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THE MONGE-AMPERE EQUATION AND AFFINE MAXIMAL SURFACES

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Abstract. Improper affine spheres are affine maximal surfaces. Examples are given as solutions of the Monge-Ampère equation. Three improper affine spheres are thus described by exact differentials.

1. Introduction. A wide class of affine hypersurfaces is formed by improper affine spheres. This class is of particular interest because it forms a subclass of the affine maximal surfaces.

In the theory of equiaffine surfaces an elliptic paraboloid has vanishing mean curvature. The question whether the converse is true leads to the affine Berstein problem; i.e. are complete affine hypersurfaces necessarily elliptic paraboloids?

2. Preliminaries. Suppose that A is an $(n+1)$-dimensional real affine space with associated vector space V and denote by M an n-dimensional differentiable manifold without boundary. A pair (M, x) where $x : M \to A$ is an immersion, is called a *hypersurface of* A. Choosing a fixed origin in A we identify A with its vector space V. A *relative normalization* of a hypersurface (M, x) is a mapping $y : M \to V$ such that in every point of M

(i) the vectors $y, x_1, \dots, x_n$ are linearly independent;

(ii) the vector y_i is contained in the span of $x_1, \dots, x_n$ for every $i = 1, 2, \dots, n$.

As usual an index denotes partial differentiation with respect to a local coordinate system on M. The triple (M, x, y) is called a *relative nomalized hypersurface*. Denote by V^* the dual vector space and by $\langle\ ,\ \rangle : V^* \times V \to R$ the nondegenerate scalar product. Then the differentiable mapping $X : M \to V^*$ such that in every point of M

$$\text{(i)}\quad \langle X, x_i\rangle = 0 \quad \text{for every} \quad i = 1, \dots, n \quad \text{and} \quad \text{(ii)}\ \langle X, y\rangle = 1$$

AMS Subject Classification (1985): Primary 53 A 15 is called the *induced conormal* of the tangential plane. It is uniquely determined by the hypersurface (M, x, y). The immersion $x : M \to A$ is said to be *regular* when $\{X, X_i, \ldots, X_n\}$ forms a basis of V^*. A *metric* on M is induced from (M, x, y) by the formula $G_{ij} = -\langle X_i, X_j\rangle,\ i, j = 1, 2, \ldots, n$. G_{ij} is definite whenever $x : M \to A$ is nondegenerate.

The hypersurface (M, x, y) induces the *relative Weingarten tensor* B_{ij} by $B_{ij} = \langle X_i, Y_j\rangle,\ i, j = 1, 2, \ldots, n$; the *affine mean curvature* H is the first curvature function, i.e. $H = \frac{1}{n} B^i_i$.

A hypersurface (M, x, y) is called *affine maximal* if the affine mean curvature vanishes identically. An affine maximal hypersurface is an *improper affine sphere* if the relative normalization is constant.

Suppose $\| \ \| : V^n \to R$ is a fixed determinant form. Define θ_{ij}, and $g_{ij},\ i, j = 1, \ldots, n$, by

$$\theta_{ij} = \|x_{ij}, x_1, \ldots, x_n\|, \qquad g_{ij} = |\det(\theta_{ij})|^{-1/(n+2)} \theta_{ij}.$$

Then g_{ij} defines a definite tensor field on M which is, by proper choice of orientation, positive definite.

The Laplacian of (M, g_{ij}) induces a relative normalization y by $y = \frac{1}{n}\Delta x$. This normalization is called the *equiaffine normalization* of (M, x).

The introduction above is given by Schneider in [SCHN], where the Weingarten tensor differs by sign from the above definition.

Suppose that Ω is a region in the plane and that an affine surface $\Sigma : \Omega \to A_3$ is given by a differentiable function z as a graph over Ω.

$$\Sigma(x, y) = \begin{bmatrix} x \\ y \\ z(x, y) \end{bmatrix}.$$

Denote by d the determinant of the Hessian of z:

$$(1) \qquad d = z_{xx} z_{yy} - z_{xy}^2.$$

$\Sigma : \Omega \to A_3$ is an affine maximal surface (cf. [CAL], [SCHN], [SI]) if $z : \Omega \to R$ satisfies the Euler-Lagrange equation

$$(2) \qquad d\{z_{xx} d_{yy} + z_{yy} d_{xx} - 2 z_{xy} d_{xy}\} = \tfrac{7}{4}\{z_{xx} d_y^2 + z_{yy} d_x^2 - 2 z_{xy} d_x d_y\}.$$

We assume that $\Sigma : \Omega \to A_3$ only consists of elliptic points, i.e. d is positive everywhere in Ω. $\Sigma : \Omega \to A_3$ is an improper affine sphere if $z : \Omega \to R$ satisfies the Monge-Ampère equation

$$(3) \qquad z_{xx} z_{yy} - z_{xy}^2 = 1.$$

3. Main results. A well-known solution of (1) is given by

$$z = (x^2 + y^2)/2 \tag{4}$$

and describes an elliptic paraboloid.

Whenever d is constant, the equation (2) is satisfied. Nontrivial solutions of (3) are given by the exact differentials

$$dz = \mathrm{tg}x\sqrt{\cos^2 x + y^2}dx + \mathrm{arsh}[y/\cos x]dy,$$
$$dz = \frac{-y\sqrt{1-[y^{-1}\sinh 2x - \cosh 2x]^2}}{\sinh 2x}dx + \frac{1}{2}\arccos[y^{-1}\sinh 2x - \cosh 2x]dy,$$
$$dz = \Big[\frac{1}{2}\{\cos 2x - \mathrm{ctg}2x\sqrt{\sin^2 2x + 4y^2}\} + \sin^2 x\Big]dx + \frac{1}{2}\mathrm{arsh}\Big[\frac{2y}{\sin 2x}\Big]dy.$$

The last differential is defined in a region with $\sin 2x \neq 0$. Suppose $f : R^+ \to R$ is defined by the integral $f(r) = \int_0^r \sqrt{1+\tau^2}d\tau$, and $z : R^2\backslash\{0\} \to R$ is given by $z = f(r)$ with $r^2 = x^2+y^2$. Then z is also a solution of the Monge-Ampère equation (3). A solution of (2) is given by the following surface

$$z = -\tfrac{9}{2}x^{2/3} + \tfrac{1}{2}y^2, \qquad x > 0,\ y \in R. \tag{5}$$

From (3) we get for the Berwald-Blaschke metric and its inverse

$$G = \mathrm{Hess}(z) = \begin{bmatrix} z_{xx} & z_{xy} \\ z_{xy} & z_{yy} \end{bmatrix}, \quad G^{-1} = \begin{bmatrix} z_{yy} & -z_{xy} \\ -z_{xy} & z_{xx} \end{bmatrix}.$$

Like $z : \Omega \to R$ the determinant of the Hessian $d : \Omega \to R$ also induces an affine surface $\Theta : \Omega \to A_3$

$$\Theta(x,y) = \begin{bmatrix} x \\ y \\ d(x,y) \end{bmatrix}.$$

We consider here the elliptic case, i.e. $d : \Omega \to R$ is strictly positive and $\Theta : \Omega \to A_3$ lies in the "upper" half plane. For example consider the elliptic paraboloid. Then $d \equiv 1$, i.e. $\Theta : \Omega \to A_3$ is a plane.

A solution of the Euler-Lagrange equation (2) is given by

$$z(x,y) = -\frac{9}{2}x^{2/3} + \frac{1}{2}y^2, \qquad x > 0,\ y \in R.$$

For this solution one gets $\quad d = z_{xx}z_{yy} - z_{xy^2} = x^{-4/3}$.

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