PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 53 (67), 1993, 122–132

CURVATURE PINCHING FOR ODD-DIMENSIONAL MINIMAL SUBMANIFOLDS IN A SPHERE

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Abstract. Using Gauchman's method, we have improved Simons' pinching constant (for codimension $p \geq 3 - 2/(n-1)$) and Ejiri's Ricci curvature pinching constant for odd-dimensional minimal submanifolds in a sphere.

0. Introduction Let M^n be an n-dimensional compact minimal subman ifold in an $(n+p)$-dimensional Riemannian manifold N^{n+p}. Let h be the second fundamental form of M^n and $f(u) = \|h(u,u)\|^2$ for any $u \in UM$. In [**2,4,5**], Gauchman developed a method which is different from that of Ros [**10,11**], but influenced by Ros' method. By use of this method, Gauchman studied the $f(u)$-pinching problems for minimal submanifolds in S^{n+p} [**4**], totally real minimal submanifolds in $CP^{n+p}(c)$ [**5**], and totally real minimal submanifolds in $HP^{n+p}(1)$ [**2**], respectively. In this paper, we find that Gauchman's method can be used for a study of curvature pinching problems of minimal submanifolds. We apply Gauchman's method and some other techniques to curvature pinching problems of minimal submanifolds in a sphere S^{n+p}. For odd-dimensional minimal submanifolds in a sphere, we have improved Simons' pinching constant (for codimension $p \geq 3 - 2/(n-1)$) (Theorem 2.2) and we have improved Ejiri's Ricci curvature pinching constant (Theorem 3.2). We also obtained a Ricci curvature pinching theorem which generalizes Shen's result for 3-dimensional minimal submanifolds in a sphere (Theorem 3.3). This paper is a part of my Ph.D. thesis (see [**9**]), which includes various results on curvature pinching theorems for minimal submanifolds in a sphere S^{n+p}, totally real minimal submanifolds in a complex projective space $CP^{n+p}(c)$ and totally real minimal submanifolds in a quaternion projective space $HP^{n+p}(1)$, respectively (see also [**7,8**]).

The author would like to express his thanks to Prof. dr. Muharem Avdispahić for his guidances and encouragements. He also thanks Prof. dr. Neda Bokan for her many valuable suggestions and discussions.

AMS Subject Classification (1985): Primary 53 C 40, Secondary 53 C 20

1. Preliminaries Let M be an n-dimensional compact Riemannian manifold which is immersed isometrically in an $(n+p)$-dimensional Riemannian manifold N^{n+p}. We choose a local field of orthonormal frames $e_1, \ldots, e_{n+p}$ in N^{n+p} in such a way that, when restricted to M, vectors $e_1, \ldots, e_n$ are tangent to M. The following conventions for the range of indices will be used

$$1 \le A, B, C, \ldots \le n+p; \qquad 1 \le i, j, k, \ldots \le n;$$
$$n+1 \le \alpha, \beta, \gamma, \ldots \le n+p.$$

Let ω_A be the field of dual frames with respect to the frame field of N^{n+p} chosen above. Then, if they are restricted to M, we have

$$\omega_\alpha = 0, \quad \omega_{\alpha i} = \sum_j h^\alpha_{ij} \omega_j, \quad h^\alpha_{ij} = h^\alpha_{ji}.$$

The second fundamental form of M in N^{n+p} is

$$h(X, Y) = \sum_{\alpha, i, j} h^\alpha_{ij} \omega_i(X) \omega_j(Y) e_\alpha, \quad \text{for } X, Y \in TM. \tag{1.1}$$

Let $UM = \bigcup_{x \in M} UM_x$ and $UM_x = [u \in TM_x : \ \|u\| = 1]$. Thus $UM \to M$ is the unit tangent bundle over M. We define $f(u) = \|h(u,u)\|^2$ for $u \in UM$. Setting $u = \sum_i u^i e_i$, from (1.1) we have

$$f(u) = \sum_\alpha \left(\sum_{ij} h^\alpha_{ij} u^i u^j \right)^2. \tag{1.2}$$

$f(u)$ may be considered as a measure of the degree at which an immersion fails to be totally geodesic.

