PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 53 (67), 1993, 153–156

STRONGLY QUASICONVEX QUADRATIC FUNCTIONS

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Abstract. A quadratic function is quasiconvex in R^n if and only if it is convex [**3**]. However, this is not true in R^n_+. We prove that a quadratic function is strongly quasiconvex in a convex cone K ($0 \in K$, $\operatorname{int} K \neq \emptyset$) if and only if it is strongly convex.

1. Introduction. Let C be a nonempty convex subset of the Euclidean n-space R^n. For $x, y \in R^n$ we denote by $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$ the scalar product, and by $\|x\| = \sqrt{\langle x, x\rangle}$ the norm of x. If $x, y \in R^n$, then $x \leq y$ means that $x_i \leq y_i$ for all $i = \overline{1, n}$. We write $Q \leq 0$ for a real matrix Q of order n if all its entries are non-positive. $Q < 0$ means $Q \leq 0$ and $Q \neq 0$.

Definition 1. [**4**] A real valued function f is said to be strongly convex on C if there exists some real number $r > 0$ such that for each $x, y \in C$ and $\lambda \in [0,1]$

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) - r\lambda(1-\lambda)\|x-y\|^2. \tag{1}$$

Definition 2. [**2**,**5**] A function $f : C \to R$ is strongly quasiconvex on C if there exists an $s > 0$ such that for any $x, y \in C$, $\lambda \in [0,1]$

$$f(\lambda x + (1-\lambda)y) \leq \max\Big\{f(x), f(y)\Big\} - s\lambda(1-\lambda)\|x-y\|^2. \tag{2}$$

If $r = 0$ in (1), then f is convex, and if $s = 0$ in (2), then f is quasiconvex.

Let $\nabla f(x)$ and $\nabla^2 f(x)$ denote the gradient and the Hessian of f at x. The following theorem will be useful.

THEOREM 1. *A continuously differentiable function $f : C \to R$ is strongly quasiconvex if and only if there exists an $s > 0$ such that*

$$f(x) \leq f(y) \Rightarrow \langle \nabla f(y), y - x\rangle \geq s\|x-y\|^2. \tag{3}$$

This is a particular case of Theorems (1) and (6) from [**6**].

AMS Subject Classification (1980): Primary 90 C 30

THEOREM 2. [**4**] *A twice differentiable function $f: C \to R$ is strongly convex on C* ($\operatorname{int} C \neq \emptyset$) *if and only if there exists an $r > 0$ such that*

$$(\forall x \in C)(\forall v \in R^n)\langle \nabla^2 f(x)v, v\rangle \geq r\|v\|^2. \tag{4}$$

Remark. Clearly, every (strongly) convex function is (strongly) quasiconvex. The function $f(x) = x$ is strongly quasiconvex on $[0, 1]$, but not strongly convex.

2. (Strongly) quasiconvex quadratic functions. Let $q(x) = \frac{1}{2}\langle Qx, x\rangle + \langle p, x\rangle$ be a quadratic function of $x \in R^n$ (Q is a symmetric real matrix of order $n, p \in R^n$). It is well known that q is convex on any C ($\operatorname{int} C \neq \emptyset$) if and only if Q is a positive semidefinite matrix, and q is strongly convex if and only if Q is a positive definite matrix (by Theorem 2). A criterion for the quasiconvexity of a quadratic function on $R^n_+ = \{x \in R^n \mid x \geq 0\}$ was given by Martos:

THEOREM 3. [**3**, p.150] *A nonconvex quadratic function q is quasiconvex in R^n_+ if and only if*

$$Q < 0 \text{ and } p \leq 0. \tag{5}$$

$$Q \text{ has exactly one negative eigenvalue.} \tag{6}$$

$$(\exists v \in R^n) \quad p = Qv \ \wedge \ \langle p, v\rangle \leq 0. \tag{7}$$

This was extended in [**1**]. We shall mention here a necessary condition for quasiconvexity of a nonconvex quadratric function on a convex cone. Let C^0 be the polar cone of a set C, i.e. $C^0 = \{v \in R^n \mid (\forall x \in C) \ \langle v, x\rangle \leq 0\}$ and let $u^i = Ue^i$, $i = \overline{1, n}$, where U is an orthogonal matrix such that U^tQU is diagonal, while e^i is a unit vector.

