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GENERALIZATION OF THE MERCER THEOREM

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Abstract. It was shown that the bilinear series of the given operator $A = H(I + S)$ $(H = H^*, S = S^*)$ uniformly converges to its kernel.

1. Introduction

In this paper we will prove some results related to possibilities of representation of integral operator kernels (close to the selfadjoint ones) by bilinear series. Let A be an integral operator acting on $L^2[a,b]$ with a continuous symetric kernel $K(\cdot,\cdot)$ on $[a,b] \times [a,b]$ $(b - a < \infty)$. Let λ_i be the eigenvalues and ϕ_i the corresponding eigenvectors, forming an orthonormal system.

It is well known [**3**] the following classical Mercer theorem: If all eigenvalues of the operator A are positive, then

$$K(x,y) = \sum_{i=1}^{\infty} \lambda_i \phi_i(x) \cdot \overline{\phi_i(y)}$$

where the convergence is uniform on $[a,b] \times [a,b]$.

In [**1**] a similar statement is proved for operators of the form $A = H(I + S)$, where H and S are integral operators with symetric continuous kernels such that $H > 0$ and the spectar of the operator S lies within $(-1, \|S\|)$. We shall prove that a similar statement is valid for operators S for which $I + S$ is not necessarily positive.

2. Main result

We shall consider operators $A = H(I + S)$, where H and S are selfadjoint integral operators with continuous kernels $H(\cdot,\cdot)$ and $S(\cdot,\cdot)$ on $[a,b] \times [a,b]$. It is

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obvious that the operator A is compact. Let λ_i and $\overline{\lambda_i}$ be eigenvalues of A and A^*, and ϕ_i and ψ_i appropriate eigenvectors normed such that $(\phi_i, \psi_j) = \delta_{ij}$. (By $(\cdot,\cdot)$ we denote the inner product in $L^2[a,b]$ space, i.e. $(f,g) = \int_a^b f(x)\overline{g(x)}\,dx$, $f, g \in L^2[a,b]$.) Let $\mathcal{A}\,(\cdot,\cdot)$ be the kernel of the integral operator $A = H(I+S)$. The main result of the paper is the following

THEOREM. *If H is a positive operator such that* $\mathrm{Ker}H = \{0\}$ *and -1 is a regular point of S, then*

$$\mathcal{A}(x,y) = \sum_{i=1}^{\infty} \lambda_i \phi_i(x) \cdot \overline{\psi_i(y)} \tag{1}$$

where convergence is uniform on $[a,b] \times [a,b]$.

Proof. Since the operator S is selfadjoint with continuous kernel on $[a,b] \times [a,b]$, it is also compact, so that the following expansion holds

$$S = \sum_{\nu=1}^{\infty} \mu_\nu (\cdot, e_\nu) e_\nu$$

where $\{e_\nu\}$ is an orthonormal base of the space $L^2[a,b]$, formed by the eigenvectors of S. (Some of μ_ν are equal to zero.) Let $\mu_1, \ldots \mu_k < -1 < \mu_{k+1}, \mu_{k+2}, \ldots$ We write

$$I + S = \sum_{\nu=1}^{\infty} (1 + \mu_\nu)(\cdot, e_\nu) e_\nu.$$

Let us introduce an operator in the following way

$$|I + S| = \sum_{\nu=1}^{\infty} |1 + \mu_\nu| (\cdot, e_\nu) e_\nu.$$

The operator $|I+S|$ is positive and invertible since $-1 \in \rho(S)$. Thus, the operators $|I+S|^{\frac{1}{2}} = \sum_{\nu=1}^{\infty} |1+\mu_\nu|^{\frac{1}{2}} (\cdot, e_\nu) e_\nu$ and $|I+S|^{-\frac{1}{2}} = \sum_{\nu=1}^{\infty} |1+\mu_\nu|^{-\frac{1}{2}} (\cdot, e_\nu) e_\nu$ are positive and invertible. We shall consider the operators $A_1 = |I+S|^{\frac{1}{2}} A |I+S|^{-\frac{1}{2}}$ and $J = (I+S)|I+S|^{-1}$, where $A = H(I+S)$. It is clear that

$$J = -\sum_{\nu=1}^{k} (\cdot, e_\nu) e_\nu + \sum_{\nu=k+1}^{\infty} (\cdot, e_\nu) e_\nu.$$