Let $x \in M$, suppose that $v \in UM_x$ satisfies $f(v) = \max_{u \in UM_x} f(u)$. We shall call v a maximal direction at x (see [**4,5**]). Assume that $e_1 = v$ is a maximal direction; we have at the point x, for any $t, x_2, \ldots, x_n \in R$

$$\left\| h\big(e_1 + t \sum_{k \ne 1} x^k e_k, \ e_1 + t \sum_{k \ne 1} x^k e_k\big) \right\|^2 \le \left[1 + t^2 \sum_{k \ne 1} (x^k)^2 \right]^2 \|h_{11}\|^2. \tag{1.3}$$

Expanding this in term of t, we obtain

$$4t \sum_{\alpha, k \ne 1} x^k h^\alpha_{11} h^\alpha_{1k} + O(t^2) \le 0.$$

It follows that

$$\sum_\alpha h^\alpha_{11} h^\alpha_{1k} = 0, \quad (k \ne 1)$$

which implies that $v = e_1$ is an eigenvector of the $(n \times n)$-matrix $(\sum_\alpha h^\alpha_{11} h^\alpha_{ij})$ at x. Hence, we can choose $e_2, \ldots, e_n$ such that the matrix $(\sum_\alpha h^\alpha_{11} h^\alpha_{ij})$ is diagonalized at x. Therefore we have

$$\sum_\alpha h^\alpha_{11} h^\alpha_{ij} = 0, \quad (i \ne j). \tag{1.4}$$

Once more expanding (1.3) in terms of t, we obtain

$$2t^2\Big[\sum_{\alpha,k\neq 1}\big((h_{11}^\alpha)^2 - h_{11}^\alpha h_{kk}^\alpha - 2(h_{1k}^\alpha)^2\big)(x^k)^2 - 2\cdot\sum_{\alpha,i\neq j,i\neq 1,j\neq 1} h_{1i}^\alpha h_{1j}^\alpha x^i x^j\Big] + O(t^3) \geq 0. \tag{1.5}$$

Since (1.5) must hold for any real x^i, we obtain the following variational inequality

$$\sum_\alpha\big[(h_{11}^\alpha)^2 - h_{11}^\alpha h_{kk}^\alpha - 2(h_{1k}^\alpha)^2\big] \geq 0, \quad (k\neq 1). \tag{1.6}$$

Let M be a Riemannian manifold and L be a covariant tensor field on M of the type $(0,k)$. At any $x\in M$, L can be considered as a multilinear mapping $L:\ T_xM\times\ldots\times T_xM\to R$. Suppose that $v\in UM_x$ satisfies $L(v,\ldots,v) = \max_{u\in UM_x} L(u,\ldots,u)$. We shall call v a maximal direction at x with respect to L. For any $x\in M$, we set $f_L(x) = L(v,\ldots,v)$, where v is a maximal direction at x with respect to L. We have the following generalized Bochner's lemma.

Lemma 1.1 (Proposition 3.1 of [**5**]). *Let M be a compact Riemannian manifold and L be a covariant tensor field on M of the type $(0,k)$. If $(\Delta L)(v,\ldots,v)\geq 0$ for any maximal direction v with respect to L, where Δ denotes the Laplace operator, then $f_L =$ constant on M and $(\Delta L)(v,\ldots,v) = 0$ for any maximal direction v.*

Let M be an n-dimensional compact submanifold in N^{n+p}. For any point $x\in M$, let $e_1,\ldots,e_{n+p}$ be a frame chosen above at x such that $e_1 = v$ is a maximal direction at x, and $\sum_\alpha h_{11}^\alpha h_{ij}^\alpha = 0$ for $i\neq j$. Let us define a 4-covariant tensor field L on M by the formula

$$L(X,Y,Z,W) = \langle h(X,Y), h(Z,W)\rangle, \tag{1.7}$$

where $X,Y,Z,W\in T_x(M)$, $x\in M$. It is clear that $f(u) = L(u,u,u,u) = \|h(u,u)\|^2$ for any $u\in UM$. We shall write $(\Delta L)_{ijkl} = (\Delta L)(e_i,e_j,e_k,e_l)$.

