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ON THE LAURENT COEFFICIENTS OF CERTAIN DIRICHLET SERIES

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Abstract. Explicit expressions for the coefficients of the Laurent (Taylor) expansions of certain Dirichlet series are given. These are used for the evaluation of certain integrals containing the error terms of some well-known problems of multiplicative number theory.

1. Expressions for the Laurent and Taylor coefficients

Let $0 = \lambda_1 < \lambda_2 < \dots$, $\lim_{n\to\infty} \lambda_n = +\infty$, and let

$$f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s} = \sum_{n=1}^{\infty} a_n \ell_n^{-s} \qquad (\ell_n = e^{\lambda_n}) \tag{1.1}$$

be a (general) Dirichlet series ($s = \sigma + it$; $\sigma, t \in \mathbf{R}e$) such that the counting function

$$A(x) = \sum_{\ell_n \le x} a_n \tag{1.2}$$

may be written in the form

$$A(x) = x^a (\sum_{m=0}^{M} c_m \log^m x) + u(x) \tag{1.3}$$

with $c_M > 0$ and

$$u(x) = O(x^b) \qquad (0 \le b < a), \tag{1.4}$$

where $u(x)$ is integrable for $x > 0$. Thus $u(x)$ may be thought of as the error term in the asymptotic formula for the counting function $A(x)$. This type of situation often occurs in many problems of analytic number theory, and some examples will be discussed in the sequel. It can be shown without difficulty that $f(s)$ can be analytically continued to a function which is regular for $\mathrm{Re}\, s > b$, except for a pole

AMS Subject Classification (1985): Primary 11 M 41, Secondary 11 N 37, 11 M 06 at $s = a$ of order $M+1$. We are interested in explicit expressions for the coefficients of the Laurent (Taylor) expansion of $f(s)$ for $\operatorname{Re} s > b$. Such problems were investigated, under various hypotheses, by several authors such as Balakrishnan [**1**], [**2**] and Briggs-Buschmann [**3**]. It appears that the previous authors focused their attention on the Laurent expansion near the pole of the Dirichlet series in question, whille we want to provide expressions for the Taylor coefficients at other points as well. Our first result is

THEOREM 1. *If $f(s)$ satisfies the above hypotheses, then its Laurent expansion at $s = a$ is*

$$f(s) = \sum_{m=1}^{M+1} D_m (s-a)^{-m} + \sum_{k=0}^{\infty} E_k (s-a)^k, \tag{1.5}$$

where

$$D_m = (m-1)!ac_{m-1} + m!c_m \quad (m = 1, \ldots, M), \;\; D_{M+1} = M!ac_M, \tag{1.6}$$

$$E_k = \frac{(-1)^k}{k!} \int_{1-0}^{\infty} x^{-a} (\log x)^x \, du(x) \qquad (k = 0, 1, \ldots). \tag{1.7}$$

The Taylor series of $f(s)$ at $s = s'$ $(s' \neq a, \ \operatorname{Re} s' > b)$ is

$$f(s) = \sum_{k=0}^{\infty} E_k(s')(s-s')^k \qquad (|s-s'| < |s'-a|, \ \operatorname{Re} s > b), \tag{1.8}$$

where for $k = 0, 1, \ldots$,

$$\begin{aligned} E_k(s') = \frac{(-1)^k}{k!} \Bigg\{ &\sum_{m=1}^{M+1} m(m+1)\ldots(m+k-1) D_m (s'-a)^{-m-k} \\ &+ \int_{1-0}^{\infty} x^{-s'} (\log x)^k du(x) \Bigg\}. \end{aligned} \tag{1.9}$$

Proof. By the Stieltjes integral representation we have, for $\operatorname{Re} s > a$,

$$\begin{aligned} f(s) = \int_{1-0}^{\infty} x^{-s} dA(x) = \sum_{m=0}^{M} c_m \int_1^{\infty} & x^{a-s-1} (a \log^m x + m \log^{m-1} x) dx \\ &+ \int_{1-0}^{\infty} x^{-s} du(x), \end{aligned} \tag{1.10}$$

since $\ell_1 = e^0 = 1$. Such representations of Dirichlet series by Stieltjes integrals are very useful, and have been systematically used e.g. by Karamata [**8**], [**9**]. One can evaluate

$$\begin{aligned} \int_1^{\infty} x^{a-s-1} \log^m x \cdot dx &= \int_0^{\infty} e^{-(s-a)u} u^m du = (s-a)^{-m-1} \int_0^{\infty} e^{-t} t^m dt \\ &= (s-a)^{-m-1} \Gamma(m+1) = m!(s-a)^{-m-1}, \end{aligned} \tag{1.11}$$

and obtain by integration by parts

$$\int_{1-0}^{\infty} x^{-s} du(x) = -u(1-0) + s\int_{1}^{\infty} u(x)x^{-s-1}dx. \tag{1.12}$$