Let us introduce projectors P_+ and P_- in the following way: $P_- = \sum_{\nu=1}^{k} (\cdot, e_\nu) e_\nu$, $P_+ = \sum_{\nu=k+1}^{\infty} (\cdot, e_\nu) e_\nu$; then we obtain $J = P_+ - P_-$. It is obvious that $P_+ + P_- = I$, $J^* = J, J^2 = J$. Let us now introduce an indefinite product $[x,y] = (Jx, y)$, where $(\cdot,\cdot)$ is the inner product in $L^2(a,b)$, $x, y \in L^2(a,b)$. Using the commutativity of $|I+S|^{\pm\frac{1}{2}}$ and $I+S$ and the fact that $\mathrm{Ker}H = \{0\}$, we obtain

$$[A_1 x, y] = [x, A_1 y] \qquad \forall x, y \in L^2(a,b), \tag{2}$$

$$[A_1 x, x] > 0, \qquad \forall x \in L^2(a,b), x \neq 0. \tag{3}$$

Since the operator A_1 is compact and by (3) J is positive, there exists (see [**2**]) J orthonormal system of the eigenvectors of A_1, so that the spectral theorem is valid, i.e.:

$$A_1 \phi'_n = \lambda_n \phi'_n, \qquad [\phi'_n, \phi'_m] = \delta_{nm} \tag{4}$$

$$A_1 f = \sum_{n=1}^{\infty} \lambda_n \cdot [f, \phi'_n]\phi'_n \tag{5}$$

where the convergence in (5) is the convergence in the L^2 norm. Moreover, the system ϕ'_n is the base of the space $L^2(a,b)$. (see [**2**, p. 271]). By (4) it is clear that $\lambda_n > 0$. Really, by (4) we have $[A_1\phi'_n, \phi'_n] = \lambda_n \cdot [\phi'_n, \phi'_n] = \lambda_n$, so that, by (3), we have that $\lambda_n > 0$. Bearing in mind the definitions of the operators A_1 and J, we can write the relation (5) in the following way

$$|I+S|^{\frac{1}{2}} A |I+S|^{-\frac{1}{2}} f = \sum_{n=1}^{\infty} \lambda_n \cdot (Jf, \phi'_n) \cdot \phi'_n. \tag{6}$$

Certainly, the convergence in (6) is the convergence in the L^2 norm. Applying the bounded operator $|I+S|^{-\frac{1}{2}}$ to the equality (6) and introducing the vectors $\phi_n = |I+S|^{-\frac{1}{2}}\phi'_n$ we obtain $A\phi_n = \lambda_n \phi_n$ and

$$A|I+S|^{-\frac{1}{2}} f = \sum_{n=1}^{\infty} \lambda_n \cdot ((I+S) \cdot |I+S|^{-\frac{1}{2}} f, \phi_n) \cdot \phi_n. \tag{7}$$

The operator $|I+S|^{-\frac{1}{2}}$ has bounded inverse, so that

$$Af = \sum_{n=1}^{\infty} \lambda_n \cdot (f, (I+S)\phi_n) \cdot \phi_n. \tag{8}$$

where convergence in (8) is taken as the convergence in the L^2 norm. From $[\phi'_n, \phi'_n] = \delta_{nm}$, the definitions of the operator J and vectors ϕ_n it follows that $\delta_{mn} = [\phi'_n, \phi'_m] = ((I+S)\phi_n, \phi_n)$. If we introduce vectors $\psi_n = (I+S)\phi_n$, then we have that $(\phi_n, \psi_m) = \delta_{nm}$, which shows that the expansion (8) can be written as

$$Af = \sum_{n=1}^{\infty} \lambda_n (f, \psi_n)\phi_n. \tag{9}$$

The expansion (9) converges in the L^2 norm and is valid for all $f \in L^2(a,b)$. Since the system ϕ'_n is the base of L^2, (because of invertibility of $|I+S|^{-\frac{1}{2}}$ and $I+S$) the systems ϕ_n and ψ_n are also bases of $L^2(a,b)$. Continuity of kernels of the operators A and $A\phi_n = \lambda_n\phi_n$ imply that the functions ϕ_n are continuous on $[a,b]$. That implies that the functions $\psi_n = (I+S)\phi_n$ are also continuous on $[a,b]$ (because the integral operator S has continuous kernel). The expansion (9) implies

$$A^* f = \sum_{n=1}^{\infty} \lambda_n (f, \phi_n)\psi_n \tag{10}$$

so that $A^*\psi_n = \lambda_n \cdot \psi_n$ holds. Since $A = H(I+S)$, we have $A^* = (I+S)H$ so, (because $\psi_n = (I+S)\phi_n$) the expansion (10) can be written as

