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GENERALIZED HERMITE POLYNOMIALS

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Abstract. We consider a new generalization of the classical Hermite polynomials and prove the basic characteristics of such polynomials $h_{n,m}^{\lambda}(x)$ (the generating function, an explicit representation, some recurrence relations, and the corresponding differential equation). For $m=2$, the polynomial $h_{n,m}^{\lambda}(x)$ reduces to $H_n(x,\lambda)/n!$, where $H_n(x,\lambda)$ is the Hermite polynomial with a parameter. For $\lambda=1$, $h_{n,2}^{l}(x)=H_n(x)/n!$, where $H_n(x)$ is the classical Hermite polynomial. Taking $\lambda=1$ and $n=mN+q$, where $N=[n/m]$ and $0\le q\le m-1$, we introduce the polynomials $P_N^{(m,q)}(t)$ by $h_{n,m}^{l}(x)=(2x)^q P_N^{(m,q)}((2x)^m)$, and prove that they satisfy an $(m+1)$-term linear recurrence relation.

1. Polynomials $\mathbf{h_{n,m}^{\lambda}(x)}$.

At the beginning, we define polynomials $h_{n,m}^{\lambda}(x)$ in the following manner.

Definition 1.1. The polynomials $h_{n,m}^{\lambda}(x)$, $\lambda\in R^+$, $n,m\in N$, are defined by the generating function

$$F(x,t)=e^{\lambda(2xt-t^m)}=\sum_{n=0}^{\infty}h_{n,m}^{\lambda}(x)t^n. \tag{1.1}$$

From above we get

$$F(x,t)=e^{\lambda(2xt-t^m)}=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{[n/m]}(-1)^k\frac{\lambda^n(2x)^{n-mk}}{\lambda^{(m-1)k}k!(n-mk)!}\right)t^n.$$

Thus, we obtain the following explicit representation

$$h_{n,m}^{\lambda}(x)=\lambda^n\sum_{k=0}^{[n/m]}(-1)^k\frac{(2x)^{n-mk}}{\lambda^{(m-1)k}k!(n-mk)!}. \tag{1.2}$$

Starting from (1.1) we can prove the following theorem.

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THEOREM 1.1. *The polynomials $h_{n,m}^{\lambda}(x)$ satisfy the three-term recurrence relation*

$$nh_{n,m}^{\lambda}(x) = \lambda(2xh_{n-1,m}^{\lambda}(x) - mh_{n-m,m}^{\lambda}(x)), \quad n \geq m \tag{1.3}$$

with initial values : $h_{n,m}^{\lambda}(x) = (2\lambda x)^n/n!, \quad 0 \leq n \leq m-1.$

Now, we prove the following theorem :

THEOREM 1.2. *The polynomials $h_{n,m}^{\lambda}(x)$ satisfy the following relations:*

$$2nh_{n,m}^{\lambda}(x) = (2x)Dh_{n,m}^{\lambda}(x) - mDh_{n+1-m,m}^{\lambda}(x); \tag{1.4}$$

$$D^k h_{n,m}^{\lambda}(x) = (2\lambda)^k h_{n-k,m}^{\lambda}(x); \tag{1.5}$$

$$\frac{(2x)^n}{n!} = \sum_{k=0}^{[n/m]} \frac{1}{k!} h_{n-mk,m}^{l}(x) \qquad (m \geq 2); \tag{1.6}$$

$$u^n h_{n,m}^{l}\left(\frac{x}{u}\right) = \sum_{k=0}^{[n/m]} \frac{(1-u^m)^k}{k!} h_{n-mk,m}^{l}(x); \tag{1.7}$$

$$h_{n,m}^{l}(x+y) = \sum_{k=0}^{n} \frac{(2y)^k}{k!} h_{n-k,m}^{l}(x), \tag{1.8}$$

where $D = d/dx$ is the differentiation operator.

Proof. Differentiating (1.1) with respect to x and t we find the next equalities:

(i) $\partial F(x,t)/\partial x = 2\lambda t e^{\lambda(2xt-t^m)}$, (ii) $\partial F(x,t)/\partial t = \lambda(2x - mt^{m-1})e^{\lambda(2xt-t^m)}$.

Combining these equalities we obtain (1.4).

Differentiating the polynomials $h_{n,m}^{\lambda}(x)$ given by (1.2) k-times we get (1.5).

The generating function (1.1) for $\lambda = 1$ reduces to

$$e^{2xt-t^m} = \sum_{n=0}^{\infty} h_{n,m}^{l}(x)t^n, \quad \text{i.e. to } e^{2xt} = e^{t^m} \sum_{n=0}^{\infty} h_{n,m}^{\lambda}(x)t^n.$$

Developing both sides of the last equality in t, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(2x)^n}{n!} t^n &= \left(\sum_{n=0}^{\infty} \frac{t^{mn}}{n!}\right)\left(\sum_{n=0}^{\infty} h_{n,m}^{l}(x)t^n\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{[n/m]} \frac{1}{k!} h_{n-mk,m}^{l}(x)\right) t^n.\end{aligned}$$

Now, comparing coefficients of t^n in the last equality we get (1.6).