In view of (1.4) it follows that the integral on the right-hand side of (1.12) converges absolutely for $\operatorname{Re} s > b$, and in that region it represents a regular function of s. Thus from (1.10) and (1.11) we have, for $\operatorname{Re} s > b$,

$$f(s) = \sum_{m=1}^{M+1} D_m(s-a)^{-m} + \int_{1-0}^{\infty} x^{-s} du(x) \tag{1.13}$$

with D_m given by (1.6). The relation (1.13) provides analytic continuation of $f(s)$ to the region $\operatorname{Re} s > b$, where $f(s)$ is regular except for a pole of order $M+1$ at $s = a$. To obtain (1.7) note that

$$\begin{aligned}\int_{1-0}^{\infty} x^{-s} du(x) &= \int_{1-0}^{\infty} x^{-a} e^{-(s-a)\log x} du(x) \\ &= \int_{1-0}^{\infty} x^{-a} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \log^k x \cdot (s-a)^k du(x) \\ &= \sum_{k=0}^{\infty} \Big\{ \frac{(-1)^k}{k!} \int_{1-0}^{\infty} x^{-a} \log^k x \cdot du(x) \Big\} (s-a)^k.\end{aligned} \tag{1.14}$$

To justify the inversion of summation and integration note that

$$\int_{1-0}^{\infty} x^{-a} \log^k x \cdot du(x) = \varepsilon_k u(1-0) - \int_{1}^{\infty} u(x)(-a\log^k x + k\log^{k-1} x)x^{-a-1}dx$$

($\varepsilon_0 = 1$ and $\varepsilon_k = 0$ for $k \geq 1$), so that by (1.4) the integral on the right-hand side is absolutely convergent. Inserting (1.14) in (1.13) and comparing with (1.15) we obtain (1.7), since Laurent expansions of analytic functions are unique.

To obtain (1.9) note that for $\operatorname{Re} s > b,\ s \neq a$ and $k = 0, 1, \dots$ we have from (1.13)

$$\begin{aligned}f^{(k)}(s) = \sum_{m=1}^{M+1} & D_m(-1)^k m(m+1)\dots(m+k-1)(s-a)^{-m-k} \\ &+(-1)^k \int_{1-0}^{\infty} x^{-s} \log^k x \cdot du(x),\end{aligned} \tag{1.15}$$

since we may use (1.12) in (1.13) and differentiate under the integral sign in view of the absolute convergence of the integral in question. On the other hand, for $\operatorname{Re} s' > b,\ s' \neq a$,

$$f(s) = \sum_{k=0}^{\infty} \frac{f^{(k)}(s')}{k!}(s-s')^k \qquad (|s-s'| < |s'-a|, \operatorname{Re} s > b). \tag{1.16}$$

The condition $|s - s'| < |s' - a|$ in (1.16) stems from the fact $f(s)$ has a pole at $s = a$. Replacing s by s' in (1.15) we obtain (1.8) with $E_k(s')$ given by (1.9). This finishes the proof of Theorem 1.

Before we proceed further some remarks are in order. No attempts have been made to make Theorem 1 as general as possible, and there are several ways in which it could be generalized. First, the condition (1.3) can be naturally generalized to read

$$A(x) = \sum_{\ell_n \le x} a_n = \sum_{i=1}^{k} \sum_{j=0}^{M_i} c_{i,j} x^{\alpha_i} \log^j x + u(x), \tag{1.17}$$

where the $c_{i,j}$'s are real constants ($c_{1,m_1} > 0$),

$$\alpha_1 > \alpha_2 > \ldots > \alpha_k > 0, \tag{1.18}$$

and the constant b in (1.4) is to satisfy now $0 \le b < \alpha_k$. The analysis made in the proof of Theorem 1 remains valid with the obvious changes. Namely, there will be poles as $s = \alpha_i$ of order $M_i + 1$, and (1.13) will be replaced by

$$f(s) = \sum_{i=1}^{k} \sum_{j=1}^{M_i+1} D_{i,m}(s - \alpha_i)^{-j} + \int_{1-0}^{\infty} x^{-s} du(x) \qquad (\mathrm{Re}\, s > b) \tag{1.19}$$

with suitable constants $D_{i,m}$.

