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ON THE CONTINUITY OF INTERNAL FUNCTIONS

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Abstract. A modification of the S-continuity is studied. Our concept clarifies the relationship between S-continuity and almost S-continuity introduced by N. Vakil. Moreover we give a standard description of almost S-continuity of a standard family of internal functions solving a problem posed by N. Vakil.

Introduction and Terminology. Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be uniform spaces and let x,y be elements in the nonstandard model ${}^*X$. As usual, $y \approx x$ is defined to mean that $(x,y) \in {}^*U$ for every $U \in \mathcal{U}$ and $\mu[x] := \{y \in {}^*X : \ y \approx x\}$ is called the *monad* of $x \in {}^*X$. The union of all monads $\mu[{}^*x]$ with $x \in X$ is the set $\mathrm{ns}\,{}^*X$ of all *nearstandard points*. Let $f\colon {}^*X \to {}^*Y$ be an internal function. Recall that f is S-continuous at $x \in {}^*X$ if for every $y \in {}^*X$ with $y \approx x$ we have $f(y) \approx f(x)$. It is a matter of fact that this concept is too restrictive for non-locally compact spaces. We discuss the following modified notion: let M be an arbitrary subset of ${}^*X$. Call f to be *M-continuous* if $y \approx x$ with $y,x \in M$ implies $f(y) \approx f(x)$. Obviously f is $\mu[x]$-continuous [$\mathrm{ns}\,{}^*X$-continuous resp.] if and only if f is S-continuous at $x \in {}^*X$ [at each $x \in X$ resp.]. We call f *strongly M-continuous* if for every $a \in M$, $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $({}^*x, a) \in {}^*U$ implies $(f({}^*x), f(a)) \in {}^*V$ for all $x \in X$. The last definition is due to N. Vakil who called f to be *almost S-continuous* on M. The reason for adopting a new notion is the simple result that strong M-continuity implies M-continuity. The first section contains general results about the relationship between the M-continuity and the strong M-continuity. In the second section we prove the main result characterizing almost S-continuity of a standard family. We always assume that the nonstandard model is polysaturated.

1. Strong M-continuity. Recall that $\mathrm{pns}\,X := \cap_{U\in\mathcal{U}} \cup_{x\in X} {}^*U[x]$ is the set of all *prenearstandard points* of X. It is obvious that the condition of strong continuity is trivially satisfied for each $a \in {}^*X \setminus \mathrm{pns}\,{}^*X$. Hence we should restrict ourselves to the case that $M \subset \mathrm{pns}\,{}^*X$. By ${}^\sigma X$ we denote the set $\{{}^*x : \ x \in X\}$.

PROPOSITION 1.1 *Let $M \subset \mathrm{pns}\,X$. If $f\colon {}^*X \to {}^*Y$ is strongly M-continuous, then it is M-continuous. If M is closed under the operation $\approx$, then the converse is also true.*

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Proof. Let $y_1, y_2 \in M$ with $y_1 \approx y_2$ and let $V \in \mathcal{V}$. Choose V_1 symmetric with $V_1 \circ V_1 \subset V$. Then there exists $U \in \mathcal{U}$ such that $({}^*x, y_i) \in {}^*U$ implies $(f({}^*x), f(y_i)) \in {}^*V_1$ for all $x \in X$. Choose $x_1, x_2 \in X$ with $y_i \in {}^*U[x_i]$. Then $(f(y_2), f(y_1)) \in {}^*V_1 \circ {}^*V_1 \subset {}^*V$. For the converse let $a \in M$ and $V \in \mathcal{V}$. Since M is closed under $\approx$ the following statement is true: $(\forall x \in {}^*X)(x \approx a \Rightarrow (f(a), f(x)) \in {}^*V)$. Then $I := \{x \in {}^*X : (f(a), f(x)) \in {}^*V\}$ and $U_a := \{x \in {}^*X : (a, x) \in {}^*U\}$ are internal sets satisfying the relation $\cap_{U \in \mathcal{U}} U_a \subset I$. By a saturation argument there exists U_a with $U_a \subset I$. Hence there exists $U \in \mathcal{U}$ such that $(a, x) \in {}^*U$ implies $(f(a), f(x)) \in {}^*V$.