Therefore we have proved the following lemma ensuing from (1.2), (1.4), (1.6), (1.7) and Lemma 1.1.

Lemma 1.2 *Let M be a compact n-dimensional submanifold in an $(n+p)$--dimensional Riemannian manifold N^{n+p}. Let $b_{ij} = \sum_\alpha h_{11}^\alpha h_{ij}^\alpha$. With respect to the frame field chosen above, we have at any point $x\in M$*

$$f(v) = b_{11} = \sum_\alpha (h_{11}^\alpha)^2 = \max_{u\in UM_x}[\|h(u,u)\|^2], \tag{1.8}$$

$$\frac{1}{2}(\Delta L)_{1111} = \sum_{\alpha,k}(h_{11k}^\alpha)^2 + \sum_{\alpha,k} h_{11}^\alpha h_{11kk}^\alpha, \tag{1.9}$$

$$b_{ij} = 0 \quad (i\neq j), \tag{1.10}$$

$$2\sum_{\alpha}(h_{1k}^{\alpha})^2 + b_{kk} - f(v) \leq 0, \quad (k \neq 1). \tag{1.11}$$

If $(\Delta L)_{1111} \geq 0$ *for any maximal direction* $e_1 = v$, *then* $f(v) = b_{11} =$ *constant on* M *and* $(\Delta L)_{1111} = 0$ *for any maximal direction* $e_1 = v$.

2. Scalar curvature pinching for odd-dimensional minimal submanifolds in $\mathbf{S}^{n+p}$. Now we let ambient space N^{n+p} be a unit sphere S^{n+p} of dimension $n+p$. Let M^n be an n-dimensional compact minimal submanifold in S^{n+p}. Gauss-Codazzi-Ricci equations of M^n are

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_{\alpha}(h_{ik}^{\alpha}h_{jl}^{\alpha} - h_{il}^{\alpha}h_{jk}^{\alpha}), \tag{2.1}$$

$$h_{ijk}^{\alpha} = h_{ikj}^{\alpha}, \tag{2.2}$$

$$R_{\alpha\beta ij} = \sum_{k}(h_{ik}^{\alpha}h_{jk}^{\beta} - h_{jk}^{\alpha}h_{ik}^{\beta}), \tag{2.3}$$

where R_{ijkl} and $R_{\alpha\beta ij}$ are the respective curvature tensors for tangent connection and the normal connection of M^n and h_{ijk}^{α} is the covariant derivative of h_{ij}^{α}.

By (2.1) the Ricci curvature and scalar curvature of M^n are

$$R_{ij} = (n-1)\delta_{ij} - \sum_{\alpha,k} h_{ik}^{\alpha}h_{kj}^{\alpha}, \tag{2.4}$$

$$R = n(n-1) - \|\sigma\|^2, \tag{2.5}$$

where $\|\sigma\|^2 = \sum_{\alpha,i,j}(h_{ij}^{\alpha})^2$.

It is well known [**1,13**] that if the lenght square $\|\sigma\|^2$ of the second fundamental form on M^n satisfies

$$\|\sigma\|^2 \leq \frac{n}{2-1/p}$$

everywhere, then either $\|\sigma\|^2 = 0$ (i.e. M^n is totally geodesic) or

$$\|\sigma\|^2 = \frac{n}{2-1/p}.$$

In the latter case M^n is either a Clifford hypersurface or a Veronese surface in S^4. In [**8**], we have improved Simons' pinching constant for higher codimension. In fact, we have established

THEOREM 2.1 [**8**]. *Let* M^n *be an* n*-dimensional* $(n \geq 2)$ *compact minimal submanifold in* S^{n+p}. *If*

$$\|\sigma\|^2 \leq \frac{n(3n-2)}{5n-4}, \tag{2.6}$$

then M^n *is either a totally geodesic submanifold or a Veronese surface in* S^4.