$$(I+S)Hf = \sum_{n=1}^{\infty} \lambda_n (f, \phi_n)(I+S)\phi_n. \tag{11}$$

The operator $(I+S)^{-1}$ is bounded, so that (11) implies

$$Hf = \sum_{n=1}^{\infty} \lambda_n (f, \phi_n)\phi_n. \tag{12}$$

The convergence in (12) is the convergence in the L^2 norm. The expansion (12) will play the main role in proving the main result. Notice that (12) is not spectral expansion of the selfadjoint operator H because the functions ϕ_n do not form orthonormal system. In the case when $S = 0$ the operator A becomes H, the system ϕ_n becomes orthonormal and the expansion (12) represents the spectral theorem for H. Let us prove that

$$H(x,y) = \sum_{n=1}^{\infty} \lambda_n \phi_n(x)\overline{\phi_n(y)} \tag{13}$$

where convergence in (13) is uniform on $[a,b]\times[a,b]$. Let us introduce a continuous function $H_n(x,y) = H(x,y) - \sum_{k=1}^{n} \lambda_k \phi_k(x)\overline{\phi_k(y)}$. Because of (12) we have that, for $f \in L^2(a,b)$

$$\int_a^b H_n(x,y)f(y)\,dy = \sum_{k=n+1}^{\infty} (f, \phi_k)\phi_k. \tag{14}$$

Let $\mathcal{H}_n$ denotes an integral operator whose kernel is $H_n(\cdot,\cdot)$. From (14) we obtain

$$(\mathcal{H}_n f, f) \geq 0. \tag{15}$$

Let us prove that $H_n(x,x) \geq 0$ on $[a,b]$. Suppose contrary, i.e. that there is $x_0 \in [a,b]$ so that $H_n(x_0,x_0) < 0$. The definition of the function H_n implies

$$\overline{H_n(y,x)} = H_n(x,y) \tag{16}$$

Let us write the function H_n as $H_n(x,y) = P_n(x,y) + iQ_n(x,y)$, where the functions $P_n(\cdot,\cdot)$ and $Q_n(\cdot,\cdot)$ are real and continuous. Now, by (16) we have

$$P_n(x,y) = P_n(y,x), Q_n(x,y) = -Q_n(y,x), \quad \forall x,y \in [a,b]. \tag{17}$$

That implies $Q_n(x_0, x_0) = 0$ so that the condition $H_n(x_0, x_0) < 0$ means nothing but $P_n(x_0, x_0) < 0$. Since $P_n(\cdot, \cdot)$ is a real and continuous function, there is a neighbourhood $U \times U$, where $U = (x_0 - \delta, x_0 + \delta)$, of the point (x_0, x_0) so that any $(x, y) \in U \times U$ satisfies $P_n(x, y) < 0$. We shall introduce a function

$$f_\delta(x) = \begin{cases} 0, & x \notin U \\ 1, & x \in U. \end{cases}$$

The function f_δ is from $L^2(a, b)$ and

$$(\mathcal{H}_n f_\delta, f_\delta) = \iint_{U \times U} H_n(x, y)\, dxdy =$$

$$= \iint_{U \times U} P_n(x, y)\, dxdy + i \iint_{U \times U} Q_n(x, y)\, dxdy.$$

Since (by (17)) the integral $\iint_{U \times U} Q_n(x, y)\, dxdy$ is equal to zero, we have

$$(\mathcal{H}_n f_\delta, f_\delta) = \iint_{U \times U} P_n(x, y)\, dxdy.$$

The function $P(\cdot, \cdot)$ is negative on the square $U \times U$, so that $(\mathcal{H}_n f_\delta, f_\delta) < 0$ what contradics (15). Thus, $H_n(x, x) \geq 0$ for all $x \in [a, b]$ is proved. Now, we can obtain, because of the definition of the function H_n, that

$$\sum_{k=1}^{n} \lambda_k |\phi_k(x)|^2 \leq H(x, x) \tag{18}$$

is true. Since H is continuous, we have $H(x, x) \leq M < \infty$ for $x \in [a, b]$, and $\sum_{k=1}^{\infty} \lambda_k |\phi_k(x)|^2 \leq M$.

