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THE EXISTENCE PROBLEM OF HYPERBOLIC STRUCTURES ON VECTOR BUNDLES

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Abstract. Hyperbolic structures are defined on differentiable manifolds by Libermann [**7**] and then studied by many other authors. We extend these structures to arbitrary vector bundles and give necessary and sufficient conditions for the existance of such structures.

1. Hyperbolic structures on vector bundles. In this section (E, π, M) denotes a real vector bundle of rank n over M.

Definition 1.1. A hyperbolic complex (paracomplex) structure of the vector bundle E is a product structure P (i.e. P is a section of the vector bundle $L(E, E)$, with $P^2 = I,\ P \neq \pm I$), such that the vector subbundles corresponding to the eigenvalues ± 1 of P have the same rank. If $E = TM$, then P is called an almost hyperbolic complex (almost paracomplex) structure of M. A differentiable manifold endowed with an almost hyperbolic complex structure P is called a hyperbolic complex manifold if one of the following equivalent conditions is satisfied: (i) P is integrable; (ii) the Nijenhuis tensor associated to P vanishes; (iii) the eigendistributions P^+ and P^- are integrable.

PROPOSITION 1.1. *If E is endowed with a hyperbolic complex structure P, then $n = 2k$ and P can be expressed locally by*

$$P = \begin{pmatrix} 0 & I_k \\ I_k & 0 \end{pmatrix}. \tag{1.1}$$

Moreover, changes of local charts are given by nondegenerate matrices of the form

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix}, \tag{1.2}$$

where A and B are matrices of order k.

The proof is similar to that given in [**6**] for the case when $E = TM$. In that case (1.2) leads to the para-Cauchy-Riemann equations.

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Definition 1.2. A hyperbolic Hermitian (para-Hermitian) structure (P, g) of E is a product structure P which is skew symmetric with respect to the pseudo-Riemannian structure g. (P, g) is called a hyperbolic Hermitian structure provided the following conditions hold:

$$P^2 = I, \;\; P \neq \pm I, \;\; g(Ps, s') + g(s, Ps') = 0, \qquad \forall s, s' \in \Gamma(E). \tag{1.3}$$

If $E = TM$ then (P, g) is called an almost hyperbolic Hermitian (almost para-Hermitian) structure of M. An almost hyperbolic Hermitian structure on M is a hyperbolic Hermitian structure if P is integrable.

Remark that the notion of almost paracomplex (resp. almost para-Hermitian) structure on a differentiable manifold was defined in [**7**] and Definition 1.1 (resp. Definition 1.2) only extended it to arbitrary vector bundles.

In the literature there are several examples of hyperbolic Hermitian structures on manifolds [**2**,**4**,**6**,**7**,**10**]. Now we give an example of a hyperbolic Hermitian structure on a vector bundle E. As among the vector bundles, the tangent and cotangent bundles play an important role, we take $E = TM \oplus T^*M$ where (M, h) is a Riemannian manifold. We consider the musical isomorphism $^\#$ defined by h that is each tangent vector $X \in TM$ defines a cotangent vector $X^\# = h(X, .) \in T^*M$ and conversely, each cotangent vector $w \in T^*M$ defines a tangent vector $w^\# \in TM$ such that $h(w^\#, Y) = w(Y), \;\; \forall Y \in TM$. On the above vector bundle E, we define $P : E \to E$ by $P(X + w) = w^\# + X^\#$. Let g be a pseudo-Riemannian structure of E defined by $g(X + w, Y + \Theta) = h(X, Y) - h(w^\#, \Theta^\#)$. It happens that (P, g) is an example of a hyperbolic Hermitian structure on E.

Proposition 1.2. *Let (P, g) be a hyperbolic Hermitian structure of E. Then: (i) P is a hyperbolic complex structure of E and rank $E = 2k$; (ii) g is a pseudo-Riemannian structure of signature (k, k) ; (iii) the eigensubbundles P^+ and P^- (corresponding to the eigenvalues ± 1) are maximally isotropic with respect to g.*