In this section, we will improve the theorem above for odd-dimensional minimal subamnifolds in S^{n+p}. We will prove

THEOREM 2.2. *Let M^n be a compact n-dimensional $(n \geq 3)$ minimal submanifold in S^{n+p}, and let n be odd. If*

$$\|\sigma\|^2 \leq \frac{n(3n-5)}{5n-9}, \tag{2.7}$$

then M^n is either a totally geodesic submanifold or $n = 3$ and $\|\sigma\|^2 = 2$ on M^3 and the second fundamental form is given by

$$\begin{aligned} &(h^4_{ij}) = \begin{pmatrix} 1/\sqrt{2} & 0 & 0 \\ 0 & -1/\sqrt{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad (h^5_{ij}) = \begin{pmatrix} 0 & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ &(h^\alpha_{ij}) = 0, \ \ \alpha \geq 6. \end{aligned} \tag{2.8}$$

Remark 2.1. For odd-dimensional minimal submanifolds in S^{n+p}, our pinching constant $n(3n-5)/(5n-9)$ is independent of the codimension p of M^n and is not smaller than Simons' pinching constant $n/(2-1/p)$ in case of $p \geq 3 - 2/(n-1)$ (i.e. $n = 3$ and $p \geq 2$; $n \geq 5$ and $p \geq 3$).

Remark 2.2. Theorem 2.2 improves Theorem 2.1 for odd-dimensional minimal submanifolds in a sphere S^{n+p}.

COROLLARY 2.1 [**12**]. *Let M^3 be a compact 3-dimensional minimal submanifold in S^{3+p}. If*

$$\|\sigma\|^2 < 2, \tag{2.9}$$

then M^3 is a totally geodesic submanifold.

Remark 2.3. In [**4**], Gauchman obtained results (Theorem 3 and Theorem 4 of [**4**]) of kind described in Theorem 2.1 and Theorem 2.2 in which $f(u)$ was used instead of $\|\sigma\|^2$ for minimal submanifolds in a sphere, where $f(u) = \|h(u,u)\|^2$ for any $u \in UM$.

Proof of Theorem 2.2. We begin with Lemma 1.2. All the calculations below will be made at a point $x \in M$, unless otherwise stated. By Ricci identities, (2.2) and (1.10), from (1.9) we get

$$\begin{aligned} \frac{1}{2}(\Delta L)_{1111} &\geq \sum_{\alpha,i} h^\alpha_{11} h^\alpha_{1i1i} \\ &= \sum_{\alpha,i} (h^\alpha_{11} h^\alpha_{ii} R_{i11i} + (h^\alpha_{11})^2 R_{1i1i}) + \sum_{\alpha,\beta,i} h^\alpha_{11} h^\beta_{1i} R_{\beta\alpha 1i}. \end{aligned} \tag{2.10}$$

Making use of (2.1), (1.10) and (2.3), one easily sees that

$$\begin{aligned} &\sum_{\alpha,i} (h^\alpha_{11} h^\alpha_{ii} R_{i11i} + (h^\alpha_{11})^2 R_{1i1i}) \\ &= nf(v) + \sum_{\alpha,k} b_{kk} (h^\alpha_{1k})^2 - \sum_k (b_{kk})^2 - f(v) \sum_{\alpha,k} (h^\alpha_{1k})^2, \end{aligned} \tag{2.11}$$

$$\sum_{\alpha,\beta,i} h_{11}^{\alpha} h_{1i}^{\beta} R_{\beta\alpha 1i} = \sum_{\alpha,k} b_{kk}(h_{1k}^{\alpha})^2 - f(v)\sum_{\alpha,k}(h_{1k}^{\alpha})^2.$$

Substituting (2.11) and (2.12) into (2.10), we obtain

$$\begin{aligned}\frac{1}{2}(\Delta L)_{1111} \geq & nf(v) + 2\sum_{\alpha,k\neq 1} b_{kk}(h_{1k}^{\alpha})^2 \\ & - \sum_{k\neq 1}(b_{kk})^2 - 2f(v)\sum_{\alpha,k\neq 1}(h_{1k}^{\alpha})^2 - f(v)\sum_{\alpha}(h_{11}^{\alpha})^2.\end{aligned} \tag{2.13}$$