Another possibility for generalization is to consider, instead of (1.3), the case when

$$A(x) = x^a L(x) + u(x), \tag{1.20}$$

where $u(x)$ satisfies (1.4), and $L(x)$ is a slowly varying function in the sense of J. Karamata. For our purposes this will mean that $L(x)$ is a positive, continuous function for $x \ge 1$ such that

$$\lim_{x\to\infty} \frac{L(cx)}{L(x)} = 1$$

for any $c > 0$. Then one has (see e.g. E. Seneta [**12**])

$$L(x) = B(x) \exp\left(\int_1^x \eta(t) \frac{dt}{t}\right), \tag{1.21}$$

where $\lim_{x\to\infty} B(x) = B > 0$, $\lim_{x\to\infty} \eta(t) = 0$ with $B(x)$ continuous and $\eta(t)$ integrable. The sum of logarithms appearing in (1.3) is obviously a slowly varying function.

For technical reasons we have assumed in (1.1) that $\lambda_1 = 0$, and this condition may be removed with the obvious modifications in the proof. However, in the most often encountered applications in multiplicative number theory the Dirichlet series are of the form $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$, which is the case $\lambda_n = \log n$ of (1.1), where $\lambda_1 = 0$ is fulfilled.

2. Case of the Riemann zeta-function

It seems natural to consider first the classical case when $f(s)$ in (1.1) reduces to the Riemann zeta-function. We take

$$f(s) = \sum_{n=1}^{\infty} n^{-s} = \zeta(s) \qquad (\operatorname{Re} s > 1),$$

so that in the notation of Section **1**

$$A(x) = \sum_{n \le x} 1 = x + u(x),\ u(x) = [x] - x,\ a = 1,\ M = b = 0,\ D_1 = 1.$$

In this case we have

$$\zeta(s) = \frac{1}{s-1} + \sum_{k=0}^{\infty} \gamma_k (s-1)^k, \tag{2.1}$$

where (1.7) gives (the constant in (2.1) are traditionally denoted by γ_k)

$$\begin{aligned}\gamma_k =& \frac{(-1)^k}{k!} \lim_{N\to\infty} \left(\int_{1-0}^{N} x^{-1} \log^k x \cdot d([x] - x) \right) \\ =& \frac{(-1)^k}{k!} \lim_{N\to\infty} \left(\sum_{n \le N} \frac{\log^k n}{n} - \frac{\log^{k+1} N}{k+1} \right).\end{aligned} \tag{2.2}$$

For $s = s'$ and $s' \ne 1$, $\operatorname{Re} s' > 0$ we have

$$\zeta(s) = \sum_{k=0}^{\infty} \gamma_k(s')(s - s')^k, \tag{2.3}$$

where by (1.9)

$$\begin{aligned}\gamma_k(s') &= (-1)^k (s'-1)^{-k-1} + \frac{(-1)^k}{k!} \lim_{N\to\infty} \left(\int_{1-0}^{N} x^{-s'} \log^k x \cdot d([x] - x) \right) \\ &= (-1)^k (s'-1)^{-k-1} + \frac{(-1)^k}{k!} \lim_{N\to\infty} \left(\sum_{n \le N} n^{-s'} \log^k n - \int_1^N x^{-s'} \log^k x dx \right).\end{aligned}$$

Evaluating the last integrals by successive integrations by parts we obtain, for $k \ge 1$,

$$\begin{aligned}\gamma_k(s') =& (-1)^k (s'-1)^{-k-1} \\ &+ \frac{(-1)^k}{k!} \lim_{N\to\infty} \Bigg\{ \sum_{n \le N} n^{-s'} \log^k n + \frac{N^{1-s'}}{s'-1} \Big(\log^k N + \frac{k \log^{k-1} N}{s'-1} \\ &\qquad + \frac{k(k-1) \log^{k-2} N}{(s'-1)^2} + \ldots + \frac{k!}{(s'-1)^k} \Big) \Bigg\}.\end{aligned} \tag{2.4}$$

