PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 52 (66), 1992, 136–152

GENERALIZED MIRON'S d-CONNECTION IN THE RECURRENT K-HAMILTON SPACES

Irena Čomić

Abstract. For the generalized Miron's d-connection in recurrent K-Hamilton spaces, the torsion tensor and the connection coefficients are determined. For special cases the already known results are obtained.

1. Introduction. This paper is a generalization of [**8**], based on works of Miron, Anastasiei, Janus, Kirhovits and others, in the Hamilton geometry. Here the generalization is going in different directions:

a) The transformation of the coordinate system is given by (2.1), where (2.1b) is more general than in former investigations in this field.

b) In the metric tensor the blocks over and under the diagnal are not necessarily equal to zero.

c) The connection coeficients are introduced in such a way that $\nabla_X\colon T(E^*) \to T(E^*)$ by which we can obtain eight kinds of connection coefficients.

d) A consequence of **c)** is that the torsion tensor has also eight of components.

e) The field $\lambda(x,p)$, the vector field of recurrency is introduced.

The main result is that the connection coefficients for such a general case are obtained explicitly. For some special cases the already known results are obtained. Theorems 7.6 and 7.7 give the relations between the metric tensor and the nonlinear connection of such K-Hamilton spaces, which allow torsion-free Miron's d-connection.

2. Adapted bases in $T(E^*)$ and $T^*(E^*)$. Let E^* be an $(n+mK)$-dimensional differentiable manifold. If u is one point of E^*, then, in some local chart, u has the coordinates

$$u = \big((x^i),(p^1_a),(p^2_a),\dots,(p^K_a)\big) = \big((x^i),(p^\alpha_a)\big) = (x,p).$$

AMS Subject Classification (1980): Primary 53 B 40
Research supported by Science Fund of Serbia, grant 0401A, trough Mathematical Institute

where $(x^i) = (x^1, x^2, \ldots, x^n) = (x)$, $(p_a^\alpha) = (p_1^\alpha, \ldots, p_m^\alpha) = (p^\alpha)$, and $a, b, c, d, e, f = \overline{1, m}$; $i, j, h, k, l = \overline{1, n}$; $\alpha, \beta, \gamma, \delta = \overline{1, K}$.

We shall consider the following coordinate transformation. If $\left((x^{i'}), (p_{a'}^\alpha)\right) = (x', p')$ are the coordinates of the same point u in the new coordinate system, then,

$$\text{(2.1)} \qquad (a) \quad x^{i'} = x^{i'}(x^1, \ldots, x^n), \qquad \operatorname{rank}\left[\partial x^{i'}/\partial x^i\right] = n$$

$$(b) \quad p_{a'}^\alpha = \overset{(\alpha)}{M}{}_{a'}^{a}(x^1, \ldots, x^n) p_a^\alpha, \qquad \operatorname{rank}\left[\partial p_{a'}^\alpha / \partial p_a^\alpha\right] = m.$$

The Einstein summation convention will be used for all three kinds of indices, except in the case when the index appears in brackets. If (2.1) is valid, then an inverse transformation exists, i.e.

$$\text{(2.2)} \qquad (a) \quad x^i = (x^{1'}, \ldots, x^{n'}) \qquad (b) \quad p_a^\alpha = \overset{(\alpha)}{M}{}_{a}^{a'}(x^{1'}, \ldots, x^{n'}) p_{a'}^\alpha.$$

The natural basis $\overline{B} = \{(\partial_i), (\partial_1^a), \ldots, (\partial_K^a)\}$ of $T(E^*)$ is formed by n vectors of the type $\partial_i = \partial/\partial x^i$ and $m \cdot K$ vectors of the type $\partial_\alpha^a = \partial/\partial p_a^\alpha$. Any vector field $X \in T(E^*)$ may be represented in the form

$$\text{(2.3)} \qquad X = \overline{X}^i \partial_i + \overline{X}_a^\alpha \partial_\alpha^a.$$

With respect to the coordinate transformations (2.1) and (2.2), the basic vectors of $\overline{B}$ obey the following law of transformation:

$$\text{(2.4)} \qquad \begin{bmatrix} \partial_i \\ \partial_1^a \\ \vdots \\ \partial_K^a \end{bmatrix} = \begin{bmatrix} \partial x^{i'}/\partial x^i & \left(\partial_i \overset{(1)}{M}{}_{a'}^{a}\right) p_a^1 & \ldots & \left(\partial_i \overset{(K)}{M}{}_{a'}^{a}\right) p_a^K \\ 0 & \overset{(1)}{M}{}_{a'}^{a} & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & \overset{(K)}{M}{}_{a'}^{a} \end{bmatrix} \begin{bmatrix} \partial_{i'} \\ \partial_1^{a'} \\ \vdots \\ \partial_K^{a'} \end{bmatrix}$$

$$\text{(2.5)} \qquad \begin{bmatrix} \partial_{i'} \\ \partial_1^{a'} \\ \vdots \\ \partial_K^{a'} \end{bmatrix} = \begin{bmatrix} \partial x^{j}/\partial x^{i'} & \left(\partial_{i'} \overset{(1)}{M}{}_{b}^{b'}\right) p_{b'}^1 & \ldots & \left(\partial_{i'} \overset{(K)}{M}{}_{b}^{b'}\right) p_{b'}^K \\ 0 & \overset{(1)}{M}{}_{b}^{a'} & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & \overset{(K)}{M}{}_{b}^{a'} \end{bmatrix} \begin{bmatrix} \partial_{j} \\ \partial_1^{b} \\ \vdots \\ \partial_K^{b} \end{bmatrix}$$

Substituting (2.5) into (2.4), we obtain,

$$\text{(2.6)} \qquad (a) \quad \frac{\partial x^{i'}}{\partial x^i} \frac{\partial x^j}{\partial x^{i'}} = \delta_i^j \qquad (b) \quad \overset{(\alpha)}{M}{}_{a'}^{a} \overset{(\alpha)}{M}{}_{b}^{a'} = \delta_b^a$$

$$(c) \quad \left(\partial_{i'} \overset{(\alpha)}{M}{}_{b}^{b'}\right) p_{b'}^\alpha \frac{\partial x^{i'}}{\partial x^i} + \left(\partial_i \overset{(\alpha)}{M}{}_{a'}^{a}\right) p_a^\alpha \overset{(\alpha)}{M}{}_{b}^{a'} = 0,$$

where (2.6c) is the consequence of (2.4) and (2.6b).

From (2.4) and (2.5) it is obvious that ∂_i and $\partial_{i'}$ do not transform as tensors, so we introduce a new, the so-called adapted basis $B = \{(\delta_i), (\partial^a_1), \dots, (\partial^a_K)\}$ of $T(E^*)$, where by definition

$$\delta_i = \partial_i - N^{\alpha}_{a\,i}(x,p)\partial^a_{\alpha}, \tag{2.7}$$

and $N^{\alpha}_{a\,i}(x,p)$ are the coefficients of the nonlinear connection. These are the arbitrary functions, which under the coordinate transformations (2.1) and (2.2) transform in the following manner:

$$\begin{aligned} &(a) \quad N^{\alpha}_{a'\,i'}(x',p') = \overset{(\alpha)}{M}{}^{a}_{a'} \frac{\partial x^i}{\partial x^{i'}} N^{\alpha}_{a\,i}(x,p) - p^{\alpha}_a \frac{\partial \overset{(\alpha)}{M}{}^{a}_{a'}}{\partial x^i} \frac{\partial x^i}{\partial x^{i'}} \\ &(b) \quad N^{\alpha}_{b\,j}(x,p) = \overset{(\alpha)}{M}{}^{b'}_{b} \frac{\partial x^{j'}}{\partial x^j} N^{\alpha}_{b'\,j'}(x',p') + p^{\alpha}_a \overset{(\alpha)}{M}{}^{a'}_{b} \frac{\partial \overset{(\alpha)}{M}{}^{a}_{a'}}{\partial x^j}. \end{aligned} \tag{2.8}$$

Any vector field $X \in T(E^*)$ in the adapted basis B is given by

$$X = X^i \delta_i + X^{\alpha}_a \partial^a_{\alpha}. \tag{2.9}$$

The coordinates of the vector X given by (2.9) and the elements of basis B transform as tensors in the following manner:

$$\begin{aligned} \delta_i &= \frac{\partial x^{i'}}{\partial x^i} \delta_{i'} \qquad & \partial^a_{\alpha} &= \overset{(\alpha)}{M}{}^{a}_{a'}(x) \partial^{a'}_{\alpha} \\ X^i &= \frac{\partial x^i}{\partial x^{i'}} X^{i'} \qquad & X^{\alpha}_a &= \overset{(\alpha)}{M}{}^{a'}_{a}(x') X^{\alpha}_{a'}. \end{aligned} \tag{2.10}$$

From (2.3) and (2.9) we obtain the relation between coordinates of the field X, in the bases $\overline{B}$ and B. They are connected by the relations $X^i = \overline{X}^i$, $X^{\alpha}_a = \overline{X}^{\alpha}_a + N^{\alpha}_{a\,i}\overline{X}^i$. The subspace of $T(E^*)$ spanned by $\{\delta_i\}$, shall be denoted by $T_H(E^*)$ (the horizontal part) and the subspace spanned by $\{\partial^a_{\alpha}\}$, by ${}_{(\alpha)}T_V(E^*)$ (the vertical α-part). So, we have $T(E^*) = T_H(E^*) \oplus T_V(E^*)$, where,

$$T_V(E^*) = \sum_{\alpha=1}^{K} {}_{(\alpha)}T_V(E^*), \qquad \dim T_H(E^*) = n, \qquad \dim_{(\alpha)} T_V(E^*) = m.$$

Here, $X^i\delta_i$ is the horizontal and $X^{\alpha}_a\partial^a_{\alpha}$ the vertical part of the field X. Now (2.9) may be written in the from

$$X = X_H + X_V, \qquad X_H = X^i\delta_i \qquad X_V = X^{\alpha}_a \partial^a_{\alpha}.$$

Let us consider the dual tangent space of E^*, the space $T^*(E^*)$. The natural basis in $T^*(E^*)$ is

$$\overline{B}^* = \left\{dx^1, \dots, dx^n, dp^1_1, \dots, dp^1_m, \dots, dp^K_1, \dots, dp^K_m\right\} = \left\{dx^i, dp^1_a, \dots, dp^K_a\right\}.$$

From (2.1) we obtain

$$(2.11)\qquad (a)\quad dx^{i'} = \frac{\partial x^{i'}}{\partial x^i}dx^i \qquad (b)\quad dp^\alpha_{a'} = \frac{\partial \overset{(\alpha)}{M}{}^a_{a'}(x)}{\partial x^i}p^\alpha_a dx^i + \overset{(\alpha)}{M}{}^a_{a'}(x)dp^\alpha_a.$$

From (2.11b) it is obvious that dp^α_a do not transform as tensors, so we intruduce a new basis $B^* = \{(dx^1), (\delta p^1_a), \dots, (\delta p^K_a)\}$, where,

$$(2.12)\qquad \delta p^\alpha_a = dp^\alpha_a + N^\alpha_{a\,i}(x,p)dx^i.$$

Through the coordinate transformation (2.1) the bases $\overline{B}^*$ and B^* are related by (2.11a) and

$$(2.13)\qquad (a)\quad \delta p^\alpha_a = \overset{(\alpha)}{M}{}^{a'}_a(x')\delta p^\alpha_{a'} \qquad (b)\quad \delta p^\alpha_{a'} = \overset{(\alpha)}{M}{}^a_{a'}(x)\delta p^\alpha_a.$$

The proof of (2.13) is obtained by using (2.12) and (2.8). Any field $w \in T^*(E^*)$ can be written in the bases $\overline{B}^*$ and B^* in the following manner:

$$(2.14)\qquad w = \overline{w}_i dx^i + \overline{w}^a_\alpha dp^\alpha_a = w_i dx^i + w^a_\alpha \delta p^\alpha_a$$

where $w_i = \overline{w}_i - N^\alpha_{a\,i}\overline{w}^a_\alpha$, $\overline{w}^a_\alpha = w^a_\alpha$.