Corollary 1.2 *Let $g: X \to Y$ be a function. Then the following assertions are equivalent:*

a) *g is continuous.*

b) *${}^*g$ is $\mathrm{ns}\,{}^*X$-continuous.*

c) *${}^*g$ is strongly $\mathrm{ns}\,{}^*X$-continuous.*

d) *${}^*g$ is strongly ${}^\sigma X$-continuous.*

Proof. a) $\Rightarrow$ b) follows from the nonstandard characterization of continuity, Proposition 1.1 yields b $\Rightarrow$ c), and c) $\Rightarrow$ d) is trivial. d) $\Rightarrow$ a) is an immediate consequence of the definition of strong continuity and the transfer principle.

In general the converse in Proposition 1.1 is not true as the example 2.4 in [**8**] (with $M := {}^\sigma X$) or the example after Proposition 2.3 in [**6**] shows where M is equal to the set $\mathrm{cpt}\,{}^*X$ of all *compact points*, i.e. the union of all ${}^*K$ with $K \subset X$ compact. Nonetheless there exists a topological property defined in [**3**] assuring a converse in Proposition 1.1.

Definition 1.3 Let X be a topological space and α be a family of subsets of X. Define $\alpha\mathrm{pts}\,{}^*X := \cup_{A \in \alpha} {}^*A$. We call X an *α-space* if a set $U \subset X$ is open if and only if $U \cap A$ is open in every subspace $A \in \alpha$ endowed with the relative topology.

If k is the family of all compact subsets we obtain in 1.3 the well known definition of a *k-space* or *compactly generated space.* Every locally compact and every metric space is a k-space, cf. [**9**, p. 285].

Theorem 1.4 *Let X be an α-space and $\alpha\mathrm{pts}\,{}^*X \subset \mathrm{ns}\,{}^*X$. Then an internal function $f: {}^*X \to {}^*Y$ with $f({}^\sigma X) \subset \mathrm{ns}\,{}^*Y$ is strongly $\alpha\mathrm{pts}\,{}^*X$-continuous if and only if f is $\alpha\mathrm{pts}\,{}^*X$-continuous.*

Proof. For every $x \in X$ we have by assumption $f({}^*x) \in \mathrm{ns}\,{}^*Y$, hence there exists $y \in Y$ with $f({}^*x) \approx y =: h(x)$. It is easy to see that h is continuous on every set $A \in \alpha$. Since X is an α-space h is continuous on X and therefore ${}^*h$ is strongly $\mathrm{ns}\,{}^*X$-continuous. It follows that $f(x) \approx {}^*h(x)$ for all $x \in \alpha\mathrm{pts}\,{}^*X$. By Proposition 1.5 b) $\Rightarrow$ a) f is strongly $\alpha\mathrm{pts}\,{}^*X$-continuous.

Proposition 1.5 *Let $f, h: {}^*X \to {}^*Y$ be internal functions with $f({}^*x) \approx h({}^*x)$ for all $x \in X$. If $M \subset \mathrm{pns}\,X$ then the following assertions are equivalent:*

a) *f and h are strongly M-continuous.*

b) *$f(x) \approx h(x)$ for all $x \in M$ and h is strongly M-continuous.*

Proof. a) $\Rightarrow$ b). Let $a \in M$ and $V \in \mathcal{V}$. Since f, h are strongly M--continuous there exists $U \in \mathcal{U}$ such that $({}^*x, a) \in {}^*U$ implies $(f({}^*x), f(a)) \in {}^*V$

and $(h({}^*x), h(a)) \in {}^*V$. As $M \subset \operatorname{pns} X$ there exists $x \in X$ with $({}^*x, a) \in {}^*U$. Using $(h({}^*x), f({}^*x)) \in {}^*V$ we obtain $(h(a), f(a)) \in {}^*V^{-1} \circ {}^*V \circ {}^*V$. As this holds for every $V \in \mathcal{V}$ we infer b). For the converse let $a \in M$ and $V \in \mathcal{V}$. Choose $U \in \mathcal{U}$ such that $({}^*x, a) \in {}^*U$ imply $(h({}^*x), h(a)) \in {}^*V$. But $(f({}^*x), h({}^*x)) \in {}^*V$ and $(h({}^*x), h(a)) \in {}^*V$ and $(h(a), f(a)) \in {}^*V$; thus $(f({}^*x), f(a)) \in V \circ V \circ V$ for all $x \in X$ with $({}^*x, a) \in {}^*U$.