The representation (2.2) is the classical one due to Stieltjes (1885), and the constants γ_k are commonly called the Stieltjes constants. Their numerical values have been calculated by Israilov [**4**], [**5**], and the first few are

$$\gamma = \gamma_0 = 0.57721\ldots, \gamma_1 = 0.07281\ldots, \gamma_2 = -0.00485\ldots, \gamma_3 = -0.00034\ldots.$$

The constant $\gamma = \gamma_0 = -\int_0^\infty e^{-x}\log x \cdot dx$ is traditionally known as the Euler constant. It was proved by Mitrović [**11**] that each of the inequalities $\gamma_{2n} > 0,\ \gamma_{2n} < 0,$ $\gamma_{2n-1} > 0,\ \gamma_{2n-1} < 0$ holds for infinitely many n. For the expression (2.4) I have not been able to find a reference in the literature.

3. Finite zeta-products

In many problems of multiplicative number theory the generating Dirichlet series is of the form

$$f(s) = \Pi_{j=1}^J (\zeta^{(r_j)}(\alpha_j s))^{k_j} \Pi_{m=1}^M (\zeta^{(q_m)}(\beta_m s))^{-\ell_m}, \tag{3.1}$$

where $r_j, q_m \ge 0;\ \ k_j, \ell_m \ge 1$ are integers, and

$$0 < \alpha_1 < \alpha_2 < \ldots < \alpha_J,\ \ \beta_1 < \beta_2 < \ldots < \beta_M \qquad (\alpha_j \ne \beta_m)$$

are real numbers such that $\alpha_1 < \beta_1$. For $\mathrm{Re}\, s > 1/\alpha_1$

$$f(s) = \sum_{n=1}^\infty a_n n^{-s}, \tag{3.2}$$

where a_n is a suitable arithmetic function (sequence), and the series in (3.2) converges absolutely. We can write

$$\sum_{n\le x} a_n = M(x) + E(x), \tag{3.3}$$

where $M(x)$ and $E(x)$ may be thought of as the main term and the error term, respectively, for the summatory function of a_n. By applying Perron's inversion formula for Dirichlet series (see e.g. the Appendix of [**7**]) and using the residue theorem it is seen that we may assume $M(x)$ to be of the form

$$M(x) = \sum_{j=1}^{J'} x^{1/\alpha_j} Q_{r_j+k_j-1}(\log x), \tag{3.4}$$

where $Q_p(t)$ denotes a suitable polynomial of degree p in t, and J' is the largest integer such that $\alpha_{J'} > \beta_1/2$, since $\zeta^{(q_1)}(\beta_1 s)$ certainly has zeros in the region $\mathrm{Re}\, s \ge \beta_1/2$. We also suppose that we can obtain, by elementary or analytic arguments,

$$E(x) = O(x^\eta), \qquad 0 \le \eta < 1/\alpha_{J'}. \tag{3.5}$$

Then we have

THEOREM 2. *If $f(s)$ is generated by the finite zeta-product* (3.1), *then for* $\operatorname{Re} s' > \eta$ *and any integer* $k \geq 0$

$$\int_1^\infty x^{-s'-1}\left(\log^k x\right)E(x)dx \tag{3.6}$$

can be expressed as an explicit finite sum involing the coefficients γ_k *and* $\gamma_k(s')$ *given by* (2.2) *and* (2.4), *respectively.*

Proof. At the point $s = s'$ the k-th Taylor coefficient (or Laurent coefficient, if $s' = \alpha_j$ for some $1 \leq j \leq J'$) of $f(s)$ can be expressed as an explicit finite sum involving the constants γ_k or $\gamma_k(s')$. This follows directly from the product representation (3.1), since the Taylor (Laurent) series for $\zeta^{(q)}(s)$ may be obtained by differentiating the Taylor (Laurent) series of $\zeta(s)$ term by term q times. The coefficients of $1/\zeta^{(q)}(s)$ may be found from the relation $1 = \zeta^{(q)}(s) \cdot 1/\zeta^{(q)}(s)$ in terms of the γ_k's or $\gamma_k(s')$'s.