The subspace of $T^*(E^*)$ spanned by $\{(dx^i)\}$ shall be denoted by $T^*_H(E^*)$ and the subspace spanned by $\{(\delta p^\alpha_a)\}$ by ${}_{(\alpha)}T^*_V(E^*)$. So, we have

$$T^*(E^*) = T^*_H(E^*) \oplus T^*_V(E^*), \quad \text{where} \quad T^*_V(E^*) = \sum_{\alpha=1}^K \oplus_{(\alpha)}T^*_V(E^*).$$

.

Now, (2.14) may be written in the form

$$w = w_H + w_V, \qquad w_H = w_i dx^i, \qquad w_V = w^a_\alpha \delta p^\alpha_a.$$

If $\{(dx^i), (\delta p^1_a), \dots, (\delta p^K_a)\}$ and $\{(dx^{i'}), (\delta p^1_{a'}), \dots, (\delta p^K_{a'})\}$ are two bases in $T^*(E^*)$, related by (2.11a) and (2.13), then any $w \in T^*(E^*)$ satisfies the relation

$$(2.15)\qquad w = w_i dx^i + w^a_\alpha \delta p^\alpha_a = w_{i'}dx^{i'} + w^{a'}_\alpha \delta p^\alpha_{a'}.$$

Substituting $dx^{i'}$ from (2.11a) and $\delta p^\alpha_{a'}$ from (2.13b) into (2.15) and comparing the coefficients of the basis vectors, we obtain

$$(2.16)\qquad w_i = w_{i'}\partial x^{i'}/\partial x^i, \qquad w^a_\alpha = \overset{(\alpha)}{M}{}^a_{a'} w^{a'}_\alpha.$$

By a straightforward calculation, we can prove the following

Proposition 2.1. *The adapted bases* $\{(\delta_i), (\partial^a_1), \dots, (\partial^a_K)\}$ *and* $\{(dx^i), (\delta p^1_a), \dots, (\delta p^K_a)\}$ *are dual to each other, i.e.*

$$(2.17)\qquad \left\langle \delta_i, dx^j \right\rangle = \delta^j_i, \qquad \left\langle \delta_i, \delta p^\alpha_a \right\rangle = 0,$$
$$\left\langle \partial^a_\alpha, dx^j \right\rangle = 0, \qquad \left\langle \partial^a_\alpha, \delta p^\beta_b \right\rangle = \delta^a_b \delta^\beta_\alpha.$$

3. Tensor Fields on E^*. *a*) A horizontal tensor field t_H has the local representation:

$$t_H = t^{i_1\dots i_p}j_1\dots j_q(x,p)\delta_{i_1}\otimes\dots\otimes\delta_{i_p}\otimes dx^{j_1}\otimes\dots\otimes dx^{j_q};$$

it is defined on

$$\underbrace{T_H(E^*)\otimes\dots\otimes T_H(E^*)}_{p\text{ times}}\otimes\underbrace{T^*_H(E)\otimes\dots\otimes T^*_H(E^*)}_{q\text{ times}}$$

By the coordinate transformations given by (2.1) and (2.2) the coordinates of the field t_H have the following transformation law

$$t^{i'_1\dots i'_p}{}_{j'_1\dots j'_q} = \frac{\partial x^{i'_1}}{\partial x^{i_1}}\dots\frac{\partial x^{i'_p}}{\partial x^{i_p}}\frac{\partial x^{j_1}}{\partial x^{j'_1}}\dots\frac{\partial s^{j_q}}{\partial x^{j'_q}}t^{i_1\dots i_p}{}_{j_1\dots j_q}.$$

b) The α-vertical tensor field ${}_{(\alpha)}t_V$ has the local representation

$${}_{(\alpha)}t_V = {}_{(\alpha)}t^{\ \alpha\dots\alpha\ \ a_1\dots a_r}_{\ b_1\dots b_s\ \ \alpha\dots\alpha}\,\partial^{\ b_1}_{\ \alpha}\otimes\dots\partial^{\ b_s}_{\ \alpha}\otimes\delta p^{\ \alpha}_{\ a_1}\otimes\dots\otimes\delta p^{\ \alpha}_{\ a_r}$$

(not summing over α); it is defined on

$${}_{(\alpha)}\underbrace{T_V(E^*)\otimes\dots\otimes{}_{(\alpha)}T_V(E^*)}_{s\text{ times}}\otimes{}_{(\alpha)}\underbrace{T^*_V(E^*)\otimes\dots\otimes{}_{(\alpha)}T^*_V(E^*)}_{r\text{ times}}$$

By changing of coordinats of kind (2.1) and (2.2) the coordinates of the field ${}_{(\alpha)}t_V$, given above, have the following transformation law

$${}_{(\alpha)}t^{\ a'_1\dots a'_r\alpha\ \dots\ \ \alpha}_{\ \alpha\dots\alpha\ b'_1\dots b'_s} = {}_{(\alpha)}t^{\ a_1\dots a_r\alpha\ \dots\alpha}_{\ \alpha\dots\alpha\ b_1\dots b_s}\overset{(\alpha)}{M}{}^{a'_1}_{a_1}\dots\overset{(\alpha)}{M}{}^{a'_r}_{a_r}\overset{(\alpha)}{M}{}^{b_1}_{b'_1}\dots\overset{(\alpha)}{M}{}^{b_s}_{b'_s}.$$

c) A vertical tensor field t_V on $T_V(E^*)\otimes T^*_V(E^*)$ has the form

$$t_V = t^{\ \alpha b_1}_{\ a_1\beta}\,\partial^{\ a_1}_{\ \alpha}\otimes\delta p^{\ \beta}_{\ b_1}\qquad\text{(summing over }\alpha\text{ and }\beta).$$

The coordinate transformation of the tensor t_V is given by

$$t^{\ \alpha b'_1}_{\ a'_1\beta} = t^{\ \alpha b_1}_{\ a_1\beta}\overset{(\alpha)}{M}{}^{a_1}_{a'_1}\overset{(\beta)}{M}{}^{b'_1}_{b_1}.$$

d) A tensor field t on

$$\underbrace{T_H(E^*)\otimes\times\dots\otimes T_H(E^*)}_{p\text{ times}}\otimes\underbrace{T^*_H(E^*)\otimes\dots\otimes T^*_H(E^*)}_{q\text{ times}}\otimes$$
$$\underbrace{T_V(E^*)\otimes\dots\otimes T_V(E^*)}_{s\text{ times}}\times\underbrace{T^*_V(E^*)\otimes\dots\otimes T^*_V(E^*)}_{r\text{ times}},$$

is given by

$$t = t^{i_1\dots i_p}{}_{j_1\dots j_q}{}^{\beta_1\dots\beta_s a_1\dots a_r}_{b_1\dots b_s\alpha_1\dots\alpha_r}(x,p)\delta_{i_1}\otimes\dots\otimes\delta_{ip}$$
$$\otimes dx^{j_1}\otimes\dots\otimes dx^{j_q}\otimes\partial^{\ b_1}_{\ \beta_1}\otimes\dots\otimes\partial^{\ b_s}_{\ \beta_s}\otimes\delta p^{\ \alpha_1}_{\ a_1}\otimes\dots\otimes\delta p^{\ \alpha_r}_{\ a_r}.$$

The summation is performed over all indices. The coordinate transformation of the above tensor is given by

$$t^{i'_1\dots i'_p}{}_{j'_1\dots j'_q}{}^{\beta_1\dots\beta_s a'_1\dots a'_r}_{b'_1\dots b'_s\,\alpha_1\dots\alpha_r} = t^{i_1\dots i_p}{}_{j_1\dots j_q}{}^{\beta_1\dots\beta_s a_1\dots a_r}_{b_1\dots b_s\,\alpha_1\dots\alpha_r}$$
$$\frac{\partial x^{i'_1}}{\partial x^{i_1}}\cdots\frac{\partial x^{i'_p}}{\partial x^{i_p}}\frac{\partial x^{j_1}}{\partial x^{j'_1}}\cdots\frac{\partial x^{j_q}}{\partial x^{j'_q}}\overset{(\beta_1)}{M}{}^{b_1}_{b'_1}\dots\overset{(\beta_s)}{M}{}^{b_s}_{b'_s}\overset{(\alpha_1)}{M}{}^{a'_1}_{a_1}\dots\overset{(\alpha_r)}{M}{}^{a'_r}_{a_r}.$$

The order of spaces $T_H(E^*)$, $T^*_H(E^*)$, ${}_{(\alpha)}T^*_V(E^*)$ and $T^*_V(E^*)$ can be taken arbitrarily. It has the influence on the order of the indices of the tensor t, which is defined on their tensor product.