Proof. (i) Since g is nondegenerate on an arbitrary chart of E, then there is a local section s_1 of E with $g(s_1, s_1) \neq 0$. We suppose $g(s_1, s_1) > 0$, the other case being similar. It follows that s_1 and Ps_1 are orthogonal to each other with respect to g. If $\operatorname{rank} E > 2$, then let s_2 be a local section orthogonal to both s_1 and Ps_1 with respect to g. We obtain the independent local sections s_1, Ps_1, s_2, Ps_2. Proceeding further, we get $\{s_1, Ps_1, \ldots, s_k, Ps_k\}$ to be a local orthogonal basis of E. Moreover, rank $E = 2k$. (ii) The signature of g is (k, k) since $g(s_i, s_i) > 0$ and $g(Ps_i, Ps_i) < 0, \;\; i = \overline{1, k}$. Let us denote $u_i = s_i + Ps_i$ and $v_i = s_i - Ps_i, \;\; i = \overline{1, k}$. We obtain that $\{u_i\}_{i=1,k}$ and $\{v_i\}_{i=1,k}$ are local bases respectively for P^+ and P^-. In order to prove (iii) we remark that P^+ and P^- are isotropic, that is g restricted either to P^+ or to P^- is zero. Moreover, P^+ and P^- are maximal since thay have the same rank and E splits inte direct sum $E = P^+ \oplus P^-$. $\square$

Definition 1.3. [**1**]. If w is a symplectic structure of E, then any vector subbundle T of E is isotropic with respect to w if the restriction of w to T is zero. T is called a Lagrangian vector subbundle of E if, moreover, T is maximal.

PROPOSITION 1.3. *The following assertions are equivalent:*

(i) *E admits a hyperbolic Hermitian structure (P, g);*

(ii) *E can be endowed with a structure (P, Ω), where Ω is a symplectic structure and P is a product structure which is skew symmetric with respect to Ω;*

(iii) *E can be endowed with a structure (g, Ω), where g is a pseudo-Riemannian structure and Ω is a symplectic structure such that $E = T_1 \oplus T_2$ is a Whitney sum of two vector subbundles T_1 and T_2 which are isotropic with respect to both g and Ω and the following relation holds:*

$$g(s,t) = \Omega(s,t), \ \ \forall s \in \Gamma(M, T_1), \ \ t \in \Gamma(M, T_2). \tag{1.4}$$

Proof. (i) $\Rightarrow$ (ii). We construct Ω by:

$$\Omega(s, s') = g(Ps, s'), \ \ \forall s, s' \in \Gamma(E). \tag{1.5}$$

(ii) $\Rightarrow$ (i). We define $g(s, s') = \Omega(Ps, s'), \ \ s, s' \in \Gamma(E)$.

(i) $\Rightarrow$ (iii). Defining Ω by the relation (1.5), we remark that the eigensubbundles $T_1 = P^+$ and $T_2 = P^-$ satisfy the above conditions.

(iii) $\Rightarrow$ (i). We construct on E a product structure P by having T_1 and T_2 as the eigensubbundles. For every $s = s_1 + s_2$ and $s' = s_1' + s_2'$, with $s_i, s_i' \in \Gamma(M, T_i), \ \ i = 1, 2$, we have:

$$\begin{aligned} g(Ps, s') &= g(Ps_1 + Ps_2, s_1' + s_2') = g(s_1 - s_2, s_1' + s_2') = g(s_1, s_2') - g(s_2, s_1') \\ &= \Omega(s_1, s_2') - \Omega(s_2, s_1'). \end{aligned}$$

In a similar way, we get $g(s, Ps') = \Omega(s_1', s_2) - \Omega(s_1, s_2')$. It follows that P is skew symmetric with respect to g. □

We remark that the above result gives the conditions for any two structures among P, g and Ω to imply the existence of the third. It is easily seen that T_1 and T_2 are Lagrangians with respect to Ω.

We call Ω the associated symplectic structure of (P, g). If $E = TM$, then Ω is called the associated 2-form of M.

In spite of the Hermitian case, when the existence of a commplex (resp. product) structure assures the existence of a Hermitian (resp. Riemannian product) structure of E, in the hyperbolic case, neither the existance of a pseudo-Riemannian structure g of the signature (k, k), nor the existence of a product structure P, does not imply alone the existence of a hyperbolic Hermitian structure (P, g), as for example the sphere S^2.