On the other hand, by following the proof of Theorem 1, it is seen that the k-th coefficient of $f(s)$, say c_k, may be expressed in terms of

$$\int_{1-0}^\infty x^{-s'}\log^k x \cdot dE(x) = -\,E(1-0)\varepsilon_k + (s'+1)\int_1^\infty x^{-s'-1}\log^k x \cdot E(x)dx$$
$$-\,k\int_1^\infty x^{-s'-1}\log^{k-1} x \cdot E(x)\ dx.$$

If we compare the two expressions for c_k we shall obtain a linear recurrent relation between d_k and d_{k-1}, where

$$d_k = \int_1^\infty x^{-s'-1}\log^k x \cdot E(x)\ dx,$$

and by solving this recurrent relation for d_k the assertion of the theorem follows. It is clear that in the general case the expression for d_k is complicated, so no attempt is being made to write it down explicitly. In the next section several examples of finite zeta-products will be worked out in detail.

4. Some examples of finite zeta-products

4.1. The general Dirichlet divisor problem. Consider

$$f(s) = \sum_{n=1}^\infty d_M(n)n^{-s} = \left(\sum_{n=1}^\infty n^{-s}\right)^M = \zeta^M(s) \quad (M \in \mathbf{N};\ \operatorname{Re} s > 1), \tag{4.1}$$

so that, for $M \geq 2$ fixed, the multiplicative function $d_M(n)$ denotes the number of ways n may be represented as a product of M factors. In this case we know that

$$A(x) = \sum_{n\leq x} d_M(n) = xP_{M-1}(\log x) + \Delta_M(x), \tag{4.2}$$

where $P_{M-1}(t)$ is a suitable polynomial in t of degree $M-1$. The estimation of the error term $\Delta_M(x)$ in (4.2) is known as the general Dirichlet divisor problem (the case $M = 2$ being known as the Dirichlet divisor problem). An elementary argument (see Ch.12 of Titchmarsh [**13**]) easily shows that

$$\Delta_M(x) = O(x^{1-1/M}\log^{M-2} x) \qquad (M \ge 2),$$

while for the main term we have

$$P_{M-1}(\log x) = \mathrm{Res}_{s=1} x^{s-1}\zeta^M(s)s^{-1},$$

so that the coefficients of P_{M-1} may be expressed in terms of the Stieltjes constants γ_k. In the notation of (1.3) this means that $a = 1,\ u(x) = \Delta_M(x)$, and near $s = 1$ we have the Laurent expansion

$$\zeta^M(s) = \sum_{j=1}^{M} d_{j,M}(s-1)^{-j} + \sum_{j=0}^{\infty}\gamma_{j,M}(s-1)^j, \tag{4.3}$$

where by (1.7)

$$\gamma_{j,M} = \frac{(-1)^j}{j!}\int_{1-0}^{\infty} x^{-1}\log^j x\cdot d\Delta_M(x). \tag{4.4}$$

In view of (4.2) we can write down explicitly

$$\begin{aligned}\gamma_{j,M} = \frac{(-1)^j}{j!}\lim_{N\to\infty}\Biggl(&\sum_{n\le N} d_M(n)\log^j n\cdot n^{-1}\\ &- \int_1^N x^{-1}\log^j x\cdot Q_{M-1}(\log x)\cdot dx\Biggr),\end{aligned} \tag{4.5}$$

where $Q_{M-1}(t) = P_{M-1}(t) + P'_{M-1}(t)$, and the integral in (4.5) may be easily evaluated elementarily. Of course, when $M = 1$, then $\gamma_{j,1} = \gamma_j$, as given by (2.2). On the other hand by (2.1) we have

$$\zeta^M(s) = \bigl(\frac{1}{s-1} + \sum_{j=0}^{\infty}\gamma_j(s-1)^j\bigr)^M, \tag{4.6}$$

and developing the right-hand side of (4.6) by the multinominal theorem and comparing with (4.3), it follows that each $\gamma_{j,M}$ may be expressed as a suitable finite sum of the form $\sum \gamma_{j_1}^{r_1}\ldots\gamma_{j_M}^{r_M}$ with non-negative integer exponents $r_1,\ldots,r_M$. However from (4.4) we have

$$\gamma_{j,M} = \frac{(-1)^j}{j!}(-\varepsilon_j\Delta_M(1-0) + \int_1^{\infty}\Delta_M(x)x^{-2}\log^j x\cdot dx),$$

where as before $\varepsilon_0 = 1$ and $\varepsilon_j = 0$ for $j \ge 1,\ \Delta_M(1-0) = -P_{M-1}(0)$. Hence this shows that each integral

$$\int_1^{\infty}\Delta_M(x)x^{-2}\log^j x\cdot dx \qquad (M \ge 2;\ j = 0,1,\ldots)$$

may be expressed as a finite sum involving the Stieltjes constants γ_k. In particular, this method easily shows that

$$\int_1^\infty \Delta_2(x)x^{-2}dx = (\gamma_0 - 1)^2 + 2\gamma_1, \tag{4.7}$$

$$\int_1^\infty \Delta_3(x)x^{-2}dx = (\gamma_0 - 1)^3 + 3(\gamma_1 - \gamma_2) + 6\gamma_0\gamma_1. \tag{4.8}$$

The identitites (4.7) and (4.8) were obtained, in a more complicated way, by Lavrik et al. [**10**].