4. The Metric Tensor. In the space $T^*(E^*)\otimes T^*(E^*)$, the metric tensor G, with respect to the basis $=\{(dx^i),(\delta p^1_a),\dots,(\delta p^K_a)\}$, has the form

(4.1)
$$G=[(dx^i),(\delta p^1_a),\dots,(\delta p^K_a)]\begin{bmatrix}[g_{ij}] & [g_{i1}{}^{b}] & \dots & [g_{iK}{}^{b}]\\ [g^{a}{}_{1j}] & [g^{ab}_{11}] & \dots & [g^{ab}_{1K}]\\ \vdots & \vdots & & \vdots\\ [g^{a}{}_{Kj}] & [g^{ab}_{K1}] & \dots & [g^{ab}_{KK}]\end{bmatrix}\otimes\begin{bmatrix}dx^j\\ \delta p^{\ 1}_{b}\\ \vdots\\ \delta p^{\ K}_{b}\end{bmatrix}=$$
$$g_{ij}dx^i\otimes dx^j+g^{\ a}_{\alpha j}\delta p^{\ \alpha}_{a}\otimes dx^j+g^{\ b}_{i\beta}dx^i\otimes\delta p^{\ \beta}_{b}+g^{ab}_{\alpha\beta}\delta p^{\ \alpha}_{a}\otimes\delta p^{\ \beta}_{b}.$$

The matrices $[g_{ij}]$, $\left[g^{\ b}_{i\beta}\right]$, $\left[g^{\ a}_{\alpha j}\right]$ and $\left[g^{ab}_{\alpha\beta}\right]$ have the formats $n\times n$, $n\times m$, $m\times n$ and $m\times m$. As G is a tensor, its coordinates in the new coordinate system (x',p') transform in the following manner:

$$\text{(4.2)}\qquad g_{i'j'}=g_{ij}\frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^j}{\partial x^{j'}},\qquad g^{\ a'}_{\alpha j'}=g^{\ a}_{\alpha j}\overset{(\alpha)}{M}{}^{a'}_{\ a}\frac{\partial x^j}{\partial x^{j'}},$$
$$g^{\ b'}_{i'\beta}=g^{\ b}_{i\beta}\frac{\partial x^i}{\partial x^{i'}}\overset{(\beta)}{M}{}^{b'}_{\ b},\qquad g^{a'b'}_{\alpha\beta}=g^{ab}_{\alpha\beta}\overset{(\alpha)}{M}{}^{a'}_{\ a}\overset{(\beta)}{M}{}^{b'}_{\ b}$$

We shall suppose that G is a symmetric, positive definite tensor field of rank $n+mK$. From the symmetry, it follows that $g_{ij}=g_{ji}$, $g^{\ b}_{i\beta}=g^{\ b}_{\beta i}$, $g^{ab}_{\alpha\beta}=g^{ba}_{\beta\alpha}$. The "covariant" coordinates of the field $X=X^i\delta_i+X^{\ \alpha}_{a}\partial^{\ a}_{\alpha}$ are given by

$$X_i=g_{ij}X^j+g^{\ a}_{i\alpha}X^{\ \alpha}_{a},\qquad X^{\ a}_{\alpha}=g^{\ a}_{\alpha i}X^i+g^{ab}_{\alpha\beta}X^{\ \beta}_{b}.$$

The inverse matrix of G (appearing in (4.1)) is given by

$$\begin{bmatrix}[g^{jk}] & [g^{j1}{}_{c}] & \dots & [g^{jK}{}_{c}]\\ [g^{1k}{}_{b}] & [g^{11}_{bc}] & \dots & [g^{1K}_{bc}]\\ \vdots & \vdots & & \vdots\\ [g^{Kk}{}_{b}] & [g^{K1}_{bc}] & \dots & [g^{KK}_{bc}]\end{bmatrix}$$

The matrices $\left[g^{jk}\right]$, $\left[g^{\,j\gamma}_{\;\;c}\right]$, $\left[g^{\,\gamma k}_{\;c}\right]$ and $\left[g^{\,\beta\gamma}_{\;bc}\right]$ have formats $n\times n$, $n\times m$, $m\times n$ and $m\times m$. Now we have:

$$g_{ij}g^{jk}+g_{\;i\beta}^{\;b}g^{\,\beta k}_{\;b}=\delta_i^k,\qquad g^{\;a}_{\;\alpha j}g^{\,j\gamma}_{\;\;c}+g^{\;ab}_{\;\alpha\beta}g^{\,\beta\gamma}_{\;bc}=\delta^a_c\delta^\gamma_\alpha,$$
$$g_{ij}g^{\,j\gamma}_{\;\;c}+g_{\;i\beta}^{\;b}g^{\,\beta\gamma}_{\;bc}=0,\qquad g^{\;a}_{\;\alpha j}g^{jk}+g^{\;ab}_{\;\alpha\beta}g^{\,\beta k}_{\;b}=0.$$

The contravariant coordinates $w=w_idx^i+w^a_\alpha\delta p^\alpha_a$ are given by

$$w^i=g^{ij}w_j+g^{\,i\alpha}_{\;\;a}w^a_\alpha,\qquad w^\alpha_a=g^{\,j\alpha}_{\;\;a}w_j+g^{\,\alpha\beta}_{\;ab}w^b_\beta.$$

Using (2.10), (2.16) and (4.2) the validity of the following transformation laws can be shown:

$$X_{i'}=X_i\frac{\partial x^i}{\partial x^{i'}},\quad X^{a'}_\alpha=X^a_\alpha\overset{(\alpha)}{M}{}^{a'}_a,\quad w^{i'}=w^i\frac{\partial x^{i'}}{\partial x^i},\quad w^\alpha_{a'}=w^\alpha_a\overset{(\alpha)}{M}{}^a_{a'}.$$

Definition 3.1. The differentiable maniflod E^* (in which the coordinate transformations of type (2.1) are allowed) supplied with arbitrary nonlinear connection N (which satisfies (2.8)) and the metric tensor G (given by (4.1) is called a K-Hamilton space and is denoted by (E^*,N,G).

It is a generalization of the K-Hamilton space defined in [**9**], [**10**] etc., because here the metric tensor G is not necessarily obtained from the K-Hamiltonian $H(x,p)$. If the K-Hamilton function $H(x,p)$ is given in the space (E^*), then the metric tensor G can be defined in the following way:

$$g_{ij}(x,p)=g_{ij}(x),\qquad g_{\;i\beta}^{\;b}=0,\qquad g^{\;a}_{\;\alpha j}=0,$$
$$g^{\;ab}_{\;\alpha\beta}=\frac{1}{2}\partial^{\;a}_{\;\alpha}\partial^b_\beta H^2(x,p),\qquad \text{for every } \alpha,\beta=\overline{1,K},$$

where $g_{ij}(x)$ is some metric tensor defined on M and M is the π^* projection of E^*:

$$\pi^*(E^*)=M,\quad \pi^*((x^i),(p^1_a),\dots,(p^K_a))=(x^i).$$

We can not define $g_{\;i\beta}^{\;b}(x,p)=\frac{1}{2}\delta_i\partial^b_\beta H^2(x,p)$, $g_{ij}(x,p)=\frac{1}{2}\delta_i\delta_jH^2(x,p)$, because the above quantities do not transform as tensors. Using the metric tensor G determined by (4.1), we define the scalar product (X,Y) of fields $X,Y\in T(E^*)$ by

$$(X,Y)=g_{ij}X^iY^j+g_{\;i\beta}^{\;b}X^iY^\beta_b+g^{\;a}_{\;\alpha j}X^\alpha_aY^j+g^{\;ab}_{\;\alpha\beta}X^\alpha_aY^\beta_b. \tag{4.3}$$

The length of X, $|X|$ is defined by $|X|^2=(X,X)$ and $\cos\theta$, where θ is the angle between X and Y by

$$\cos\theta=(X,Y)/(|X|\cdot|Y|). \tag{4.4}$$

When $cos\theta=0$, we say that X and Y are mutually orthogonal. For the horizontal field X_H we have $X_H=X^i\partial_i$, $|X_H|^2=g_{ij}X^iX^j$ and for the vertical vector X_V we have $X_V = X^{\alpha}_{a}\partial^{a}_{\alpha}$, $|X_V|^2 = g^{\ ab}_{\alpha\beta} X^{\alpha}_{a} X^{\beta}_{b}$. For the field ${}_{(\alpha)}X_V \in {}_{(\alpha)}T_V(E^*)$ we have

$$ {}_{(\alpha)}X_V = X^{\alpha}_{a}\partial^{a}_{\alpha} \qquad |{}_{(\alpha)}X_V|^2 = g^{\ ab}_{\alpha\alpha} X^{\alpha}_{a} X^{\alpha}_{b} \quad \text{(not summing over } \alpha\text{)}. $$

Theorem 4.1. *The subspaces $T_H(E^*)$, ${}_{(1)}T_V(E^*)$, $\dots$,${}_{(K)}T_V(E^*)$ are mutually orthogonal with respect to the metric tensor G, if and only if $[g^{b}_{ij}] = 0$, $\left[g^{\ a}_{\alpha j}\right] = 0$, $\left[g^{\ ab}_{\alpha\beta}\right] = 0$ for every $\alpha, \beta = \overline{1, K}$, $\alpha \neq \beta$.*

The proof follows from (4.3) and (4.4).

5. Generalized Miron's d-connection in $T(E^*)$.

The distinguished connection ∇, or the d-connection, in the K-Hamilton space in [**15**], [**9**], [**16**] and others is defined as a function $\nabla : (X, Y) \to \nabla_X Y$ $X, Y, \nabla_X Y \in T(E^*)$ for which, beside the usual conditions for the linear connection, the following restrictions are valied:

$$ \nabla_X Y_H \in T_H(E^*), \qquad \nabla_X Y_V \in T_V(E^*) \text{for all } X \in T(E^*), $$
$$ \text{all } Y_H \in T_H(E^*), \text{ and all } Y_V \in T_V(E^*). $$

For the generalized linear d-connection in $T(E^*)$ in he K-Hamilton space the above restrictions need not be satisfied.

Definition 5.1. The generalized linear Miron's d-connection in $T(E^*)$ is defined by

$$ \begin{aligned} \nabla_{\delta_i}\delta_j &= F_j{}^k{}_i\delta_k + F_j{}^{\gamma}_{c}{}_i\partial^{c}_{\gamma}, \\ \nabla_{\delta_i}\partial^{a}_{\alpha} &= F^{a}_{\alpha}{}^{k}{}_{i}\delta_k + F^{a}_{\alpha}{}^{\gamma}_{c}{}_{i}\partial^{c}_{\gamma}, \\ \nabla_{\partial^{a}_{\alpha}}\delta_j &= C_j{}^{ka}_{\ \alpha}\delta_k + C_{\ jc\alpha}^{\ \gamma a}\partial^{c}_{\gamma}, \\ \nabla_{\partial^{a}_{\alpha}}\partial^{b}_{\beta} &= C^{\ bka}_{\ \beta\ \alpha}\delta_k + C^{\ b\gamma a}_{\ \beta c \alpha}\partial^{c}_{\gamma}. \end{aligned} \tag{5.1} $$

Proposition 5.1. *If $X, Y \in T(E^*)$, where X is given by (2.9), and $Y = Y^j\delta_j + Y^{\beta}_{b}\partial^{b}_{\beta}$, then*

$$ \nabla_X Y = \left(Y^k_{\ |\,i}X^i + Y^k|^{a}_{\alpha}X^{\alpha}_{a}\right)\delta_k + \left(Y^{\gamma}_{c|i}X^i + Y^{\gamma}_{c}|^{a}_{\alpha}X^{\alpha}_{a}\right)\partial^{c}_{\gamma}, \tag{5.2} $$

where,

$$ \begin{aligned} Y^k_{\ |\,i} &= \delta_i Y^k + F_j{}^k{}_i Y^j + F^{\ bk}_{\ \beta\ i} Y^{\beta}_{b}, \\ Y^k|^{a}_{\alpha} &= \partial^{a}_{\alpha}Y^k + C_{\ j\ \alpha}^{\ ka} Y^j + C^{\ bka}_{\ \beta\ \alpha} Y^{\beta}_{b}, \\ Y^{\ \gamma}_{\ c|i} &= \delta_i Y^{\gamma}_{c} + F^{\ \gamma}_{\ jci} Y^j + F^{\ b\gamma}_{\ \beta c i} Y^{\beta}_{b}, \\ Y^{\gamma}_{c}|^{a}_{\alpha} &= \partial^{a}_{\alpha}Y^{\gamma}_{c} + C^{\ \gamma a}_{\ jc\alpha} Y^j + C^{\ b\gamma a}_{\ \beta c\alpha} Y^{\beta}_{b}. \end{aligned} \tag{5.3} $$

Proof. (5.2) and (5.3) are proved by using the linearity of the connection ∇ and (5.1).