Related to the assertion of Proposition 1.1 is:

PROPOSITION 1.4. *Let E be endowed with a hyperbolic Hermitian structure (P, g) and rank $E = 2k$. Then, locally, we have:*

$$P = \begin{pmatrix} 0 & I_k \\ I_k & 0 \end{pmatrix}, \ \ \textit{and} \ \ g = \begin{pmatrix} -I_k & 0 \\ 0 & I_k \end{pmatrix}. \tag{1.6}$$

as well as

$$P = \begin{pmatrix} I_k & 0 \\ 0 & -I_k \end{pmatrix}, \quad and\ g = \begin{pmatrix} 0 & I_k \\ I_k & 0 \end{pmatrix}. \tag{1.7}$$

Proof. Let s_1 be a local section of E with $g(s_1, s_1) = -1$. It follows that $\{s_1, Ps_1\}$ is orthonormal with respect to g. If $\operatorname{rank} E > 2$, then let s_2 be a local section of E with $g(s_2, s_2) = -1$ and such that $\{s_1, Ps_1, s_2\}$ is orthonormal with respect to g, [**9**]. Proceeding further, we get a local orthonormal basis $\{s_1, \ldots, s_k, Ps_1, \ldots, Ps_k\}$ such that (1.6) holds good. To prove (1.7), let $\{u_1, \ldots, u_{2k}\}$ be a local basis, where $u_i = (Ps_i + s_i)/\sqrt{2}$, $u_{k+i} = (Ps_i - s_i)/\sqrt{2}$, $i = \overline{1,k}$.

Now, by using [**5**], we obtain the following:

THEOREM 1.1. *A vector bundle E admits a hyperbolic Hermitian structure if and only if the structural group acting on its frame bundle can be reduced to the group of the matrices of the form*

$$\begin{pmatrix} S & 0 \\ 0 & ({}^tS)^{-1} \end{pmatrix}, \qquad S \in \mathrm{GL}(k, R), \text{ where rank } E = 2k. \tag{1.8}$$

We recall that a $2k$-dimensional manifold M is endowed with an almost cotangent structure if its structural group can be reduced to the matrices of the form:

$$\begin{pmatrix} S & 0 \\ B & (t_S)^{-1} \end{pmatrix}, \qquad S \in \mathrm{GL}(k, R),\ B \in \mathrm{gl}\,(k, R),$$

such that ${}^tSB = {}^tBS$, [**3**].

COROLLARY 1.1. *An almost hyperbolic Hermitian structure of a manifold M is a particular case of an almost cotangent structure.*

We recall that an almost hyperbolic Hermitian manifold (M, P, g) is almost hyperbolic Kaehlerian if its associated 2-form Ω is closed. Moreover, if P is integrable, then (M, P, g) is called hyperbolic Kaehlerian.

2. The existence problem of the hyperbolic structures. By Proposition 1.4, it follows that a trivial vector bundle of even rank admits a hyperbolic Hermitian structure and, therefore, any parallelizable manifold of even dimension can be endowed with an almost hyperbolic Hermitian structure. Particularly, every even dimensional Lie group admits an almost hyperbolic Hermitian structure which need not be integrable or almost Kaehlerian.

THEOREM 2.1. *A vector bundle E admits a hyperbolic Hermitian structure, if E can be endowed with a symplectic structure having a Lagrangian subbundle.*

Proof. It is known, [**11**] that E admits a Lagrangian subbundle if and only if the structural group of the frame bundle can be reduced to the group of the matrices of the form (1.8). Applying Theorem 1.1, we complete the proof. □

COROLLARY 2.1. *A manifold M can be endowed with an almost hyperbolic Hermitian structure if and only if it admits an almost cotangent structure.*

We need to recall now the following

Definition 2.1. [**1**] Let (E_i, ω_i), $i = 1, 2$, be two symplectic vector bundles over M and let the Whitney sum $E = E_1 \oplus E_2$ be endowed with the symplectic structure $\omega_1 - \omega_2$. Then any Lagrangian subbundle of $(E, \omega_1 - \omega_2)$ is called a linear canonical relation.

It is easy to prove

PROPOSITION 2.1. *Let (E_i, P_i, g_i), $i = 1, 2$, be two hyperbolic Hermitian vector bundles over the same manifold M, with associated 2-forms Ω_i, $i = 1, 2$. Then the Whitney sum $E = E_1 \oplus E_2$ can be endowed with a hyperbolic Hermitian structure (P, g), where*

$$P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}, \qquad g = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix},$$

such that the eigensubbundles P^+ and P^- are linear canonical relations of E with respect to the symplectic structure $\Omega_1 - \Omega_2$.