4.2. Squarefree numbers. A number n is squarefree if $n = 1$ or if $n = p_1 \dots p_r$, where the p_i's are distinct primes. Thus n is squarefree if and only if $\mu^2(n) = 1$, where $\mu(n)$ denotes the Möbius function. Hence

$$\begin{aligned} f(s) &= \sum_{n=1}^\infty \mu^2(n)n^{-s} = \zeta(s)/\zeta(2s) \qquad (\operatorname{Re} s > 1),\\ A(x) &= \sum_{n\le x} \mu^2(n) = 6\pi^{-2}x + R(x), \end{aligned} \tag{4.9}$$

where $R(x) = O(x^{1/2})$ follows by elementary arguments (see. Ch. 14 of [**7**] for a sharper result). In this case (1.7) of Theorem 1 may be directly applied to yield

$$\begin{aligned} E_k &= \frac{(-1)^k}{k!}\int_{1-0}^\infty x^{-1}(\log x)^k dR(x)\\ &= \frac{(-1)^k}{k!}\Big(-\varepsilon_k R(1-0) + \int_1^\infty x^{-2}\log^k xR(x)dx - k\int_1^\infty x^{-2}\log^{k-1} xR(x)dx\Big) \end{aligned}$$

with $\varepsilon_0 = 1$ and $\varepsilon_k = 0$ for $k \ge 1$. Since $R(1-0) = -6\pi^{-2}$, we have

$$\begin{aligned} E_0 &= 6\pi^{-2} + F_0,\\ E_k &= \frac{(-1)^k}{k!}(F_k - kF_{k-1}) = G_k + G_{k-1} \qquad (k \ge 1), \end{aligned} \tag{4.10}$$

where

$$F_k = \int_1^\infty x^{-2}\log^k x \cdot R(x)\ dx, \quad G_k = \frac{(-1)^k}{k!}F_k.$$

From (4.10) we have, since $G_0 = F_0 = E_0 - 6\pi^{-2}$,

$$G_k = \sum_{j=0}^k (-1)^{k-j}E_j + (-1)^{k+1}6\pi^{-2},$$

hence

$$F_k = k!\sum_{j=0}^k (-1)^j E_j - k!6\pi^{-2}. \tag{4.11}$$

On the other hand

$$\frac{\zeta(s)}{\zeta(2s)} = \Big(\frac{1}{s-1} + \gamma_0 + \gamma_1(s-1) + \gamma_2(s-1)^2 + \dots\Big)\Big(d_0 + d_1(s-1) + d_2(s-1)^2 + \dots\Big).$$

From (2.3) and (2.4) one has $d_k = 2^k\gamma_k(2)$, but it is perhaps more convenient to note that

$$d_k = \frac{1}{k!}\Big(\frac{1}{\zeta(2s)}\Big)^{(k)}\Big|_{s=1} = \frac{(-2)^k}{k!}\sum_{n=1}^{\infty}\mu(n)\log^k n \cdot n^{-2}. \tag{4.12}$$

Therefore

$$E_k = \sum_{j=0}^{k}\gamma_j d_{k-j} + d_{k+1}. \tag{4.13}$$

Inserting (4.13) in (4.11) we obtain

Theorem 3. *If $R(x)$ is defined by* (4.9)*, γ_k by* (2.2) *and d_k by* (4.12)*, then for $k = 0, 1, 2, \dots$*

$$\int_1^{\infty} x^{-2}R(x)\log^k x\;dx = k!\sum_{j=0}^{k}(-1)^j\left(\sum_{r=0}^{j}\gamma_r d_{j-r} + d_{j+1}\right) - k!6\pi^{-2},$$

and in particular

$$\int_1^{\infty} x^{-2}R(x)\;dx = (\gamma_0 - 1)6\pi^{-2} - 72\pi^{-4}\zeta'(2).$$

It is clear that instead of squarefree integers we may consider r-free integers (generated by $\zeta(s)/\zeta(rs)$). Also, for $\operatorname{Re} s' > 3/2$ and $j = 0, 1, \dots$ the integral $\int_1^{\infty} x^{-s'}R(x)\log^k x\;dx$ may be explicitly evaluated in terms of the Stieltjes constants γ_j and $\gamma_j(s')$'s as stated in Theorem 2.