PROPOSITION 5.2. *If* $((x^i),(p^\alpha_a))$ *and* $((x^{i'}),(p^\alpha_{a'}))$ *are two coordinate systems connected by* (2.1) *and* (2.2), *then*

$$\nabla_{X'}Y' = \nabla_X Y \tag{5.4}$$

iff $Y^k_{|i}$, $Y^k|^a_\alpha$, $Y^\gamma_{c\,|i}$ *and* $Y^\gamma_c|^a_\alpha$ *transform as tensors, i.e. if*

$$Y^{k'}_{|i'} = Y^k{}_{|i}(\partial_{i'}x^i)(\partial_k x^{k'}), \qquad Y^{k'}|^\alpha_{a'} = Y^k|^\alpha_a(\partial_k x^{k'})\overset{(\alpha)}{M}{}^a_{a'},$$
$$Y^\gamma_{c'\,|i'} = Y^\gamma_{c\,|i}\overset{(\gamma)}{M}{}^c_{c'}(\partial_{i'}x^i), \qquad Y^\gamma_{c'}|^{a'}_\alpha = Y^\gamma_c|^a_\alpha\overset{(\gamma)}{M}{}^c_{c'}\overset{(\alpha)}{M}{}^{a'}_a,$$

or, equivalently, iff the connection coefficients have the following transformation law

(5.5) (a) $F_j{}^k{}_i = F_{j'}{}^{k'}{}_{i'}(\partial_j x^{j'})(\partial_{k'}x^k)(\partial_i x^{i'}) + (\partial_i\partial_j x^{k'})(\partial_{k'}x^k)$,

(b) $F_j{}^\gamma_c{}_i = F_{j'}{}^\gamma_{c'}{}_{i'}(\partial_j x^{j'})\overset{(\gamma)}{M}{}^{c'}_c(\partial_i x^{i'})$,

(c) $F^b_\beta{}^k{}_i = F^{b'}_\beta{}^{k'}{}_{i'}\overset{(\beta)}{M}{}^b_{b'}(\partial_{k'}x^k)(\partial_i x^{i'})$,

(d) $F^b_\beta{}^\gamma_c{}_i Y^\beta_b = F^{b'}_\beta{}^\gamma_{c'}{}_{i'}\overset{(\beta)}{M}{}^b_{b'}\overset{(\gamma)}{M}{}^{c'}_c(\partial_i x^{i'})Y^\beta_b + (\partial_i\overset{(\gamma)}{M}{}^d_{c'})\overset{(\gamma)}{M}{}^{c'}_c Y^\gamma_d$,

(e) $C_j{}^k{}^a_\alpha = C_{j'}{}^{k'}{}^{a'}_\alpha(\partial_j x^{j'})(\partial_{k'}x^k)\overset{(\alpha)}{M}{}^a_{a'}$,

(f) $C_j{}^\gamma_c{}^a_\alpha = C_{j'}{}^\gamma_{c'}{}^{a'}_\alpha(\partial_j x^{j'})\overset{(\gamma)}{M}{}^{c'}_c\overset{(\alpha)}{M}{}^a_{a'}$,

(g) $C^b_\beta{}^k{}^a_\alpha = C^{b'}_\beta{}^{k'}{}^{a'}_\alpha\overset{(\beta)}{M}{}^b_{b'}(\partial_{k'}x^k)\overset{(\alpha)}{M}{}^a_{a'}$,

(h) $C^b_\beta{}^\gamma_c{}^a_\alpha = C^{b'}_\beta{}^\gamma_{c'}{}^{a'}_\alpha\overset{(\beta)}{M}{}^b_{b'}\overset{(\gamma)}{M}{}^{c'}_c\overset{(\alpha)}{M}{}^a_{a'}$.

As Y^β_b *and* Y^γ_d *are the coordinates of the arbitrary vector* Y, *from* (5.5d) *it follows that:*

$$F^{a'}_{(\alpha)}{}^\alpha_{d'}{}_{i'} = F^a_{(\alpha)}{}^\alpha_d{}_i\overset{(\alpha)}{M}{}^{a'}_a\overset{(\alpha)}{M}{}^d_{d'}(\partial_{i'}x^i) + \left(\partial_{i'}\overset{(\alpha)}{M}{}^{a'}_d\right)\overset{(\alpha)}{M}{}^d_{d'} \tag{5.5.d'}$$

$$F^{a'}_\alpha{}^\gamma_{d'}{}_{i'} = F^a_\alpha{}^\gamma_d{}_i\overset{(\alpha)}{M}{}^{a'}_a\overset{(\gamma)}{M}{}^d_{d'}(\partial_i x^i), \qquad \alpha \neq \gamma. \tag{5.5.d''}$$

Proof. The proof is obtained by a direct calculation, using (5.2)–(5.4), (2.7) and (2.10).

The formulae (5.5) may be obtained by using the transformation law of the basic vectors and the linearity of the connection ∇.

If one nonlinear connection $N^{\alpha}_{a\,i}$ is given, we have an adapted basis and using this basis, we define the linear connection ∇. Another nonlinear connection $\bar{N}^{\gamma}_{c\,i}$ may be obtained from (5.5d). If we take $Y^{\beta}_{p} = p^{\beta}_{p}$, and introduce the notation $F^{b\,\gamma}_{\beta c\,i} p^{\beta}_{b} = \bar{N}^{\gamma}_{c\,i}$, then (5.5d) becomes

$$\bar{N}^{\gamma}_{c\,i} = \bar{N}^{\gamma}_{c'\,i'} \overset{(\gamma)}{M}{}^{c'}_{c} (\partial_i x^{i'}) + \left(\partial_i \overset{(\gamma)}{M}{}^{d}_{c'}\right) \overset{(\gamma)}{M}{}^{c'}_{c} p^{\gamma}_{d},$$

which, compared to (2.8), shows, that $\bar{N}^{\gamma}_{c\,i}$ transform as a nonlinear connection.

The torsion tensor $T(X,Y)$ is, as usual, given by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]. \tag{5.6}$$

THEOREM 5.1. *The torsion tensor $T(X,Y)$ for the connection ∇ has the form*

$$T(X,Y) = T_{HHH} + T_{HHV} + T_{HVH} + T_{HVV} + T_{VHH} + T_{VHV} + T_{VVH} + T_{VVV},$$

where, for instance, $T_{VVH} = [T(X_V, Y_V)]_H$, $T_{VHH} = [T(X_V, Y_H)]_H$. The components of the torsion tensor are

(5.7)
$$\begin{aligned}
&(a) \quad T_{HHH} = (F_j{}^k{}_i - F_i{}^k{}_j) X^i Y^j \delta_k = T_j{}^k{}_i X^i Y^j \delta_k\\
&(b) \quad T_{HHV} = (F_{j\,c\,i}^{\;\gamma} - F_{i\,c\,j}^{\;\gamma} + \delta_i N^{\gamma}_{c\;\,j} - \delta_j N^{\gamma}_{c\;\,i}) X^i Y^j \partial^c_{\gamma} = T_{j\,c\,i}^{\;\gamma} X^i Y^j \partial^c_{\gamma}\\
&(c) \quad T_{HVH} = (F^{b\,k}_{\beta\;\;i} - C_i{}^{kb}_{\;\;\beta}) X^i Y^{\beta}_b \delta_k = T^{b\,k}_{\beta\;\;i} X^i Y^{\beta}_b \delta_k\\
&(d) \quad T_{HVV} = (F^{b\,\gamma}_{\beta c\,i} - C_{i\,c\,\beta}^{\;\gamma b} - \partial^b_{\beta} N^{\gamma}_{c\,i}) X^i Y^{\beta}_b \partial^c_{\gamma} = T^{b\,\gamma}_{\beta c\,i} X^i Y^{\beta}_b \partial^c_{\gamma}\\
&(e) \quad T_{VHH} = (C_j{}^{ka}_{\;\;\alpha} - F^{a\,k}_{\alpha\;\;j}) X^{\alpha}_a Y^j \delta_k = T_j{}^{ka}_{\;\;\alpha} X^{\alpha}_a Y^j \delta_k\\
&(f) \quad T_{VHV} = (C_{j\,c\,\alpha}^{\;\gamma a} - F^{a\,\gamma}_{\alpha c\,j} + \partial^a_{\alpha} N^{\gamma}_{c\,j}) X^{\alpha}_a Y^j \partial^c_{\gamma} = T_{j\,c\,\alpha}^{\;\gamma a} X^{\alpha}_a Y^j \partial^c_{\gamma}\\
&(g) \quad T_{VVH} = (C^{b\,ka}_{\beta\;\;\alpha} - C^{a\,kb}_{\alpha\;\;\beta}) X^{\alpha}_a Y^{\beta}_b \delta_k = T^{b\,ka}_{\beta\;\;\alpha} X^{\alpha}_a Y^{\beta}_b \delta_k\\
&(h) \quad T_{VVV} = (C^{b\,\gamma a}_{\beta c\,\alpha} - C^{a\,\gamma b}_{\alpha c\,\beta}) X^{\alpha}_a Y^{\beta}_b \partial^c_{\gamma} = T^{b\,\gamma a}_{\beta c\,\alpha} X^{\alpha}_a Y^{\beta}_b \partial^c_{\gamma}.
\end{aligned}$$

Proof. The proof is obtained by a direct calculation by using (5.2), (5.3) and (5.6).

*Definition*5.2. The K-Hamiltonian space (E^*, N, G), supplied with the linear connection ∇ defined by (5.1) and an arbitrary torsion tensor T, given by (5.7) is denoted by (E^*, N, G, ∇, T).