From (1.8) and (1.11) it follows that

$$2\sum_{\alpha}(h_{1k}^{\alpha})^2 \leq f(v) - b_{kk} \leq f(v) + \sqrt{\sum_{\alpha}(h_{11}^{\alpha})^2 \sum_{\alpha}(h_{kk}^{\alpha})^2} \leq 2f(v)$$

i.e. $\sum_{\alpha}(h_{1k}^{\alpha})^2 \leq f(v)$. Combining this with an elementary inequality, we find

$$\begin{aligned}2\sum_{\alpha,k\neq 1} b_{kk}(h_{1k}^{\alpha})^2 &\geq -\frac{1}{a}\sum_{k\neq 1}(b_{kk})^2 - a\sum_{k\neq 1}\left(\sum_{\alpha}(h_{1k}^{\alpha})^2\right)^2 \\ &\geq -\frac{1}{a}f(v)\sum_{\alpha,k\neq 1}(h_{kk}^{\alpha})^2 - af(v)\sum_{\alpha,k\neq 1}(h_{1k}^{\alpha})^2,\end{aligned} \tag{2.14}$$

where $a > 0$ is an arbitrary real number. On the other hand $(b_{kk})^2 \leq f(v)\sum_{\alpha}(h_{kk}^{\alpha})^2 \leq f(v)^2$, $(f(v) + b_{kk})(f(v) - b_{kk}) \geq 0$. Combining this with (1.11), we have $b_{kk} \geq -f(v)$, therefore we get the following estimate

$$2\sum_{\alpha,k\neq 1} b_{kk}(h_{1k}^{\alpha})^2 \geq -2f(v)\sum_{\alpha,k\neq 1}(h_{1k}^{\alpha})^2. \tag{2.15}$$

Combining (2.14) with (2.15), we obtain the following estimate

$$\begin{aligned}2\sum_{\alpha,k\neq 1} b_{kk}(h_{1k}^{\alpha})^2 &= b\sum_{\alpha,k\neq 1} b_{kk}(h_{1k}^{\alpha})^2 + (2-b)\sum_{\alpha,k\neq 1} b_{kk}(h_{1k}^{\alpha})^2 \\ &\geq -\frac{bf(v)}{2a}\sum_{\alpha,k\neq 1}(h_{kk}^{\alpha})^2 - (2 - b + \frac{ab}{2})f(v)\sum_{\alpha,k\neq 1}(h_{1k}^{\alpha})^2,\end{aligned} \tag{2.16}$$

where $a > 0$ and $2 \geq b \geq 0$ are arbitrary real numbers.

By (2.13) and (2.16), we have

$$\begin{aligned}\frac{1}{2}(\Delta L)_{1111} \geq & nf(v) - (4 - b + \frac{ab}{2})f(v)\sum_{\alpha,k\neq 1}(h_{1k}^{\alpha})^2 \\ & - \frac{b}{2a}f(v)\sum_{\alpha,k\neq 1}(h_{kk}^{\alpha})^2 - \sum_{k\neq 1}(b_{kk})^2 - f(v)^2.\end{aligned} \tag{2.17}$$

We can write $b_k = b_{kk} = \sum_\alpha h^\alpha_{11} h^\alpha_{kk}$. By (1.8) and minimality of the immersion, we have

$$-f(v) \le b_k \le f(v), \quad (k \ne 1). \tag{2.18}$$

$$\sum_{k=2}^n b_k = \sum_{k=2}^n b_{kk} = -f(v). \tag{2.19}$$

Because we assume that n is an odd number, it can easily be seen that the convex function $f(b_2, \dots, b_n) = \sum_{k=2}^n (b_k)^2$ of $(n-1)$ variables $b_2, \dots, b_n$ subject to the linear constraints (2.18) and (2.19) attains its maximal value when (after suitable renumbering of $e_1, \dots, e_n$) (see [**5**])

$$b_2 = \dots = b_m = -b_{m+1} = \dots = -b_{2m} = f(v); \;\; b_{2m+1} = 0,$$

where $n = 2m+1$. Therefore, we have

$$\sum_{k\ne 1} (b_{kk})^2 \le (n-2) f(v)^2. \tag{2.20}$$

We also know, by the Cauchy inequality, that

$$\sum_{k\ne 1} (b_{kk})^2 \le f(v) \sum_{\alpha, k\ne 1} (h^\alpha_{kk})^2. \tag{2.21}$$