4.3. Squarefull numbers. A number n is squarefull if $n = 1$ or if

$$n = p_1^{\alpha_1}\dots p_r^{\alpha_r} \qquad (\alpha_1 \ge 2, \dots, \alpha_r \ge 2),$$

where the p_i's are distinct primes. Let $A(x) = \sum_{n\le x} a_n$, where $a_n = 1$ if n is squarefull and zero otherwise, so that $A(x)$ is the number of squarefull integers not exceeding x. Then we have (see Ch. 14 of [**7**])

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s} = \frac{\zeta(2s)\zeta(3s)}{\zeta(6s)} \qquad (\operatorname{Re} s > 1/2), \tag{4.12}$$

$$A(x) = \frac{\zeta(3/2)}{\zeta(3)}x^{1/2} + \frac{\zeta(2/3)}{\zeta(2)}x^{1/3} + E(x), \tag{4.13}$$

where for some $C > 0$

$$E(x) = O(x^{1/6}\exp(-C\log^{3/5} x(\log\log x)^{-1/5})). \tag{4.14}$$

Here again we have an example of a finite zeta-product, and we shall give explicit formulas for $I(3/2)$ and $I(4/3)$, where

$$I(s) = \int_1^\infty x^{-s} E(x)\ dx. \tag{4.15}$$

In view of (4.14) and Theorem 2 it follows that $I(s)$ may be explicitly evaluated for $\operatorname{Re} s \geq 7/6$ in terms of the constants γ_j and $\gamma_j(s')$. In the neighborhood of $s = 1/2$ we have

$$\begin{aligned}
f(s) = \int_{1-0}^\infty x^{-s} dA(x) &= \frac{\zeta(3/2)}{2\zeta(3)} \int_1^\infty x^{-s-1/2} dx \\
&\quad + \frac{\zeta(2/3)}{3\zeta(2)} \int_1^\infty x^{-s-2/3} dx + \int_{1-0}^\infty x^{-s} dE(x) \\
&= \frac{\zeta(3/2)}{2\zeta(3)(s-1/2)} + \frac{\zeta(2/3)}{3\zeta(2)(s-1/3)} + \int_{1-0}^\infty x^{-s} dE(x) \\
&= \frac{\zeta(3/2)}{2\zeta(3)(s-1/2)} + \frac{2\zeta(2/3)}{\zeta(2)} + \int_{1-0}^\infty x^{-1/2} dE(x) + \sum_{j=1}^\infty e_j (s - \frac{1}{2})^j
\end{aligned}$$

for suitable constants e_j. On the other hand

$$\begin{aligned}
f(s) = \frac{\zeta(2s)\zeta(3s)}{\zeta(6s)} &= \Big(\frac{1}{2s-1} + \gamma + \gamma_1(2s-1) + \ldots\Big) \times \\
&\quad \times \Big(\frac{\zeta(3/2)}{\zeta(3)} + \Big(\frac{d\zeta(3s)/\zeta(6s)}{ds}\Big)\Big|_{s=\frac{1}{2}} (s - \frac{1}{2}) + \ldots\Big),
\end{aligned}$$

so that by comparing the constant terms in the two forms of the Laurent expansion for $f(s)$ near $s = 1/2$ we obtain

$$\frac{2\zeta(2/3)}{\zeta(2)} + \int_{1-0}^\infty x^{-1/2} dE(x) = \frac{\gamma\zeta(3/2)}{\zeta(3)} + \frac{3\zeta(3)\zeta'(3/2) - 6\zeta(3/2)\zeta'(3)}{2\zeta^2(3)}.$$

Since

$$\begin{aligned}
\int_{1-0}^\infty x^{-1/2} dE(x) &= -E(1-0) + \frac{1}{2}\int_1^\infty x^{-3/2} E(x)\ dx \\
&= \frac{\zeta(3/2)}{\zeta(3)} + \frac{\zeta(2/3)}{\zeta(2)} + \frac{1}{2}\int_1^\infty x^{-3/2} E(x)\ dx,
\end{aligned}$$

we obtain the explicit expression for $I(3/2)$, and in a similar way the formula for $I(4/3)$ may be found. The final formulas are contained in