Choose $x_0 \in [a, b]$. Then from

$$|\sum_{n=p}^{q} \lambda_n \phi_n(x_0)\overline{\phi_n(y)}|^2 \leq \sum_{n=p}^{q} \lambda_n |\phi_n(x_0)|^2 \cdot \sum_{n=p}^{q} \lambda_n |\phi_n(y)|^2 \leq$$

$$M \sum_{n=p}^{q} \lambda_n |\phi_n(x_0)|^2 \longrightarrow 0 \;\; (p, q \longrightarrow \infty)$$

it follows that the series

$$\sum_{n=1}^{\infty} \lambda_n \phi_n(x_0)\overline{\phi_n(y)} \tag{19}$$

uniformly converges to $B(x_0, y)$, say, which is continuous function of y. Let $f \in C[a, b]$ be an arbitrary function. The uniform convergence of the series (19) shows that

$$\begin{aligned}\int_a^b B(x_0, y) f(y)\, dy &= \sum_{n=1}^{\infty} \lambda_n \phi_n(x_0) \int_a^b f(y) \overline{\phi_n(y)}\, dy \\ &= \sum_{n=1}^{\infty} \lambda_n \phi_n(x_0)(f, \phi_n).\end{aligned} \tag{20}$$

Let us prove that

$$\int_a^b H(x_0, y) f(y)\, dy = \sum_{n=1}^{\infty} \lambda_n \phi_n(x_0)(f, \phi_n). \tag{21}$$

The systems ϕ_n and ψ_n are bases of $L^2[a, b]$ and $(\phi_n, \psi_m) = \delta_{mn}$. Then we obtain $f = \sum_{n=1}^{\infty} (f, \phi_n)\psi_n$. Because of that, we can write

$$\| f - f_n \|_{L^2(a,b)} \longrightarrow 0, \qquad (n \longrightarrow \infty) \tag{22}$$

where $f_n = \sum_{k=1}^{n} (f, \phi_k)\psi_k$. Notice that, by (12), we have $H\psi_k = \lambda_k \phi_k$ and we obtain

$$(Hf_n)(x_0) = \int_a^b H(x_0, y) \sum_{k=1}^{n} (f, \phi_k)\psi_k(y)\, dy = \sum_{k=1}^{n} \lambda_k (f, \phi_k)\phi_k(x_0).$$

Since $(Hf)(x_0) = \int_a^b H(x_0, y) f(y)\, dy$, we have the following estimation

$$\begin{aligned}\left| (Hf)(x_0) - \sum_{\nu=1}^{n} \lambda_\nu (f, \phi_\nu)\phi_\nu(x_0) \right|^2 &= \left| \int_a^b H(x_0, y)(f(y) - f_n(y))\, dy \right|^2 \\ &\le \int_a^b |H(x_0, y)|^2\, dy \cdot \| f - f_n \|^2 \longrightarrow 0 \quad (n \longrightarrow \infty)\end{aligned}$$

(because of (22)) what proves the equality (21). Now, if we combine (20) and (21), we obtain that for any $f \in L^2(a, b)$

$$\int_a^b B(x_0, y) f(y)\, dy = \int_a^b H(x_0, y) f(y)\, dy$$

i.e.

$$\int_a^b (B(x_0, y) - H(x_0, y)) f(y)\, dy = 0.$$

If we have chosen the continuous function $f(y) = \overline{B(x_0, y) - H(x_0, y)}$ instead of f, we would have obtained

$$\int_a^b |B(x_0, y) - H(x_0, y)|^2\, dy = 0$$

implying that $B(x_0, y) = H(x_0, y)$ for all $y \in [a, b]$. If we put $y = x_0$ in the preceding equality, then we will have

$$\sum_{n=1}^{\infty} \lambda_n |\phi_n(x_0)|^2 = H(x_0, x_0).$$

Since x_0, is arbitrary in the interval $[a, b]$ and the function $H(\cdot, \cdot)$ is continuous, we have, by the Dini's theorem, that the series $\sum_{n=1}^{\infty} \lambda_n |\phi_n(x)|^2$ uniformly converges on $[a, b]$ and $\sum_{n=1}^{\infty} \lambda_n |\phi_n(x)|^2 = H(x, x)$