Combining (2.20) with (2.21), we have

$$\begin{aligned} -\sum_{k\ne1}(b_{kk})^2 &= -(1 - \frac{b}{2(n-1)a}) \sum_{k\ne1} (b_{kk})^2 - \frac{b}{2(n-1)a} \sum_{k\ne1} (b_{kk})^2 \\ &\ge -(1 - \frac{b}{2(n-1)a}) f(v) \sum_{\alpha,k\ne1} (h^\alpha_{kk})^2 - \frac{(n-2)b}{2(n-1)a} f(v)^2. \end{aligned} \tag{2.22}$$

Substituing (2.22) into (2.17), we obtain

$$\begin{aligned} &\frac{1}{2}(\Delta L)_{1111} \\ &\ge f(v)\Big[n - (4 - b + \frac{ab}{2}) \sum_{\alpha,k\ne1} (h^\alpha_{1k})^2 - (1 + \frac{(n-2)b}{2a(n-1)}) \sum_{\alpha,k} (h^\alpha_{kk})^2\Big]. \end{aligned} \tag{2.23}$$

Let

$$4 - b + \frac{ab}{2} = 2\frac{1 + (n-2)b}{2(n-1)a}, \quad \text{i.e.} \quad b = \frac{4(n-1)a}{3n - 5 - (n-1)(a-1)^2}.$$

Noting that $\|\sigma\|^2 = \sum_{\alpha,i,j} (h^\alpha_{ij})^2 \ge \sum_\alpha (h^\alpha_{kk})^2 + 2\sum_{\alpha,k\ne1} (h^\alpha_{1k})^2$, choosing $a = 1$, we obtain from (2.23)

$$\frac{1}{2}(\Delta L)_{1111} \ge f(v)\big[n - \frac{5n-9}{3n-5}\|\sigma\|^2(x)\big]. \tag{2.24}$$

By (2.7), $(\Delta L)_{1111} \geq 0$. We obtain $(\Delta L)_{1111} = 0$ from Lemma 1.2. Thus, if $f(v) = 0$, then $\|h(u,u)\|^2 = 0$ for any $u \in UM$, so that M^n is totally geodesic. If $f(v) \neq 0$, then $\|\sigma\|^2(x) = n(3n-5)/(5n-9)$, so that (2.13) - (2.24) all are equalities with $a = 1$ and $b = 4(n-1)/(3n-5)$. We easily get $n = 3$, and we have $h^\alpha_{11} = -h^\alpha_{22}$, $h^\alpha_{33} = 0$, $h^\alpha_{13} = h^\alpha_{23} = 0$, $\sum_\alpha (h^\alpha_{12})^2 = f(v)$ and $\|\sigma\|^2 = 2$ on M^3. By (1.10), we can choose $e_4 = h(e_1,e_1)/\sqrt{f(v)}$ and $e_5 = h(e_1,e_2)/\sqrt{f(v)}$. Therefore we have (2.8) and that completes the proof.

3. Ricci curvature pinching for odd-dimensional minimal submanifolds in $\mathbf{S}^{n+p}$. Ejiri [**3**] obtained the following well known Ricci curvature pinching theorem

THEOREM 3.1. *Let M^n be a compact n-dimensional $(n \geq 4)$ minimal submanifold in S^{n+p}. If the Ricci curvature of M^n satisfies*

$$\mathrm{Ric}(M^n) \geq n-2, \tag{3.1}$$

then M^n is totally geodesic, or $n = 2m$ and $M^n = S^m(\sqrt{1/2}) \times S^m(\sqrt{1/2})$ or $n = 4$ and $M^4 = CP^2(4/3) \to S^7$.

It is generally considered that the above theorem is the best possible result, but, in fact, Ejiri's theorem above is only the possible best result for even-dimensional minimal submanifolds in S^{n+p}. In this section we establish the following best possible Ricci curvature pinching theorem for odd-dimensional minimal submanifolds in S^{n+p}

THEOREM 3.2. *Let M^n be a compact n-dimensional $(n \geq 5)$ minimal submanifold in S^{n+p}. Assume that n is odd. If the Ricci curvature of M^n satisfies*

$$\mathrm{Ric}(M^n) \geq n-2-1/(n-1), \tag{3.2}$$

then M^n is either a totally geodesic submanifold or $n = 5$ and $R_{11} = R_{22} = R_{33} = R_{44} = 3 - 1/4$, $R_{55} = 4$ and $\|\sigma\|^2 = 5$ on M^5.