6. Generalized Miron's d-connection in $T^*(E^*)$ Using the duality of the bases B and B^* of $T(E^*)$, and $T^*(E^*)$, respectively expressed by (2.17) one can obtain

THEOREM 6.1. *The connection on $T^*(E^*)$ acts in the following way:*

$$\begin{aligned}(6.1)\quad &\nabla_{\delta_i} dx^j = -F_k{}^j{}_i dx^k - F_{\gamma}^{c\,j}{}_i \delta p_c^{\gamma}, &\nabla_{\delta_i} \delta p_b^{\beta} = -F_{k\,b}^{\,\beta}{}_i dx^k - F_{\gamma\,b}^{c\,\beta}{}_i \delta p_c^{\gamma},\\ &\nabla_{\partial_\alpha^a} dx^j = -C_k{}^j{}_\alpha^a dx^k - C_{\gamma}^{c\,j}{}_\alpha^a \delta p_c^{\gamma}, &\nabla_{\partial_\alpha^a} \delta p_b^{\beta} = -C_{k\,b}^{\,\beta}{}_\alpha^a dx^k - C_{\gamma\,b}^{c\,\beta}{}_\alpha^a \delta p_c^{\gamma}.\end{aligned}$$

Using the properties of the linear connection ∇ and relations (4.1) and (6.1), we see that the following relations are satisfied:

$$\begin{aligned}\nabla_X G = &\left(g_{ij|k} X^k + g_{ij}|_\gamma^c X_c^\gamma\right) dx^i \otimes dx^j + \left(g_{j\beta|k}^{\,b} X^k + g_{i\beta}^{\,b}|_\gamma^c X_c^\gamma\right) dx^i \otimes \delta p_b^\beta +\\ &\left(g_{\alpha j|k}^{a} X^k + g_{\alpha j}^{a}|_\gamma^c X_c^\gamma\right) \delta p_a^\alpha \otimes dx^j + \left(g_{\alpha\beta|k}^{a\,b} X^k + g_{\alpha\beta}^{a\,b}|_\gamma^c X_c^\gamma\right) \delta p_a^\alpha \otimes \delta p_b^\beta,\end{aligned}$$

where,

$$\begin{aligned}(6.2)\quad g_{ij|k} &= \delta_k g_{ij} - g_{hj} F_i{}^h{}_k - g_{\delta j}^{d} F_{i\,d}^{\,\delta}{}_k - g_{ih} F_j{}^h{}_k - g_{i\delta}^{\,d} F_{j\,d}^{\,\delta}{}_k,\\ g_{ij}|_\gamma^c &= \partial_\gamma^c g_{ij} - g_{hj} C_i{}^h{}_\gamma^c - g_{\delta j}^{d} C_{i\,d}^{\,\delta}{}_\gamma^c - g_{ih} C_j{}^h{}_\gamma^c - g_{i\delta}^{\,d} C_{j\,d}^{\,\delta}{}_\gamma^c,\\ g_{i\beta|k}^{\,b} &= \delta_k g_{i\beta}^{\,b} - g_{h\beta}^{\,b} F_i{}^h{}_k - g_{\delta\beta}^{db} F_{i\,d}^{\,\delta}{}_k - g_{ih} F_{\beta}^{b\,h}{}_k - g_{i\delta}^{\,d} F_{\beta\,d}^{b\,\delta}{}_k,\\ g_{i\beta}^{\,b}|_\gamma^c &= \partial_c^\gamma g_{i\beta}^{\,b} - g_{h\beta}^{\,b} C_i{}^h{}_\gamma^c - g_{\delta\beta}^{db} C_{i\,d}^{\,\delta}{}_\gamma^c - g_{ih} C_{\beta}^{b\,h}{}_\gamma^c - g_{i\delta}^{\,d} C_{\beta\,d}^{b\,\delta}{}_\gamma^c,\\ g_{\alpha\beta|k}^{a\,b} &= \delta_k g_{\alpha\beta}^{a\,b} - g_{h\beta}^{\,b} F_{\alpha}^{a\,h}{}_k - g_{\delta\beta}^{db} F_{\alpha\,d}^{a\,\delta}{}_k - g_{\alpha h}^{a} F_{\beta}^{b\,h}{}_k - g_{\alpha\delta}^{a\,d} F_{\beta\,d}^{b\,\delta}{}_k,\\ g_{\alpha\beta}^{a\,b}|_\gamma^c &= \partial_\gamma^c g_{\alpha\beta}^{a\,b} - g_{h\beta}^{\,b} C_{\alpha}^{a\,h}{}_\gamma^c - g_{\delta\beta}^{db} C_{\alpha\,d}^{a\,\delta}{}_\gamma^c - g_{\alpha h}^{a} C_{\beta}^{b\,h}{}_\gamma^c - g_{\alpha\delta}^{a\,d} C_{\beta\,d}^{b\,\delta}{}_\gamma^c.\end{aligned}$$

Definition 6.1. The space (E^*, N, G, ∇, T) is called the recurrent K-Hamilton space and denoted by $(E^*, N, G, \nabla, T, \lambda)$, if tensor fields $\lambda_k(x,p)$ and $\lambda_\gamma^c(x,p)$ exist, so that

$$\begin{aligned}(6.3)\quad &g_{ij|k} = \lambda_k g_{ij}, \quad g_{ij}|_\gamma^c = \lambda_\gamma^c g_{ij}, \quad g_{i\beta|k}^{\,b} = \lambda_k g_{i\beta}^{\,b}, \quad g_{i\beta}^{\,b}|_\gamma^c = \lambda_\gamma^c g_{i\beta}^{\,b},\\ &g_{\alpha j|k}^{a} = \lambda_k g_{\alpha j}^{a}, \quad g_{\alpha j}^{a}|_\gamma^c = \lambda_\gamma^c g_{\alpha j}^{a}, \quad g_{\alpha\beta|k}^{a\,b} = \lambda_k g_{\alpha\beta}^{a\,b}, \quad g_{\alpha\beta}^{a\,b}|_\gamma^c = \lambda_\gamma^c g_{\alpha\beta}^{a\,b}.\end{aligned}$$

Definition 6.2. The recurrent K-Hamilton space will be called the metric K-Hamilton space and denoted by $(E^*, N, G, \nabla, T, 0)$, if

$$\begin{aligned}&g_{ij|k} = 0, \qquad g_{ij}|_\gamma^c = 0, \qquad g_{i\beta|k}^{\,b} = 0, \qquad g_{i\beta}^{\,b}|_\gamma^c = 0,\\ &g_{\alpha j|k}^{a} = 0, \qquad g_{\alpha j}^{a}|_\gamma^c = 0, \qquad g_{\alpha\beta|k}^{a\,b} = 0, \qquad g_{\alpha\beta}^{a\,b}|_\gamma^c = 0.\end{aligned}$$

THEOREM 6.2. *In the recurrent K-Hamilton space, the coordinates of the inverse metric tensor satisfy the following relations*

$$\begin{aligned}(6.4)\quad &g^{lk}{}_{|h} = -\lambda_h g^{lk}, \qquad g^{l\beta}{}_{b\,|h} = -\lambda_h g^{l\beta}{}_{b}, \qquad g_{db\,|h}^{\delta\beta} = -\lambda_h g_{db}^{\delta\beta},\\ &g^{lk}|_\gamma^c = -\lambda_\gamma^c g^{lk}, \qquad g^{l\beta}{}_{b}|_\gamma^c = -\lambda_\gamma^c g^{l\beta}{}_{b}, \qquad g_{db}^{\delta\beta}|_\gamma^c = -\lambda_\gamma^c g_{db}^{\delta\beta}.\end{aligned}$$

Proof. It follows, from (4.4), that

$$
\text{(6.5)}\qquad
\begin{aligned}
&(a) & \lambda_h \delta_i^k + g_{ij} g^{jk}{}_{|h} + g_{i\beta}^{b} g_{b}^{\beta k}{}_{|h} &= 0\\
&(b) & \lambda_h \delta_c^a \delta_a^\gamma + g_{\alpha j}^{a} g^{j\gamma}_{\ c|h} + g_{\alpha\beta}^{ab} g_{bc|h}^{\beta\gamma} &= 0\\
&(c) & g_{ij} g^{j\gamma}_{\ c|h} + g_{i\beta}^{b} g_{bc|h}^{\beta\gamma} &= 0\\
&(d) & g_{\alpha j}^{a} g^{jk}{}_{|h} + g_{\alpha\beta}^{ab} g_{b}^{\beta k}{}_{|h} &= 0
\end{aligned}
$$

The sum of (6.5a), multiplied by g^{il}, and (6.5d) multiplied by $g^{la}_{\ \alpha}$, yields $g^{lk}{}_{|h} = -\lambda_h g^{lk}$. The other relations in (6.4) may be obtained in a similar way.

The raising and lowering of the middle index of the connection coefficients are given by the following formulae

$$
\text{(6.6)}\qquad
\begin{bmatrix} F_{ijk} \\ F_{i\alpha k}^{\ a} \end{bmatrix} = \begin{bmatrix} g_{jh} & g_{j\delta}^{\ d} \\ g_{\alpha h}^{a} & g_{\alpha\delta}^{ad} \end{bmatrix} \begin{bmatrix} F_{i\ k}^{\ h} \\ F_{i\ dk}^{\ \delta} \end{bmatrix} \quad\Leftrightarrow\quad \begin{bmatrix} F_{ik}^{\ h} \\ F_{idk}^{\ \delta} \end{bmatrix} = \begin{bmatrix} g^{hj} & g^{h\alpha}_{\ \ a} \\ g^{\delta j}_{d} & g^{\delta\alpha}_{da} \end{bmatrix} \begin{bmatrix} F_{ijk} \\ F_{i\alpha k}^{\ a} \end{bmatrix}
$$

$$
\begin{bmatrix} F^{a}_{\alpha ik} \\ F^{ab}_{\alpha\beta k} \end{bmatrix} = \begin{bmatrix} g_{ih} & g_{i\delta}^{d} \\ g_{\beta h}^{b} & g_{\beta\delta}^{bd} \end{bmatrix} \begin{bmatrix} F^{ah}_{\alpha\ k} \\ F^{a\delta}_{\alpha dk} \end{bmatrix} \quad\Leftrightarrow\quad \begin{bmatrix} F^{ah}_{\alpha\ k} \\ F^{a\delta}_{\alpha dk} \end{bmatrix} = \begin{bmatrix} g^{hi} & g^{h\beta}_{\ \ b} \\ g^{\delta i}_{d} & g^{\delta\beta}_{db} \end{bmatrix} \begin{bmatrix} F^{a}_{\alpha ik} \\ F^{ab}_{\alpha\beta k} \end{bmatrix},
$$