The uniform convergence of the series $\sum_{n=1}^{\infty} \lambda_n |\phi_n(x)|^2 = H(x, x)$ on $[a, b]$ and

$$\psi_n(x) = (I + S)\phi_n(x) = \phi_n(x) + \int_a^b S(x, y)\phi_n(y)\, dy$$

show that series $\sum_{n=1}^{\infty} \lambda_n |\psi_n(x)|^2$ also uniformly converges on $[a, b]$. Let us now prove that the series $\sum_{n=1}^{\infty} \phi_n(x)\overline{\psi_n(y)}$ uniformly converges on $[a, b] \times [a, b]$. It follows directly from the estimation

$$|\sum_{n=p}^{q} \lambda_n \phi_n(x)\overline{\psi_n(y)}|^2 \leq \sum_{n=p}^{q} \lambda_n |\phi_n(x)|^2 \cdot \sum_{n=p}^{q} \lambda_n |\psi_n(y)|^2$$

and from the uniform convergence of the series

$$\sum_{n=1}^{\infty} \lambda_n |\psi_n(x)|^2 \quad \text{and} \quad \sum_{n=1}^{\infty} \lambda_n |\phi_n(x)|^2$$

on the interval $[a, b]$. Since the systems ϕ_n and ψ_n are bases of $L^2(a, b)$ and $(\phi_n, \psi_m) = \delta_{nm}$, the systems Φ_{np} and G_{mq}, where $\Phi_{np}(x, y) = \phi_n(x)\overline{\psi_p(y)}$ and $G_{mq}(x, y) = \psi_m(x)\overline{\phi_q(y)}$, are the bases of $L^2((a, b) \times (a, b))$ and

$$(\Phi_{np}, G_{mq})_{L^2((a,b)\times(a,b))} = \delta_{nm}\delta_{qp}$$

is valid. Because of that we have

$$\mathcal{A}(x, y) = \sum_{n,p=1}^{\infty} (\mathcal{A}, G_{n,p})_{L^2((a,b)\times(a,b))} \Phi_{np}(x, y) \tag{23}$$

where the convergence in (23) is taken as the convergence in $L^2((a, b)\times(a, b))$ norm.

Since $(\mathcal{A}, G_{np})_{L^2((a,b)\times(a,b))} = (A\phi_n, \psi_n)_{L^2(a,b)} = \lambda_n$, the expansion (23) can be written as

$$\mathcal{A}(x, y) = \sum_{n=1}^{\infty} \lambda_n \phi_n(x)\overline{\psi_n(y)}$$

where the convergence in $L^2((a,b)\times(a,b))$ norm. As it is already shown that the series $\sum_{n=1}^{\infty}\lambda_n\phi_n(x)\overline{\psi_n(y)}$ uniformly converges on $[a,b]\times[a,b]$, we obtain

$$\mathcal{A}(x,y)=\sum_{n=1}^{\infty}\lambda_n\phi_n(x)\overline{\psi_n(y)}$$

where the convergence is uniform on $[a,b]\times[a,b]$. That proves the theorem.

Remark 1 The theorem also holds for operators whose kernels are functions $H(\cdot,\cdot),S(\cdot,\cdot):R^{2k}\longrightarrow R$ if those kernels satisfy condition of the theorem.

Remark 2 The theorem can be used to establish the corresponding expansion theorems with respect to system of eigenfunctions for some nonselfadjoint differential operators whoose inverse operators have the form $H(I+S)$, where $H>0$, $\mathrm{Ker}H=\{0\}$, $S=S^*$ and $-1\in\rho(S)$.

Example Let L_1 and L_2 be selfadjoint differential operators generated by differential expresions

$$l_1(y)=\sum_{i=0}^{n}\alpha_i y^{(i)},\quad \alpha_i=\alpha_i(x)\qquad l_2(y)=\sum_{i=0}^{m}\beta_i y^{(i)},\quad \beta_i=\beta_i(x)$$

and by selfadjoint boundary condition

$$U_i^{(1)}(y)=0,\ i=1,2,...,n\qquad U_j^{(2)}(y)=0,\ j=1,2,...,m.$$

Suppose that L_1 is a positive differential operator. Let $\mathrm{Ker}L_2=\mathrm{Ker}L_1=\{0\}$ and -1 be regular point of the operatorL_2. Denote by $G(\cdot,\cdot)$ Green function of the operator L_2+I. We consider the operator

$$Ly=\int_0^1 G(x,\xi)l_2(l_1(y(\xi)))d\xi$$

with boundary condition

$$\begin{cases} U_i^{(1)}(y)=0,\ i=1,2,...,n \\ U_j^{(2)}(l_1(y))=0,\ j=1,2,...,m. \end{cases}\tag{*}$$

Then every function $f\in C^{m+n}[a,b]$ satisfying boundary conditions $(*)$ can be expanded in a uniform convergent series with respect to a system of eigenfunctions of the operator L.

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