Starting from $e^{2xt-t^m u^m} = e^{2xt-t^m} \circ e^{t^m - u^m t^m}$, we get (1.7).

Finally, from the equality $e^{2(x+y)t-t^m} = e^{2xt-t^m} \circ e^{2yt}$, we get

$$\sum_{n=0}^{\infty} t^n \sum_{k=0}^{[n/m]} (-1)^k \frac{(2x+2y)^{n-mk}}{k!(n-mk)!} =$$

$$\left(\sum_{n=0}^{\infty} t^n \sum_{k=0}^{n} (-1)^k \frac{(2x)^{n-k} t^{(m-1)k}}{k!(n-k)!}\right) \left(\sum_{n=0}^{\infty} \frac{t^n (2y)^n}{n!}\right),$$

wherefrom, after some calculations, we obtain (1.8).

COROLLARY 1.1. *For $m = 2$ and $\lambda = 1$ the equalities* (1.4)–(1.8) *reduce to the corresponding relations for the classical Hermite polynomials.*

At the end of this section we prove that the polynomials $h_{n,m}^{\lambda}(x)$ have an interesting property.

THEOREM 1.3. *The polynomial $h_{n,m}^{\lambda}(x)$ is a particular solution of linear homogeneous equation of m-th order given by*

$$L_n(y) = y^{(m)} - 2^m m^{-1} \lambda^{m-1} (xy - ny) = 0. \tag{1.9}$$

Proof. Using (1.5) and the recurrence relation (1.3) we get

$$\begin{aligned} L_n[h_{n,m}^{\lambda}(x)] &= (2\lambda)^m h_{n-m,m}^{\lambda}(x) - 2^m m^{-1} \lambda^{m-1} x (2\lambda) h_{n-1,m}^{\lambda}(x) \\ &\quad + \ 2^m m^{-1} \lambda^{m-1} n h_{n,m}^{\lambda}(x) \\ &= 2^m m^{-1} \lambda^{m-1} (n h_{n,m}^{\lambda}(x) - 2\lambda x h_{n-1,m}^{\lambda}(x) + m\lambda h_{n-m,m}^{\lambda}(x)) = 0. \end{aligned}$$

2. Polynomials $\mathbf{P_N^{m,q}(t)}$. In this section we introduce a class of polynomials $\{P_N^{m,q}(t)\}_{N=0}^{\infty}$. Let us suppose that $n = mN + q$, where $N = [n/m]$ and $0 \le q \le m-1$. Starting from (1.2) and taking $\lambda = 1$, we have

$$\begin{aligned} h_{n,m}^{l}(x) &= (2x)^q \sum_{k=0}^{N} (-1)^k \frac{(2x)^{mN-mk}}{k!(mN+q-mk)!} \\ &= (2x)^q \sum_{k=0}^{N} (-1)^k \frac{((2x)^m)^{N-k}}{k!(q+m(N-k))!} \\ &= (2x)^q P_N^{(m,q})(t), \text{ where } t = (2x)^m. \end{aligned}$$

In this way we come to

$$P_N^{m,q}(t) = \sum_{k=0}^{N} (-1)^k \frac{t^{N-k}}{k!(q+m(k+1))!}. \tag{2.1}$$

In fact, the polynomials $P_N^{m,q}(t)$ depend on two parameters: $m \in N$ and $q \in \{0, 1, \ldots, m-1\}$.

Using (1.3) for $\lambda = 1$, i.e., $n h_{n,m}^{l}(x) = 2x h_{n-1,m}^{l}(x) - m h_{n-m,m}^{l}(x)$, where $n \ge m \ge 1$, we can prove the following theorem:

THEOREM 2.1 *The polynomials $P_N^{m,q}(t)$ satisfy the next recurrence relations:*

$$\begin{aligned} (mN+q) P_N^{(m,q)}(t) &= P_N^{(m,q-1)}(t) - m P_{N-1}^{(m,q)}(t), \quad \textit{for } 1 \le q \le m-1, \\ mN P_N^{(m,0)}(t) &= t P_{N-1}^{(m,m-1)}(t) - m P_{N-1}^{(m,0)}(t), \quad \textit{for } q = 0. \end{aligned}$$

It is interesting to find a recurrence relation for the polynomials $P_N^{(m,q)}(t)$ where the parameters m and q are fixed.

Using the same method as in [**3**] we can prove the following result:

THEOREM 2.2. *The polynomials $P_N^{(m,q)}(t)$ satisfy an $(m+1)$-term recurrence relation of the form*

$$\sum_{i=0}^{m} A_{i,N,q} P_{N+1-i}^{(m,q)}(t) = B_{N,q} t P_N^{(m,q)}(t),$$

where $B_{N,q}$ and $A_{i,N,q}$ $(i = 0, 1, \dots, m)$ are constants depending only on N, m and q.

According to the explicit representation of polynomials $P_N^{(m,q)}(t)$ given by (2.1), we get:

PROPOSITION 2.3 *The polynomials $P_N^{(m,q)}(t)$ have no negative real zeros.*

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