$$
\begin{bmatrix} C_{ij\gamma}^{\ \ c} \\ C_{i\alpha\gamma}^{\ ac} \end{bmatrix} = \begin{bmatrix} g_{jh} & g_{j\delta}^{\ d} \\ g_{\alpha h}^{a} & g_{\alpha\delta}^{ad} \end{bmatrix} \begin{bmatrix} C_{i\ \gamma}^{\ hc} \\ C_{id\gamma}^{\ \delta c} \end{bmatrix} \quad\Leftrightarrow\quad \begin{bmatrix} C_{i\ \gamma}^{\ hc} \\ C_{id\gamma}^{\ \delta c} \end{bmatrix} = \begin{bmatrix} g^{hj} & g^{h\alpha}_{\ \ a} \\ g^{\delta j}_{d} & g^{\delta\alpha}_{da} \end{bmatrix} \begin{bmatrix} C_{ij\gamma}^{\ \ c} \\ C_{i\alpha\gamma}^{\ ac} \end{bmatrix},
$$

$$
\begin{bmatrix} C^{a\ c}_{\alpha i\gamma} \\ C^{abc}_{\alpha\beta\gamma} \end{bmatrix} = \begin{bmatrix} g_{ih} & g_{i\delta}^{d} \\ g_{\beta h}^{b} & g_{\beta\delta}^{bd} \end{bmatrix} \begin{bmatrix} C^{ahc}_{\alpha\ \gamma} \\ C^{a\delta c}_{\alpha d\gamma} \end{bmatrix} \quad\Leftrightarrow\quad \begin{bmatrix} C^{ahc}_{\alpha\ \gamma} \\ C^{a\delta c}_{\alpha d\gamma} \end{bmatrix} = \begin{bmatrix} g^{hi} & g^{h\beta}_{\ \ b} \\ g^{\delta i}_{d} & g^{\delta\beta}_{db} \end{bmatrix} \begin{bmatrix} C^{a\ c}_{\alpha i\gamma} \\ C^{abc}_{\alpha\beta\gamma} \end{bmatrix}
$$

By using (6.1) and the above notations, (6.2) may be written in the following form:

$$
\text{(6.7)}\qquad
\begin{aligned}
g_{ij|k} &= \delta_k g_{ij} - F_{ijk} - F_{jki}, & g_{ij}|^{c}_{\gamma} &= \partial^{c}_{\gamma} g_{ij} - C_{ij\gamma}^{\ \ c} - C_{ji\gamma}^{\ \ c},\\
g_{i\beta|k}^{b} &= \delta_k g_{i\beta}^{b} - F_{i\beta k}^{\ b} - F^{b}_{\beta ik}, & g_{i\beta}^{b}|^{c}_{\gamma} &= \partial^{c}_{\gamma} g_{i\beta}^{b} - C_{i\beta\gamma}^{\ bc} - C^{b\ c}_{\beta i\gamma},\\
g_{\alpha\beta|k}^{ab} &= \delta_k g_{\alpha\beta}^{ab} - F^{ab}_{\alpha\beta k} - F^{ba}_{\beta\alpha k}, & g_{\alpha\beta}^{ab}|^{c}_{\gamma} &= \partial^{c}_{\gamma} g_{\alpha\beta}^{ab} - C^{abc}_{\alpha\beta\gamma} - C^{bac}_{\beta\alpha\gamma}.
\end{aligned}
$$

7. The Connection Coefficients in Recurrent k-Hamilton Spaces.

THEOREM 7.1. *In a recurrent K-Hamilton space $(E^*, N, G, \nabla, T, \lambda)$, the connection coefficients are determined by*

$$
\text{(7.1)}\qquad
\begin{aligned}
(a)\quad 2F_{ijk} &= (\delta_k g_{ij} + \delta_i g_{jk} - \delta_j g_{ki}) - (\lambda_k g_{ij} + \lambda_i g_{jk} - \lambda_j g_{ki})\\
&\quad + (F^*_{kij} + F^*_{ikj} + F^*_{kji}),\\
(b)\quad F^*_{kij} &= F_{kij} - F_{jik} = g_{ih}(F_{k\ j}^{\ h} - F_{j\ k}^{\ h}) + g_{i\delta}^{\ d}(F_{k\ dj}^{\ \delta} - F_{j\ dk}^{\ \delta})\\
&= g_{ih} T_{k\ j}^{\ h} + g_{i\delta}^{\ d} T_{k\ dj}^{\ \delta}, \qquad\qquad \text{(see 5.7)}
\end{aligned}
$$

$$(7.2)\quad (a)\quad 2F_{i\,\alpha k}^{\ a} = (\delta_k g_{i\,\alpha}^{\ a} + \delta_i g_{\alpha k}^{a} - \partial_\alpha^a g_{ik}) - (\lambda_k g_{i\,\alpha}^{\ a} + \lambda_i g_{\alpha k}^{a} - \lambda_\alpha^a g_{ik}) + A$$

$$\begin{aligned} (b)\quad A &= (F_{i\,\alpha k}^{\ a} - F_{k\,\alpha i}^{\ a}) - (F_{\alpha ki}^{a} - C_{ik\,\alpha}^{\ \ a}) - (F_{\alpha ik}^{a} - C_{ki\,\alpha}^{\ \ a}) \\ &= g_{\alpha h}^{a} T_{i\ k}^{\ h} + g_{\alpha\delta}^{a\,d} T_{idk}^{\delta} - g_{ih} T_{\alpha}^{a\,hk} - g_{kh} T_{\alpha\ \,i}^{a\,h} \\ &\quad - g_{i\,\delta}^{\ d}(T_{\alpha dk}^{a\,\delta} + \partial_\alpha^a N_{dk}^{\delta}) - g_{k\,\delta}^{\ d}(T_{\alpha di}^{a\,\delta} + \partial_\alpha^a N_{di}^{\delta}) \\ &\quad + g_{\alpha\delta}^{a\,d}(\delta_i N_{dk}^{\delta} - \delta_k N_{di}^{\delta}), \end{aligned}$$

$$(7.3)\qquad 2F_{\alpha ik}^{a} = (\delta_k g_{\alpha i}^{a} + \partial_\alpha^a g_{ik} - \delta_i g_{k\,\alpha}^{\ a}) - (\lambda_k g_{\alpha i}^{a} + \lambda_\alpha^a g_{ik} - \lambda_i g_{k\,\alpha}^{\ a}) - A,$$

$$\begin{aligned} (7.4)\quad (a)\quad 2F_{\alpha\beta k}^{a\,b} &= (\delta_k g_{\alpha\beta}^{a\,b} + \partial_\alpha^a g_{\beta k}^{b} - \partial_\beta^b g_{k\,\alpha}^{\ a}) \\ &\quad - (\lambda_k g_{\alpha\beta}^{a\,b} + \lambda_\alpha^a g_{\beta k}^{b} - \lambda_\beta^b g_{k\,\alpha}^{\ a}) + B, \end{aligned}$$

$$\begin{aligned} (b)\quad B &= (F_{\alpha\beta k}^{a\,b} - C_{k\,\beta\alpha}^{\ b\,a}) - (F_{\beta\alpha k}^{b\,a} - C_{k\,\alpha\beta}^{\ a\,b}) + (C_{\alpha k\beta}^{a\ \,b} - C_{\beta k\alpha}^{b\ \,a}) \\ &= g_{\beta\delta}^{b\,d}\partial_\alpha^a N_{dk}^{\delta} - g_{\alpha\delta}^{a\,d}\partial_\beta^b N_{dk}^{\delta} + g_{\beta h}^{b} T_{\alpha\ \,k}^{a\,h} + g_{\beta\delta}^{b\,d} T_{\alpha dk}^{a\,\delta} \\ &\quad - g_{\alpha h}^{a} T_{\beta\ \,k}^{b\,h} - g_{\alpha\delta}^{a\,d} T_{\beta dk}^{b\,\delta} + g_{kh} T_{\alpha\ \ \beta}^{a\,hb} + g_{k\,\delta}^{\ d} T_{\alpha d\beta}^{a\,\delta b}, \end{aligned}$$

$$(7.5)\quad (a)\quad 2C_{ij\,\alpha}^{\ \ a} = (\partial_\alpha^a g_{ij} + \delta_i g_{j\,\alpha}^{\ a} - \delta_j g_{\alpha i}^{a}) - (\lambda_\alpha^a g_{ij} + \lambda_i g_{j\,\alpha}^{\ a} - \lambda_j g_{\alpha i}^{a}) + C,$$

$$\begin{aligned} (b)\quad C &= (F_{i\,\alpha j}^{\ a} - F_{j\,\alpha i}^{\ a}) + (F_{\alpha ij}^{a} - C_{ji\,\alpha}^{\ \ a}) - (F_{\alpha ji}^{a} - C_{ij\,\alpha}^{\ \ a}) \\ &= g_{\alpha h}^{a} T_{i\ j}^{\ h} + g_{\alpha\delta}^{a\,d}(T_{idj}^{\delta} + \delta_i N_{dj}^{\delta} - \delta_j N_{di}^{\delta}) + g_{ih} T_{\alpha\ \,j}^{a\,h} \\ &\quad - g_{jh} T_{\alpha\ \,i}^{a\,h} + g_{i\,\delta}^{\ d}(T_{\alpha dj}^{a\,\delta} + \partial_\alpha^a N_{dj}^{\delta}) - g_{j\,\delta}^{\ d}(T_{\alpha di}^{a\,\delta} + \partial_\alpha^a N_{di}^{\delta}) \end{aligned}$$

$$\begin{aligned} (7.6)\quad (a)\quad 2C_{k\,\alpha\beta}^{\ a\,b} &= (\partial_\beta^b g_{k\,\alpha}^{\ a} + \delta_k g_{\alpha\beta}^{a\,b} - \partial_\alpha^a g_{\beta k}^{b}) - (\lambda_\beta^b g_{k\,\alpha}^{\ a} + \lambda_k g_{\alpha\beta}^{a\,b} - \lambda_\alpha^a g_{\beta k}^{b}) \\ &\quad - D - E, \end{aligned}$$

$$\begin{aligned} (b)\quad D &= (F_{\alpha\beta k}^{a\,b} - C_{k\,\beta\alpha}^{\ b\,a}) + (F_{\beta\alpha k}^{b\,a} - C_{k\,\alpha\beta}^{\ a\,b}) \\ &= g_{\beta h}^{b} T_{\alpha\ \,k}^{a\,h} + g_{\alpha h}^{a} T_{\beta\ \,k}^{b\,h} + g_{\beta\delta}^{b\,d}(T_{\alpha dk}^{a\,\delta} + \partial_a^\alpha N_{dk}^{\delta}) \\ &\quad + g_{\alpha\delta}^{ad}(T_{\beta dk}^{b\delta} + \partial_\beta^b N_{dk}^{\delta}), \end{aligned}$$