In [**8**] N. Vakil has given a description of the nearstandard points of the set $C(X,Y)$ of all continuous functions endowed with the topology τ_α of uniform convergence on the sets $A \in \alpha$: if ${}^\sigma X \subset \alpha\mathrm{pts}\,{}^*X \subset \mathrm{ns}\,{}^*X$ then

$$\mathrm{ns}_{\tau_\alpha}{}^*C(X,Y) = \{f \text{ strongly } \alpha\mathrm{pts}\,{}^*X\text{-cont. and } f({}^\sigma X) \subset \mathrm{ns}\,{}^*Y\}. \tag{1}$$

Recall that $f \in {}^*C(X,Y)$ is infinitesimal near to $g \in C(X,Y)$ with respect to τ_α iff $f(x) \approx {}^*g(x)$ for all $x \in \alpha\mathrm{pts}\,{}^*X$. Combining Corollary 1.2 and Proposition 1.5 with $h := {}^*g$ and $M := \alpha\mathrm{pts}\,{}^*X \subset \mathrm{ns}\,{}^*X$ one obtains a proof of formula (1). If $\alpha = k$ and X is a k-space we can replace strong $\mathrm{cpt}\,{}^*X$-continuity by $\mathrm{cpt}\,{}^*X$-continuity, cf. Theorem 1.4. We note that formula (1) is not true for the class of all families α with $\alpha\mathrm{pts}\,{}^*X \subset \mathrm{pns}\,X$: the validity of (1) implies that for every $f \in C(X,Y)$ the function ${}^*f\colon {}^*X \to {}^*Y$ is $\alpha\mathrm{pts}\,{}^*X$-continuous. If X is a totally bounded space and $\alpha = \{X\}$ then this means that every continuous function must be ${}^*X$-continuous, i.e, that f is uniformly continuous; but this statement is in general not true.

Our next proposition generalizes the well known result that every uniformly continuous function $f\colon X \to Y$ maps prenearstandard points to prenearstandard points.

Proposition 1.6 *Let $M \subset \mathrm{pns}\,{}^*X$ and $f\colon {}^*X \to {}^*Y$ be strongly M--continuous with $f({}^*x) \in \mathrm{pns}\,{}^*Y$ for every $x \in X$. Then $f(M) \subset \mathrm{pns}\,{}^*Y$.*

Proof. Let $a \in M$ and $V \in \mathcal{V}$. Choose $V_1 \in \mathcal{V}$ with $V_1 \circ V_1 \subset V$ and $U \in \mathcal{U}$ such that $(f({}^*x), f(a)) \in {}^*V_1$ for all $x \in X$ with $({}^*x, a) \in {}^*U$. Choose $x \in X$ with $a \in {}^*U[x]$. Since $f({}^*x) \in \mathrm{pns}\,Y$ there exists $y \in Y$ with $({}^*y, f({}^*x)) \in {}^*V_1$. Then $f(a) \in {}^*V_1 \circ {}^*V_1[y] \subset {}^*V[y]$. The proof is complete.

Proposition 1.1 shows that the $\mathrm{pns}\,{}^*X$-continuity is equivalent to the strong $\mathrm{pns}\,{}^*X$-continuity. Then Theorem 8.4.30 in [**7**] can be read as follows:

Theorem 1.7 *A function $f\colon X \to Y$ is strongly $\mathrm{pns}\,X$-continuous iff there exists a continuous extension $\bar{f}\colon \bar{X} \to \bar{Y}$ where $\bar{X}$ and $\bar{Y}$ are the completions of X and Y.*

2. Simply even continuity. It is clear by the nonstandard characterization of compactness that every description of the nearstandard points leads to a characterization of compact sets; hence (1) implies that $H \subset C(X,Y)$ is relative compact for τ_α iff every $f \in {}^*H$ is strongly $\alpha\mathrm{pts}\,{}^*X$-continuous and satisfies $f({}^\sigma X) \subset \mathrm{ns}\,{}^*Y$, cf. [**8**, Theorem 3.4]. In this section we give a standard description of the first property answering a question in [**8**]. Recall that the monad of a filter $\mathcal{G}$ is just the set $\mathrm{monad}(\mathcal{G}) := \cap_{G \in \mathcal{G}}{}^*G$.