Remark 3.1. Our Ricci curvature pinching constant $(n-2-1/(n-1))$ is better than Ejiri's $(n-2)$ for odd-dimensional minimal submanifold M^n in S^{n+p}.

Proof of Theorem 3.2 By (2.13), (2.15) and (2.20), we get

$$\frac{1}{2}(\Delta L)_{1111} \geq nf(v) - 4f(v)\sum_{\alpha,k\neq 1}(h^\alpha_{1k})^2 - (n-1)f(v)^2. \tag{3.3}$$

From (2.4), our assumption (3.2) and from: $R_{11} = (n-1) - f(v) - \sum_{\alpha,k\neq 1}(h^\alpha_{1k})^2$, we have

$$\sum_{\alpha,k\neq 1}(h^\alpha_{1k})^2 \leq \frac{n}{n-1} - f(v). \tag{3.4}$$

Substituting (3.4) into (3.3), we get

$$\begin{aligned}\frac{1}{2}(\Delta L)_{1111} &\geq nf(v) - 4f(v)\Big(\frac{n}{n-1} - f(v)\Big) - (n-1)f(v)^2 \\ &= (n-5)f(v)\Big(\frac{n}{n-1} - f(v)\Big).\end{aligned} \tag{3.5}$$

By (3.4) we know that $n/(n-1) - f(v) \geq 0$. Thus $(\Delta L)_{1111} \geq 0$. By Lemma 1.2, $(\Delta L)_{1111} = 0$ and $f(v) =$ constant on M^n. Therefore it follows that $f(v) = 0$, or $f(v) = n/(n-1)$, or $n = 5$.

(1) Case $f(v) = 0$. M^n is totally geodesic.

(2) Case $f(v) = n/(n-1)$. In this case (2.20) is an equality. Thus for all α we get (after suitable renumbering of $e_1, \ldots, e_n$)

$$h^{\alpha}_{11} = \ldots = h^{\alpha}_{mm} = -h^{\alpha}_{m+1\ m+1} = \ldots = -h^{\alpha}_{2m\ 2m},\ h^{\alpha}_{nn} = 0. \tag{3.6}$$

On the other hand, by (3.4), we have $h^{\alpha}_{1k} = 0,\ k \neq 1,\ \alpha = n+1, \ldots, n+p$. Since by (3.6), directions $e_1, \ldots, e_{2m}$ all are maximal, it follows that

$$h^{\alpha}_{ij} = 0, \quad i \neq n,\ j \neq i,\ \alpha = n+1, \ldots, n+p. \tag{3.7}$$

This implies $h^{\alpha}_{ij} = 0,\ i \neq j,\ \alpha = n+1, \ldots, n+p$, i.e., M^n is a submanifold with a flat normal connection. From (3.6) and (3.7), we have

$$\|\sigma\|^2 = \sum_{\alpha,i,j} (h^{\alpha}_{ij})^2 = \sum_{\alpha,k} (h^{\alpha}_{kk})^2 = n. \tag{3.8}$$

By Kenmotsu's theorem [**6**], we have $M^n = S^k(\sqrt{k/n})\ S^{n-k}(\sqrt{(n-k)/n})$ and $p = 1$. But it contradicts the following

$$h_{11} = \ldots = h_{mm} = -h_{m+1\ m+1} = \ldots = -h_{2m\ 2m} = \sqrt{n/(n-1)},\ h_{nn} = 0. \tag{3.9}$$

Thus $f(v) = n/(n-1)$ is false. We have $f(v) = 0$, i.e. M^n is totally geodesic.