$$(c)\quad E = C_{\alpha k\beta}^{a\ \,b} - C_{\beta k\alpha}^{b\ \,a} = g_{kh} T_{\alpha\ \ \beta}^{a\,hb} + g_{k\,\delta}^{\ d} T_{\alpha d\beta}^{a\,\delta b},$$

$$\begin{aligned} (7.7)\qquad 2C_{\alpha k\beta}^{a\ \,b} &= (\partial_\beta^b g_{\alpha k}^{a} + \partial_\alpha^a g_{k\,\beta}^{\ b} - \delta_k g_{\beta\alpha}^{b\,a}) - (\lambda_\beta^b g_{\alpha k}^{a} + \lambda_\alpha^a g_{k\,\beta}^{\ b} - \lambda_k g_{\beta\alpha}^{b\,a}) \\ &\quad + D + E, \end{aligned}$$

$$\begin{aligned} (7.8)\quad (a)\quad 2C_{\alpha\beta\gamma}^{a\,b\,c} &= (\partial_\gamma^c g_{\alpha\beta}^{a\,b} + \partial_\alpha^a g_{\beta\gamma}^{b\,c} - \partial_\beta^b g_{\gamma\alpha}^{c\,a}) - (\lambda_\gamma^c g_{\alpha\beta}^{a\,b} + \lambda_\alpha^a g_{\beta\gamma}^{b\,c} - \lambda_\beta^b g_{\gamma\alpha}^{c\,a}) \\ &\quad + (C^{*}{}_{\gamma\alpha\beta}^{c\,a\,b} + C^{*}{}_{\alpha\gamma\beta}^{a\,c\,b} - C^{*}{}_{\gamma\beta\alpha}^{c\,b\,a}), \end{aligned}$$

$$(c)\quad C^{*}{}_{\gamma\alpha\beta}^{c\,a\,b} = C_{\gamma\alpha\beta}^{c\,a\,b} - C_{\beta\alpha\gamma}^{b\,a\,c} = g_{\alpha h}^{a} T_{\gamma\ \ \beta}^{c\,hb} + g_{\alpha\delta}^{a\,d} T_{\gamma d}^{c\,\delta h}.$$

Proof. The proof follows from (6.7), (6.6), (6.3) and (5.7).

*Definition*7.1. The recurrent K-Hamilton space, in which $T(X,Y) = 0$, for all $X, Y \in T(E^*)$, is called torsion free recurrent K-Hamilton space and is denoted by $(E^*, N, G, \nabla, 0, \lambda)$.

THEOREM 7.2. *In the forsion-free recurrent K-Hamilton space $(E^*, N, G, \nabla, 0, \lambda)$, the connection coefficients are determined by*

$$
\begin{aligned}
(7.9)\qquad 2F_{ijk} &= (\delta_k g_{ij} + \delta_i g_{jk} - \delta_j g_{ki}) - (\lambda_k g_{ij} + \lambda_i g_{jk} - \lambda_j g_{ki}),\\
2F_{i}{}^{a}_{\alpha}{}_{k} &= (\delta_k g_{i}{}^{a}_{\alpha} + \delta_i g^{a}_{\alpha k} - \partial^a_\alpha g_{ik}) - (\lambda_k g_{i}{}^{a}_{\alpha} + \lambda_i g^{a}_{\alpha k} - \lambda^a_\alpha g_{ik})\\
&\quad - (g_{i}{}^{d}_{\delta}\partial^a_\alpha N^\delta_{dk} + g_{k}{}^{d}_{\delta}\partial^a_\alpha N^\delta_{di}) + g^{a\,d}_{\alpha\delta}(\delta_i N^\delta_{dk} - \delta_k N^\delta_{di}),\\
2F^{a}_{\alpha ik} &= (\delta_k g^a_{\alpha i} + \partial^a_\alpha g_{ik} - \delta_i g^a_\alpha) - (\lambda_k g^a_{\alpha i} + \lambda^a_\alpha g_{ik} - \lambda_i g_{k}{}^{a}_{\alpha})\\
&\quad + (g_{i}{}^{d}_{\delta}\partial^a_\alpha N^\delta_{dk} + g_{k}{}^{d}_{\delta}\partial^a_\alpha N^\delta_{di}) - g^{a\,d}_{\alpha\delta}(\delta_i N^\delta_{dk} - \delta_k N^\delta_{di})\\
2F^{a\,b}_{\alpha\beta k} &= (\delta_k g^{a\,b}_{\alpha\beta} + \partial^a_\alpha g^b_{\beta h} - \partial^b_\beta g_{k}{}^{a}_{\alpha}) - (\lambda_k g^{a\,b}_{\alpha\beta} + \lambda^a_\alpha g^b_{\beta k} - \lambda^b_\beta g_{k}{}^{a}_{\alpha})\\
&\quad + g^{b\,d}_{\beta\delta}\partial^a_\alpha N^\delta_{dk} - g^{a\,d}_{\alpha\delta}\partial^b_\beta N^\delta_{dk},\\
2C_{ij}{}^{a}_{\alpha} &= (\partial^a_\alpha g_{ij} + \delta_i g_{j}{}^{a}_{\alpha} - \delta_j g^a_{\alpha i}) - (\lambda^a_\alpha g_{ij} + \lambda_i g_{j}{}^{a}_{\alpha} - \lambda_j g^a_{\alpha i})\\
&\quad + g^{a\,d}_{\alpha\delta}(\delta_i N^\delta_{dj} - \delta_j N^\delta_{di}) + g_{i}{}^{d}_{\delta}\partial^a_\alpha N^\delta_{dj} - g_{j}{}^{d}_{\delta}\partial^a_\alpha N^\delta_{di},\\
2C_{k}{}^{a\,b}_{\alpha\beta} &= (\partial^b_\beta g_{k}{}^{a}_{\alpha} + \delta_k g^{a\,b}_{\alpha\beta} - \partial^a_\alpha g^b_{\beta k}) - (\lambda^b_\beta g_{k}{}^{a}_{\alpha} + \lambda_k g^{a\,b}_{\alpha\beta} - \lambda^a_\alpha g^b_{\beta k})\\
&\quad - (g^{b\,d}_{\beta\delta}\partial^a_\alpha N^\delta_{dk} + g^{a\,d}_{\alpha\delta}\partial^b_\beta N^\delta_{dk}),\\
2C^{a}_{\alpha}{}_{k}{}^{b}_{\beta} &= (\partial^b_\beta g^a_{\alpha k} + g^a_\alpha g_{k}{}^{b}_{\beta} - \delta_k g^{b\,a}_{\beta\alpha}) - (\lambda^b_\beta g^a_{\alpha k} - \lambda^a_\alpha g_{k}{}^{b}_{\beta} - \lambda_k g^{b\,a}_{\beta\alpha})\\
&\quad + (g^{b\,d}_{\beta\delta}\partial^a_\alpha N^\delta_{dk} + g^{a\,d}_{\alpha\delta}\partial^b_\beta N^\delta_{dk}),\\
2C^{a\,b\,c}_{\alpha\beta\gamma} &= (\partial^c_\gamma g^{a\,b}_{\alpha\beta} + \partial^a_\alpha g^{b\,c}_{\beta\gamma} - \partial^b_\beta g^{c\,a}_{\gamma\alpha}) - (\lambda^c_\gamma g^{a\,b}_{\alpha\beta} + \lambda^a_\alpha g^{b\,c}_{\beta\gamma} - \lambda^b_\beta g^{c\,a}_{\gamma\alpha}).
\end{aligned}
$$

Definition 7.2. The metric K-Hamilton space in which $T(X,Y) = 0$ for all $X, Y \in T(E^*)$, is called torsion-free metric K-Hamilton space and denoted by $(E^*, N, G, \nabla, 0, 0)$.

THEOREM 7.3. *In the torsion free metric K-Hamilton space $(E^*, N, G, \nabla, 0, 0)$, the connection coefficients are given by* (7.9) *if we substitute in them $\lambda_i = 0$, $\lambda_j = 0$, $\lambda_k = 0$, $\lambda^a_\alpha = 0$, $\lambda^b_\beta = 0$ and $\lambda^c_\gamma = 0$.*

Definition 7.3. The torsion-free, metric K-Hamilton space, in which $[g_{k}{}^{a}_{\alpha}] = 0$ for all $\alpha = \overline{1,K}$ (i.e. $T_H(E^*)$ is orthogonal to ${}_{(\alpha)}T_V(E^*)$, for all $\alpha = \overline{1,K}$, or equivalently, $T_H(E^*)$ is orthogonal to $T_V(E^*)$), will be denoted by $(E^*, N, G_H, G_V, \nabla, 0, 0)$.

THEOREM 7.4. *In $(E^*, N, G_H, G_V, \nabla, 0, 0)$ the connection coefficients are determined by*

$$
\begin{aligned}
(7.10)\qquad 2F_{ijk} &= \delta_k g_{ij} + \delta_i g_{jk} - \delta_j g_{ki},\\
2F_{i}{}^{a}_{\alpha}{}_{k} &= -\partial^a_\alpha g_{ik} + g^{a\,d}_{\alpha\delta}(\delta_i N^\delta_{dk} - \delta_k N^\delta_{di}),\\
2F^a_{\alpha ik} &= \partial^a_\alpha g_{ik} - g^{a\,d}_{\alpha\delta}(\delta_i N^\delta_{dk} - \delta_k N^\delta_{di}),\\
2F^{a\,b}_{\alpha\beta k} &= \delta_k g^{a\,b}_{\alpha\beta} + g^{b\,d}_{\beta\delta}\partial^a_\alpha N^\delta_{dk} - g^{a\,d}_{\alpha\delta}\partial^b_\beta N^\delta_{dk},
\end{aligned}
$$

$$
\begin{aligned}
2C_{ij\,\alpha}^{\ \ a} &= \partial^a_\alpha g_{ij} + g^{a\,d}_{\alpha\delta}(\delta_i N^\delta_{dj} - \delta_j N^\delta_{di}),\\
2C_{k\,\alpha\beta}^{\ a\,b} &= \delta_k g^{a\,b}_{\alpha\beta} - (g^{b\,d}_{\beta\delta}\partial^a_\alpha N^\delta_{dk} + g^{a\,d}_{\alpha\delta}\partial^b_\beta N^\delta_{dk}),\\
2C^{a\ \ b}_{\alpha k\beta} &= -\delta_k g^{b\,a}_{\beta\alpha} + (g^{b\,d}_{\beta\delta}\partial^a_\alpha N^\delta_{dk} + g^{a\,d}_{\alpha\delta}\partial^b_\beta N^\delta_{dk}),\\
2C^{a\,b\,c}_{\alpha\beta\gamma} &= \partial^c_\gamma g^{a\,b}_{\alpha\beta} + \partial^a_\alpha g^{b\,c}_{\beta\gamma} - \partial^b_\beta g^{c\,a}_{\gamma\alpha}.
\end{aligned}
$$