Definition 2.1 Let A be a subset of X. A subset $H \subset C(X,Y)$ is called *simply A-equicontinuous* if for every ultrafilter $\mathcal{G}$ on the product space $H \times A$ and every $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that for all $x \in X$ there exists $G_x \in \mathcal{G}$ with

$$(\forall (f,a) \in G_x)((a,x) \in U \Rightarrow (f(a), f(x)) \in V). \tag{2}$$

If $A = \{x_0\}$ we call H *simply equicontinuous* in $x_0 \in X$, cf. [**2**]. H is *simply equicontinuous* if H is simply equicontinuous for every $x_0 \in X$.

THEOREM 2.2 *Let α be a family of subsets of X. Then $H \subset C(X,Y)$ is simply A-equicontinuous for every $A \in \alpha$ iff every $f \in {}^*H$ is strongly* $\alpha\mathrm{pts}\,{}^*X$*-continuous.*

Proof. Let $a \in \alpha\mathrm{pts}\,{}^*X$, $f \in {}^*H$ and $V \in \mathcal{V}$. We show that f is strongly $\{a\}$-continuous. Choose $A \in \alpha$ with $a \in {}^*A$. Observe that $\mathcal{G} := \{G \subset H \times A : (f,a) \in {}^*G\}$ is an ultrafilter on H with $(f,a) \in \operatorname{monad}\mathcal{G}$. Choose $U \in \mathcal{U}$ and for every $x \in X$ choose G_x as in Definition 2.1. Then $(f,a) \in \operatorname{monad}\mathcal{G} \subset {}^*G_x$. The transfer principle applied to (2) shows that $(a, {}^*x) \in {}^*U$ implies $(f(a), f({}^*x)) \in {}^*V$. For the converse let $\mathcal{G}$ be an ultrafilter on $H \times A$ and $V \in \mathcal{V}$. Choose $(f,a) \in \operatorname{monad}\mathcal{G} \subset {}^*H \times {}^*A$. It is easy to see that $\mathcal{G} = \{G \subset H \times A : (f,a) \in {}^*G\}$. Since f is strongly $\{a\}$-continuous there exists $U \in \mathcal{U}$ such that $(a, {}^*x) \in {}^*U$ implies $(f(a), f({}^*x)) \in {}^*V$. For $x \in X$ consider $G_x := \{(g,y) \in H \times A : (y,x) \in U \Rightarrow (g(y), g(x)) \in V\}$. Since $(f,a) \in {}^*G_x$ we have $G_x \in \mathcal{G}$. By construction G_x satisfies (2).

Note that Theorem 2.2 and the above remarks yields a nonstandard proof of the following well known result [**2**]: $H \subset C(X,Y)$ is relatively compact for the pointwise topology iff H is pointwise bounded and simply equicontinuous.

THEOREM 2.3 *Let X be a k-space and $H \subset C(X,Y)$. Then the following assertions are equivalent:*

a) *H is equicontinuous.*

b) *Every $f \in {}^*H$ is* $\mathrm{ns}\,{}^*X$*-continuous.*

c) *Every $f \in {}^*H$ is strongly* $\mathrm{cpt}\,{}^*X$*-continuous.*

d) *H is simply K-equicontinuous for every compact set K.*

e) *Every $f \in {}^*H$ is* $\mathrm{cpt}\,{}^*X$*-continuous.*

f) *H is equicontinuous on compacta.*

Proof. a) $\Leftrightarrow$ b) is a well known nonstandard characterization. b) $\Rightarrow$ c) and c) $\Leftrightarrow$ d) and c) $\Rightarrow$ e) are clear by Proposition 1.1 and Theorem 2.2. The equivalence of e) and f) is straightforward. Since we do not know a reference for e) $\Rightarrow$ b) (or f) $\Rightarrow$ a)) (unless H is pointwise bounded) we give here a short proof: consider the so-called diagonal function $\Delta : X \to C(H,Y)$ defined by $\Delta(x)(f) = f(x)$. If $C(H,Y)$ is endowed with the topology of uniform convergence, then Δ is continuous iff every $f \in {}^*H$ is $\mathrm{ns}\,{}^*X$-continuous. Since X is a k-space it suffices to show that Δ is continuous on every compact set $K \subset X$. But this is equivalent to the condition e). The proof is complete.

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