(3) Case $n = 5$ and $f(v) \neq n/(n-1)$. By Lemma 1.2, $f(v) =$ constant on M^5 and (3.5) is an equality. Thus, (2.13), (2.15), (2.20), (3.3) - (3.5) all are identities and $R_{11} = 3 - 1/4$. By (2.20), we have for all α

$$h^{\alpha}_{11} = h^{\alpha}_{22} = -h^{\alpha}_{33} = -h^{\alpha}_{44},\ h^{\alpha}_{55} = 0. \tag{3.10}$$

By (2.4) (in this case), for all α we have $h^{\alpha}_{15} = 0$. Because (3.10) implies that the directions e_1, e_2, e_3 and e_4 are all maximal, we have $h^{\alpha}_{k5} = 0$ and

$$R_{11} = R_{22} = R_{33} = R_{44} = 3 - 1/4,\ R_{55} = 4. \tag{3.11}$$

Thus $R = 15$ and $\|\sigma\|^2 = 5$ on M^5. By (1.11) and (3.4), we find that $5/12 \leq f(v) < 5/4$. From (2.15), we also know that $h^{\alpha}_{12} = h^{\alpha}_{34} = 0$ and the proof is completed.

Neither Theorem 3.1 nor Theorem 3.2 yields any results for 3-dimensional minimal submanifolds in a sphere. For that case we establish the following theorem

Theorem 3.3. *Let M^3 be a 3-dimensional compact minimal submanifold in S^{3+p}. If the Ricci curvature of M^3 satisfies*

$$\mathrm{Ric}(M^3) \geq 1, \tag{3.12}$$

then M^3 is either totally geodesic, or $R_{11} = R_{22} = 1,\ R_{33} = 2$ and $\|\sigma\|^2 = 2$ on M^3 and the second fundamental form is given by (2.8).

COROLLARY 3.1 [**12**]. *Let M^3 be a 3-dimensional compact minimal submanifold in S^{3+p}. If the Ricci curvature of M^3 satisfies*

$$Ric(M^3) > 1, \tag{3.13}$$

then M^3 is totally geodesic.

Proof of Theorem 3.3. By $b_{kk} \geq -f(v)$ and the 3-dimensional minimality, we can see that

$$b_{22} \leq 0,\ b_{33} \leq 0,\ \sum_{k\neq 1}(b_{kk})^2 \leq \left(\sum_{k\neq 1} b_{kk}\right)^2 = (b_{11})^2. \tag{3.14}$$

By the definition of b_{ij} (see Lemma 1.2), we have from (2.4)

$$-\sum_{\alpha,k\neq 1}(h^\alpha_{1k})^2 = R_{11} - 2 + b_{11}.$$

From (3.14) and (1.11), we get

$$\sum_{\alpha,k\neq 1} b^\alpha_{kk}(h^\alpha_{1k})^2 \geq \frac{1}{2}\sum_{k\neq 1} b_{kk}(b_{11} - b_{kk}) = -\frac{1}{2}\sum_k (b_{kk})^2. \tag{3.16}$$

Substituting (3.15) into (2.13) in case of $n = 3$ and using (3.14), we come to

$$\frac{1}{2}(\Delta L)_{1111} \geq -f(v) + 2\sum_{\alpha,k\neq 1} b_{kk}(h^\alpha_{1k})^2 + 2f(v)R_{11}. \tag{3.17}$$

Applying (2.15) and (3.16) on (3.17), by (3.14)

$$\begin{aligned}\frac{1}{2}(\Delta L)_{1111} &\geq 2f(v)R_{11} - f(v) + f(v)(R_{11} - 2 + b_{11}) - \frac{1}{2}\sum_k (b_{kk})^2 \\ &\geq 3f(v)(R_{11} - 1).\end{aligned} \tag{3.18}$$

By Lemma 1.2, (3.12) and (3.18) imply that either $f(v) = 0$, i.e. M^3 is totally geodesic, or $R_{11} = 1$. In the latter case, (3.14) - (3.18) all are identities. By a similar argument as in the proof of Theorem 3.2, we have

$$R_{11} = R_{22} = 1,\ \ R_{33} = 2. \tag{3.19}$$

Thus $\|\sigma\|^2 = 6 - R = 2$ on M^3. So, we complete the proof of Theorem 3.3 from Theorem 2.2.

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(Received 24 07 1992)
(Revised 03 03 1993)