THEOREM 4. *If γ is Euler's constant and $E(x)$ is given by* (4.13), *$I(s)$ by* (4.15), *then*

$$\begin{aligned}
I(\frac{3}{2}) &= 2(\gamma - 1)\frac{\zeta(3/2)}{\zeta(3)} - \frac{6\zeta(2/3)}{\zeta(2)} + \frac{3\zeta(3)\zeta'(3/2) - 6\zeta(3/2)\zeta'(3)}{\zeta^2(3)}, \\
I(\frac{4}{3}) &= 3(\gamma - 1)\frac{\zeta(2/3)}{\zeta(2)} - \frac{6\zeta(3/2)}{\zeta(3)} + \frac{2\zeta(2)\zeta'(2/3) - 6\zeta(2/3)\zeta'(2)}{\zeta^2(2)}.
\end{aligned}$$

4.4. The generalized von Mangoldt function. This is the function defined by

$$\Lambda_k(n) = \sum_{d|n} \mu(d)(\log \frac{n}{d})^k,$$

where k is a natural number. For $\operatorname{Re} s > 1$,

$$\sum_{n=1}^{\infty} \Lambda_k(n) n^{-s} = (-1)^k \cdot \frac{\zeta^{(k)}(s)}{\zeta(s)} = f_k(s).$$

It is known (see [**6**] or Ch. 12 of [**7**]) that

$$\sum_{n \le x} \Lambda_k(n) = x P_{k-1}(\log x) + R_k(x),$$

where $P_{k-1}(t)$ is a suitable polynomial of degree $k-1$ in t with the leading coefficient equal to k, and

$$R_k(x) = O(x \cdot \exp(-C_k \log^{3/5} x (\log\log x)^{-1/5})) \quad (C_k > 0).$$

Thus near $s = 1$ we have, by Theorem 1,

$$f_k(s) = \sum_{j=1}^{k} B_{j,k}(s-1)^{-j} + \sum_{j=0}^{\infty} C_{j,k}(s-1)^j,$$

where

$$C_{j,k} = \frac{(-1)^j}{j!} \int_{1-0}^{\infty} x^{-1} \log^j x \cdot dR_k(x). \tag{4.16}$$

Now we have, near $s = 1$,

$$(-1)^k \zeta^{(k)}(s) = k!(s-1)^{-k-1} + \sum_{j=0}^{\infty} D_{j,k}(s-1)^j \tag{4.17}$$

and

$$\frac{1}{\zeta(s)} = \sum_{j=1}^{\infty} a_j (s-1)^j. \tag{4.18}$$

By Th. 3.13 of Titchmarsh [**13**]

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \mu(n) n^{-s} \qquad (\operatorname{Re} s \ge 1),$$

so that

$$a_j = \frac{1}{j!} \Big(\frac{1}{\zeta(s)}\Big)^{(j)} \Big|_{s=1} = \frac{(-1)^j}{j!} \sum_{n=2}^{\infty} \mu(n) n^{-1} \log^j n. \tag{4.19}$$

Therefore from (4.17)-(4.19) we obtain

$$C_{0,k} = \frac{(-1)^{k+1}}{k+1} \sum_{n=2}^{\infty} \mu(n) n^{-1} \log^{k+1} n,$$

but since (4.16) gives

$$C_{0,k} = -R_k(1-0) + \int_1^{\infty} x^{-2} R_k(x)\ dx = P_{k-1}(0) + \int_1^{\infty} x^{-2} R_k(x)\ dx,$$

we obtain

THEOREM 5.

$$\int_1^{\infty} x^{-2} R_k(x) dx = \frac{(-1)^{k+1}}{k+1} \sum_{n=2}^{\infty} \mu(n) n^{-1} \log^{k+1} n - P_{k-1}(0). \tag{4.20}$$

It may be remarked that the integral in (4.20) could be also expressed by the Stieltjes constants γ_j, but the expression would be complicated, whereas the right-hand side of (4.20) has simple form. In the case $k = 1$ it is not difficult to see that (4.20) reduces to the known identity

$$\int_1^{\infty} x^{-2} \left(\sum_{n \le x} \Lambda(n) - x \right) dx = -1 - \gamma,$$

where $\Lambda(n) \equiv \Lambda_1(n)$ is the classical von Mangoldt function.

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