It is obvious that in (7.10), $F^a_{\alpha ik} = -F_{i\,\alpha k}^{\ a}$, $C_{k\,\alpha\beta}^{\ a\,b} = -C^{a\ \ b}_{\alpha k\beta}$. In the recurrent K-Hamilton space in which $T_H(E^*)$ is orthogonal to $T_V(E^*)$, i.e. where the metric tensor has the property $[g_{i\,\alpha}^{\ a}] = 0$ for all $\alpha \in \overline{1,K}$ (in space $(E^*, N, G_H, G_V, \nabla, T, \lambda)$) we obtain from (6.6):

$$
\begin{aligned}
(7.11)\qquad & F_i{}^h{}_k = g^{hj}F_{ijk}, \qquad F_{i\,dk}^{\ \delta} = g^{\alpha\delta}_{a\,d}F_{i\,\alpha k}^{\ a}, \qquad F^{a\,h}_{\alpha\ \ k} = g^{hi}F^a_{\alpha ik},\\
& F^{a\,\delta}_{\alpha dk} = g^{\delta\beta}_{db}F^{a\,b}_{\alpha\beta k}, \quad C_i{}^{hc}{}_\gamma = g^{hj}C_{ij\gamma}^{\ \ c}, \qquad C_{i\,d\gamma}^{\ \delta c} = g^{\delta\alpha}_{da}C_{i\,\alpha\gamma}^{\ a\,c},\\
& C^{a\,kc}_{\alpha\ \ \gamma} = g^{ki}C^{a\ \ c}_{\alpha i\gamma}, \qquad C^{a\,\delta\,c}_{\alpha d\gamma} = g^{\delta\beta}_{db}C^{a\,b\,c}_{\alpha\beta\gamma}.
\end{aligned}
$$

The connection coeffcients, which appear on the right-hand side of (7.11) for $(E^*, N, G_H, G_V, \nabla, 0, 0)$, are determined by (7.10).

THEOREM 7.5. *The necessary and sufficient conditons that in* $(E^*, N, G_H, G_V, \nabla, 0, 0)$ *the connection coefficients satisfy the relations*

$$
\begin{aligned}
(7.12)\qquad & F_{i\,\alpha k}^{\ a} = 0 \quad \Leftrightarrow \quad F_{i\,dk}^{\ \delta} = 0, \qquad F^a_{\alpha ik} = 0 \quad \Leftrightarrow \quad F^{a\,h}_{\alpha\ \ k} = 0,\\
& C^{a\ \ c}_{\alpha i\gamma} = 0 \quad \Leftrightarrow \quad C^{a\,hc}_{\alpha\ \ \gamma} = 0, \qquad C_{i\,\alpha\gamma}^{\ a\,c} = 0 \quad \Leftrightarrow \quad C_{i\,d\gamma}^{\ \delta c} = 0,
\end{aligned}
$$

(*for the nonlinear connection* N) *are*

$$
(7.13)\ \ \partial^a_\alpha g_{ik} - g^{a\,d}_{\alpha\delta}(\delta_i N^\delta_{dk} - \delta_k N^\delta_{di}) = 0, \quad \delta_k g^{b\,a}_{\beta\alpha} - (g^{b\,d}_{\beta\delta}\partial^a_\alpha N^\delta_{dk} + g^{a\,d}_{\alpha\delta}\partial^b_\beta N^\delta_{dk}) = 0.
$$

Equations (7.12) *and* (7.13) *should be satisfied for all* $a, b, d \in \overline{1,m}$, *all* $h, i \in \overline{1,n}$, *all* $\alpha, \beta, \delta \in \overline{1,K}$.

Proof. The proof follows from (7.10) and (7.11).

THEOREM 7.6. *In* $(E^*, N, G_H, G_V, \nabla, 0, 0)$ *the generalized connection* ∇, *defined by* (5.1), *reduces to the Miron's d-connection defined by*

$$
\begin{aligned}
(7.14)\qquad & \nabla_{\delta_i}\delta_j = F_j{}^k{}_i\delta_k, \qquad \nabla_{\delta_i}\partial^a_\alpha = F^{a\,\gamma}_{\alpha c\,i}\partial^c_\gamma,\\
& \nabla_{\partial^a_\alpha}\delta_j = C_j{}^{ka}{}_\alpha\delta_k, \quad \nabla_{\partial^a_\alpha}\partial^b_\beta = C^{b\,\gamma\,a}_{\beta c\,\alpha}\partial^c_\gamma,
\end{aligned}
$$

iff the nonlinear connection N *and the meric tensor* G *are connected by* (7.13). *The connection coefficients of the d-connection are determined by*

$$
\begin{aligned}
(7.15)\qquad 2F_i{}^h{}_k &= g^{kj}(\delta_k g_{ij} + \delta_i g_{jk} - \delta_j g_{ki}),\\
2F^{a\,\gamma}_{\alpha c\,k} &= g^{\gamma\beta}_{c\,b}\delta_k g^{a\,b}_{\alpha\beta} + g^{\gamma\beta}_{c\,b}(g^{b\,d}_{\beta\delta}\partial^a_\alpha N^\delta_{dk} - g^{a\,d}_{\alpha\delta}\partial^b_\beta N^\delta_{dk}),\\
2C_i{}^{ha}{}_\alpha &= g^{hj}(\partial^a_\alpha g_{ij} + g^{a\,d}_{\alpha\delta}(\delta_i N^\delta_{di} - \delta_j N^\delta_{di})),\\
2C^{a\,\beta\,c}_{\alpha b\,\gamma} &= g^{\beta\delta}_{b\,d}(\partial^c_\gamma g^{a\,b}_{\alpha\beta} + \partial^a_\alpha g^{b\,c}_{\beta\gamma} - \partial^b_\beta g^{c\,a}_{\gamma\alpha}).
\end{aligned}
$$

Proof. (7.14) follows from Theorems (7.5) and (5.1). (7.15) follows from (7.10), (7.11) and (7.13).

From (7.14) it is obvious that the d-connection is the linear connection, for which $\nabla_X : T_H(E^*) \to T_H(E^*)$ and $\nabla_X : T_V(E^*) \to T_V(E^*)$ for all $X \in T(E^*)$.

THEOREM 7.7. *The space $(E^*, N, G_H, G_V, 0, 0)$, with integrable nonlinear connection N:*

$$\delta_i N^{\delta}_{dk} - \delta_k N^{\delta}_{di} = 0, \tag{7.16}$$

allows a d-connection iff

$$(a) \quad \partial^a_\alpha g_{ik} = 0, \qquad (b) \quad \delta_k g^{b\,a}_{\beta\alpha} = g^{b\,d}_{\beta\delta} \partial^a_\alpha N^{\delta}_{dk} + g^{a\,d}_{\alpha\delta} \partial^b_\beta N^{\delta}_{dk}. \tag{7.17}$$

((7.17a) means that the horizontal metric tensor is a function only of x).

The connection coefficients of such a d-connection are given by

$$F_i{}^h{}_k = 2^{-1} g^{hj} (\partial_k g_{ij} + \partial_i g_{jk} - \partial_j g_{ki}), \qquad F^{a\,\gamma}_{\alpha\,c}{}_k = \partial^a_\alpha N^{\gamma}_{c\,k},$$
$$C^{a\,\delta\,c}_{\alpha\,d\gamma} = 2^{-1} g^{\beta\delta}_{b\,d} (\partial^c_\gamma g^{a\,b}_{\alpha\beta} + \partial^a_\alpha g^{b\,c}_{\beta\gamma} - \partial^b_\beta g^{c\,a}_{\gamma\alpha}). \qquad C_i{}^{h\,a}{}_\alpha = 0,$$

Proof. From Theorem 7.6, it follows that the space $(E^*, N, G_H, G_V, 0, 0)$ allows a d-connection iff the relations (7.13) are satisfied. (7.13) and (7.16) result (7.17). Relation (7.17b) can be obtained if in (5.7d) we substitute $T_{HVV} = 0$ (torsion equal to zero) and $C_i{}^{\gamma\,b}_{c\,\beta} = 0$ (for the case of a d-connection).

REFERENCES

[**1**] M. Anastasiei, *Models of Finsler and Lagrange geometry*, Proc. Fourth National Seminar on Finsler and Lagrange spaces, Univ. Brasov, (R.S. Romania) (1986), 43–56.

[**2**] A.K. Aringazin, G.S. Asanov, *Problems of Finslerian theory of gauge fields and gravittion*, Rep. Math. Physics **25** (1988), 35–93.

[**3**] I. Čomić, *Generalized d-connection*, Tensor (N.S) **48** (1989), 199–209.

[**4**] I. Čomić, *Curvature theory of generalized Miron's d-connection*, Differential Geometry Appl., Proc. Cont. (1989), Brno, 17–26.

[**5**] I. Čomić, *Recurrent K-Lagrangian spaces with a strongly distinguished connection*, to appear

[**6**] I. Čomić, Sz.M. Kirhovits, *Generalized Miron's d-connection in a K-Lagrangian space*, Proc. Conf. Finsler geometry, Brasov, (1990), 39-58.

[**7**] I. Čomić, Sz.M. Kirhovits, *Curvature theory of generalized Miron's d-connection*, to appear.

[**8**] I. Čomić, *Recurrent Hamilton spaces wiht generalized Miron's d-connection*, to appear

[**9**] Sz. M. Kirhovits, *On K-Hamilton geometry I*, Math. Pannonica **2/1** (1991), 77–94.

[**10**] Sz.M. Kirhovits, *On K-Hamilton geometry II*, Math. Pannonica, to appear.

[**11**] R. Miron, M. Anastasiei, *Vibrate vectoriale spatii Lagrange aplicatii*, Bucuresti, 1987.

[**12**] R. Miron, *On the Finslerian theory of relativity*, Tensor (N.S) **44** (1987), 63–81.

[**13**] R. Miron, *Hamilton geometry*, Seminarul de Mecanica, Univ. Timisoara (R.S. Romania) Preprint No. 3. 1987, 54 pp.

[**14**] R. Miron, M. Anastasiei, *Vector bundles, Lagrange spaces, Applications to Reltivity*, Acad. R.S. Romania, Bucuresti, 1987.

[**15**] R. Miron, S. Janus, M. Anastasiei, *The geometry of the dual of a vector bundle*, Publ. Inst. Math. (Beograd) **46(60)** (1989), 145–162.

[**16**] D. Opris, *Lagrange-Hamilton geometry*, Seminarul de Mecanica, Univ. din Timisoare, 1988, 26 pp.

[**17**] T. Sakaguchi, *Subspaces in Lagrange spaces*, Ph. D. Thesis, Univ. Iasi (R.S. Romania), 1988, 167 pp.

Fakultet tehničkih nauka
21000 Novi Sad
Yugoslavia

(Received 02 12 1991)