rate estimate of FDS (21), (5), (22)

$$\|\overline{z}\|_{C_\tau(W_{2,h}^1)} \le C\, h^{\frac{4}{5}(s-1)} \, \|u_0\|_{W_2^s} \,, \qquad 1 \le s \le 6 \,. \tag{26}$$

5. The exact scheme

Set $\tau = h$ ($m = [T/h + 1/2]$), and approximate equation (1) by the explicit FDS

$$v_{t\bar{t}} = v_{x\bar{x}} \,. \tag{27}$$

The solution of the IBVP (1) can be represented by the series

$$u(x,t) = \sum_{k=1}^{\infty} a_k \, \cos k\pi t \, \sin k\pi x \,.$$

It could easily be verified that $u(x,t)$ satisfies equation (27). The error $z = u - v$ also satisfies (27), and the a priori estimate

$$\max_{t \in \omega_\tau^-} \llbracket \overline{z}_x \|_h \le \llbracket z_x^0 \|_h \,,$$

holds. Hence, the convergence rate depends only on the approximation of the initial conditions.

If the initial conditions are approximated by (6), then the relations (15) and (18) hold; so we have

$$\|\overline{z}\|_{C_\tau(W^1_{2,h})} \le C\, h^2\, \|u_0\|_{W^3_2}\,,$$

and

$$\|\overline{z}\|_{C_\tau(W^1_{2,h})} \le C\, \|u_0\|_{W^1_2}\,. \tag{28}$$

By interpolation we obtain

$$\|\overline{z}\|_{C_\tau(W^1_{2,h})} \le C\, h^{s-1}\, \|u_0\|_{W^s_2}\,, \qquad 1 \le s \le 3\,. \tag{29}$$

If initial conditions are approximated by (22), then (24) holds and

$$\|\overline{z}\|_{C_\tau(W^1_{2,h})} \le C\, h^4\, \|u_0\|_{W^5_2}\,. \tag{30}$$

By interpolation, from (28) and (30) we obtain the estimate in the form (29), for $1 \le s \le 5$. The estimate (29) is compatible with the smoothness of data.

The obtained results can be transferred, without difficulties, to the IBVP with nonhomogeneous second initial condition

$$\partial u(x,0)/\partial t = u_1(x)\,.$$

Let the conditions

$$u_1 \in W^{s-1}_2(0,1)\,, \qquad s \ge 1\,,$$
$$u_1^{(2j)}(0) = u_1^{(2j)}(1) = 0\,, \qquad j = 0,1,\ldots,\left[\frac{s-2}{2}\right], \qquad \text{for} \quad s \ge 2$$

hold. Then, we substitute the initial conditions (6) and (22) by

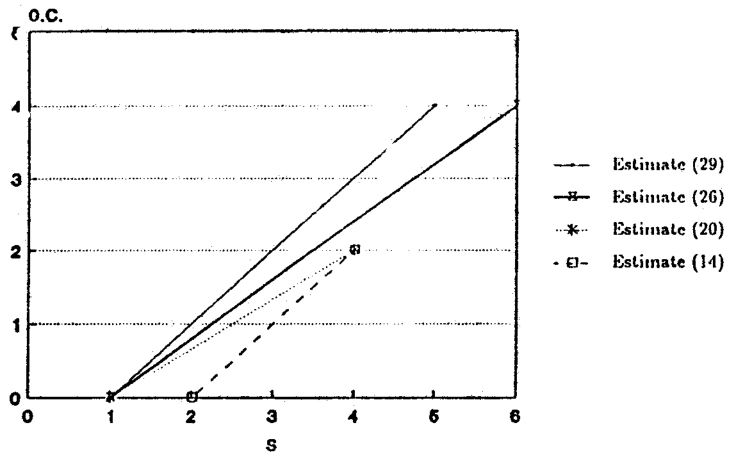
$$v^0 = u_0 - \frac{\tau}{2}\, S_x^2 u_1\,, \qquad v^1 = u_0 + \frac{\tau}{2}\, S_x^2 u_1\,, \qquad x \in \omega_h\,,$$

and

$$v^0 = u_0 - \frac{\tau}{2}\, S_x^2 u_1 + \frac{\tau^2}{8}\, u_{0,x\bar{x}} - \frac{\tau^3 - 2h^2\tau}{48}\, S_x^4 u_1''\,,$$
$$v^1 = u_0 + \frac{\tau}{2}\, S_x^2 u_1 + \frac{\tau^2}{8}\, u_{0,x\bar{x}} + \frac{\tau^3 - 2h^2\tau}{48}\, S_x^4 u_1''\,.$$

Hence, the estimates of the forms (20), (26) and (29) hold, where on the right-hand-side $\|u_0\|_{W^s_2}$ is replaced by $\|u_0\|_{W^s_2} + \|u_1\|_{W^{s-1}_2}$.

The following diagram graphically represents the relation between the smoothness of initial data (s) and the order of convergence (o.c.) in estimates (14), (20), (26) and (29).



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