THEOREM 4. [**1**] *Let a nonconvex quadratic function q be quasiconvex on a convex cone K with nonempty interior. Then*

$$(\forall x \in K) \ Qx \in K^0, \tag{8}$$

$$\text{either } u^i \in K^0, \text{ or } -u^i \in K^0, \tag{9}$$

$$p \in K^0, \tag{10}$$

$$(\forall v \in R^n) \quad Qv = p \Rightarrow \langle p, v\rangle \leq 0. \tag{11}$$

From a practical point of view, it is more difficult to recognize quasiconvexity than convexity. It is, therefore, useful to know when the classes of the functions mentioned above are identical. So, in [**3**, p.147] Martos proved that q is quasiconvex in R^n if and only if it is convex in R^n. However, there are quadratic functions quasiconvex in R^n_+ but not convex, for example $q(x) = -x_1x_2$ ($x \in R^2_+$).

We shall prove that q is strongly quasiconvex on a convex cone K ($0 \in K$, $\operatorname{int} K \neq \emptyset$) if and only if q is strongly convex. In the proof we shall need the result given in the following Lemma.

LEMMA. *Let C ($\operatorname{int} C \neq \emptyset$) be an unbounded convex set in R^n and suppose that $f : C \to R$ is bounded above in C. Then f is not strongly quasiconvex on C.*

Proof. For some fixed $x^0 \in \operatorname{int} C$ there exists a nonzero vector z such that $\{x^0 + \alpha z \,|\, \alpha \geq 0\} \subseteq \operatorname{int} C$. Assume that (2) holds. Let $x^n = x^0 + nz$, $n \in N$. We have $x^n \in C$ and

$$\begin{aligned} f(x^1) &= f((1-\frac{1}{n})x^0 + \frac{1}{n}(x^0+nz)) \\ &\leq \max\{f(x^0), f(x^n)\} - s\frac{1}{n}(1-\frac{1}{n})\|x^0 - x^n\|^2, \text{ i.e.,} \\ s(n-1)\|z\|^2 &\leq \sup_C f - f(x^1) \qquad \text{for each } n > 1, \end{aligned}$$

which contradicts $s > 0$. Thus our assumption is false and f is not strongly quasiconvex.

Let K be a convex cone such that $0 \in K \subseteq R^n$ and $\operatorname{int} K \neq \emptyset$.

THEOREM 5. *A quadratic function q is strongly quasiconvex on K if and only if q is strongly convex.*

Proof. We show that strong quasiconvexity implies strong convexity. The converse is obvious. Suppose that q is a strongly quasiconvex function. If q is not convex, by (8) and (10) we have $\langle Qx, x\rangle \leq 0$ and $\langle p, v\rangle \leq 0$, so that $q(x) \leq 0$ for each $x \in K$. This is impossible by Lemma. Therefore $(\forall v \in R^n)\langle Qv, v\rangle \geq 0$. Suppose now that 0 is eigenvalue from Q; then there exists a nonzero vector v^0 such that $Qv^0 = 0$. Since $0 \in K$ and $\operatorname{int} K \neq \emptyset$, we have $K - K = R^n$.

Hence there are $v^1, v^2 \in K$ such that $v^0 = v^1 - v^2$ and $\langle Qv^1, v^0\rangle = \langle v^1, Qv^0\rangle = 0$, $\langle Qv^2, v^0\rangle = 0$, Now $q(v^1) \leq q(v^2)$ or $q(v^2) \leq q(v^1)$ implies $\langle p, -v^0\rangle \geq s\|v^0\|^2$ or $\langle p, v^0\rangle \geq s\|v^0\|^2$, by (3). Since $q(v^1) - q(v^2) = \langle p, v^0\rangle$ we get $s\|v^0\|^2 \leq |\langle p, v^0\rangle| \leq \|p\|\,\|v^0\|$, i.e., $\|v^0\| \leq \|p\|/s$. Replacing v^0, v^1, v^2 by $\alpha v^0, \alpha v^1, \alpha v^2$ $\alpha > 0$, (since K is cone), we obtain $(\forall \alpha > 0)$ $\alpha \leq \|p\|/s\|v^0\|$; it is impossible. Hence Q has positive eigenvalues, so that q is strongly convex.

Remark. Particulary, strong quasiconvexity of q on R^n_+ is equivalent to strong convexity. This is not true for $C \subset R^n_+$. For example, $q(x) = -x^2 - x$ is strongly quasiconvex on $[0,1]$, but is not even convex.

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(Received 04 08 1992)