Our purpose now is to establish a relation between the vector bundles endowed with hyperbolic complex (resp. hyperbolic Hermitian) structures, on one side, and the vector bundles endowed with complex (resp. Hermitian) structures, on the other side, in order to obtain conditions for the existence of hyperbolic structures.

Definition 2.2. If (E, π, M) is a real vector bundle endowed with a complex or hyperbolic complex structure F, then a vector subbundle S of E is anti-invariant with respect to F, if $S \cap FS = \{0\}$, i.e. $S_x \cap FS_x = \{0\}$ for every $x \in M$.

From the definition above we may notice that $\operatorname{rank} E = 2k$. If $\operatorname{rank} S = k$, then E splits into direct sum $E = S \oplus FS$.

PROPOSITION 2.2. *Let (E, π, M) be a real vector bundle endowed with a complex structure J (resp. a hyperbolic complex structure P) and $\operatorname{rank} E = 2k$. Then J (resp. P) induces a hyperbolic complex structure P (resp. a complex structure J) provided there is a vector subbundle S of E, anti-invariant by J (resp. by P) with $\operatorname{rank} S = k$.*

Proof. At the first case let E be endowed with a complex structure J. Since $E = S \oplus JS$ (direct sum), then we define $P(u + Jv) = v + Ju$, for every $u, v \in \Gamma(M, S)$. At the second case, let E be endowed with a hyperbolic complex structure P. Since $E = S \oplus PS$ (direct sum), then we define $J(u + Pv) = -v + Pu$, for every $u, v \in \Gamma(M, S)$. □

COROLLARY 2.2. *Under the assumptions of Proposition 2.2, the vector bundle (E, π, M) admits an anti-quaternionic structure.*

Proof. Let J (resp. P) be a complex (resp. hyperbolic complex) structure of E and let P (resp. J) be the hyperbolic complex (resp. complex) structure induced by J (resp. P) of E, as in Proposition 2.2. In both cases, we have: $P^2 = I$, $J^2 = -I$ and $PJ = -JP$. If we define now $K = PJ$, then we obtain (P, K) satisfying $P^2 = K^2 = I$ and $PK = -KP$. Thus we get an anti-quaternionic structure of E. □

THEOREM 2.2. *Let* (E, π, M) *be a real vector bundle, with* $\operatorname{rank} E = 2k$. (i) *If* (P, g) *is a hyperbolic Hermitian structure of* E*, then it induces a Hermitian structure* (J, h) *and the associated symplectic structures of* (P, g) *and* (J, h) *are equal.* (ii) *Conversely, if* (J, h) *is a Hermitian structure of* E *with the associated symplectic structure* w *and* S *is an anti-invariant vector subbundle of* E *with respect to* J*, having* $\operatorname{rank} S = k$*, then* (J, h) *induces a hyperbolic Hermitian structure* (P, g) *of* E*. If moreover* S *is Lagrangian with respect to* w*, then* w *is also a symplectic structure associated to* (P, g)*.*

Proof. (i) Let Ω be the associated symplectic structure of (P, g). It is known [**11**] that there is a Hermitian structure (J, h) of the symplectic vector bundle (E, Ω), having the above properties. (ii) In the same way as in Proposition 2.2, we have $E = S \oplus JS$ and let P be a product structure of E defined by $P(u + Jv) = v + Ju$, for every $u, v \in \Gamma(M, S)$. We also define a pseudo-Riemannian structure g of E by $g(u + Jv, s + Jt) = h(u, s) - h(v, t)$, for every $u, v, s, t \in \Gamma(M, S)$. $\square$

COROLLARY 2.3. *Let* M *be a differentiable manifold.* (i) *If there is an almost hyperbolic Kaehlerian structure on* M*, then it induces an almost Kaehlerian structure and the associated* 2*-forms are equal.* (ii) *Conversely, If* M *admits an almost Kaehlerian structure* (J, h) *and a Lagrangian distribution with respect to its associated* 2*-form, which is anti-invariant with respect to* J*, then we can construct an almost hyperbolic Kaehlerian structure having the same associated* 2*-form.*

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