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SPACES WITH EXCEPTIONAL FUNDAMENTAL GROUPS

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Abstract. The geometric interpretations of all real exceptional simple Lie groups of classes G_2, F_4, E_6, E_7 and E_8 are described. In particular, we describe the interpretations of the four last classes as groups of motions of elliptic and hyperbolic planes over algebras of octaves and split octaves and over tensor products of them and algebras of usual and split complex numbers, quaternions and octaves. The explicite expressions of motions of these planes are found. The symmetry figures and parabolic figures of all considered spaces and geometric interpretations of all fundamental linear representations of real exceptional simple Lie groups are found.

1. Spaces with classical fundamental groups

The geometric meaning of the complex simple Lie groups of the infinite series A_n, B_n, C_n, D_n ("classical simple Lie groups") was well known to the founder of the theory of these groups Sophus Lie. Lie knew that: the groups A_n are the groups of collineations of the complex projective n-spaces $\mathbf{C}P^n$, the groups B_n are the groups of motions of the complex non-Euclidean $2n$-spaces $\mathbf{C}S^{2n}$, the groups C_n are the groups of symplectic transformations of the complex symplectic $(2n-1)$-spaces $\mathbf{C}Sy^{2n-1}$, and the groups D_n are the groups of motions of the complex non-Euclidean $(2n-1)$-spaces $\mathbf{C}S^{2n-1}$. The spaces $\mathbf{C}S^N$ are the spaces $\mathbf{C}P^N$ in which the quadrics $\sum_i (x^i)^2 = 0$ are given; the motions of these spaces are the collineations preserving these quadrics called the absolutes of these spaces; the spaces $\mathbf{C}Sy^{2n-1}$ are the spaces $\mathbf{C}P^{2n-1}$ in which the linear complex of lines $\sum_i p^{2i,2t+1} = 0$ is given ($p^{ij} = x^i y^j - y^i x^j$ are the Plückerian coordinates of the lines), the symplectic transformations are collineations preserving this linear complex, defined just as the spaces $\mathbf{C}S^{2n}$. All complex simple Lie groups were found by Killing [1888-1890] and by Cartan [1894].

All real simple Lie groups were found by Cartan [1914]. The geometric meaning of some of these groups was known in 19th century: certain non-compact real groups A_n and C_n are the groups of collineations of the real projective spaces P^n and the groups of symplectic transformations of the real symplectic spaces Sy^{2n-1} defined just as the complex ones; the compact real groups B_n and D_n are the groups

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AMS Subject Classification (1985): Primary 51 F 99, 53 C 35, 22 E 46 of motions of elliptic spaces S^{2n} and S^{2n-1} with imaginary absolutes having the same equations as the absolutes of the complex non-Euclidean spaces. It was also known that the non-compact real groups B_n and D_n are the groups of motions of hyperbolic (Lobachevskian) spaces H^{2n} and H^{2n-1} whose absolutes are real oval quadrics, and that the compact real groups A_n are the groups of motions of complex Hermitian elliptic spaces $\mathbf{C}\bar{S}^n$ defined by Study [1905]. The spaces $\mathbf{C}S^n$ are the spaces $\mathbf{C}P^n$ in which the absolute is an imaginary Hermitian quadric $\sum_i \bar{x}ix^i$; the motions of this space are the collineations preserving this absolute. Study [1905] also defined the complex Hermitian hyperbolic spaces $\mathbf{C}\bar{H}^n$ whose group of motions is a non-compact real group of the same class.

The geometric interpretations of many real simple Lie groups were found by Cartan in his extended translation [1915] of Fano's paper in the *Encyclopaedia of mathematical sciences*, the first 21 pages of which were published in 1914, but the whole paper was not published then because of World War I and was published only in *Cartans Collected works* in 1955. Some results of this paper were published in Cartan [1931] and in some Cartan's papers. The non-compact real groups B_n and D_n, besides the groups of motions of H^{2n} and H^{2n-1}, are the groups of motions of real pseudo-elliptic and pseudo-hyperbolic spaces S_l^{2n}, S_l^{2n-1}, H_l^{2n} and H_l^{2n-1} whose absolutes are quadrics $-\sum_\alpha (x^\alpha)^2 + \sum_i (x^i)^2 = 0$ ($1 \leq \alpha < l$ for pseudo-elliptic spaces and $1 < \alpha \leq l$ for pseudo-hyperbolic spaces) and the groups of symplectic transformations of the quaternion Hermitian symplectic spaces $\mathbf{H}\bar{S}y^{n-1}$, that is the spaces $\mathbf{H}P^{n-1}$ in which the absolute $\sum_i \bar{x}^i ix^i$ is given. The non-compact real groups A_n, besides the groups of collineations of P_n and the groups of motions of $\mathbf{C}\bar{H}^n$, are the groups of motions of complex Hermitian pseudo-elliptic and pseudo-hyperbolic spaces $\mathbf{C}\bar{S}_l^n$ and $\mathbf{C}\bar{H}_l^n$ whose absolutes are Hermitian quadrics $-\sum_\alpha x^\alpha \bar{x}^\alpha + \sum_i \bar{x}^i x^i = 0$ ($1 \leq \alpha < l$ and $1 < \alpha \leq l$ respectively) and the groups of collineations of quaternion projective spaces $\mathbf{H}P^{(n-1)/2}$. The compact real groups C_n, as Cartan [1927] established, are the groups of motions of quaternion Hermitian elliptic spaces $\mathbf{H}\bar{S}^{n-1}$ defined just as the spaces $\mathbf{C}\bar{S}^{n-1}$ (earlier Cartan used more complicated geometric interpretations of these groups by means of the spaces $\mathbf{C}\bar{S}^{2n-1}$ with linear complex. This interpretation is equivalent to the interpretation of $\mathbf{H}\bar{S}^n$ by the *paratactic congruence of lines* in $\mathbf{C}\bar{S}^{2n+1}$ analogous to the interpretation of $\mathbf{C}\bar{S}^n$ by the paratactic congruence of lines in S^{2n+1}).

These compact groups are often called "unitary symplectic groups" or "quaternion symplectic groups". The non-compact real groups C_n, besides the groups of symplectic transfomations of Sy^{2n-1}, are the groups of motions of quaternion Hermitian hyperbolic, pseudo-elliptic and pseudo-hyperbolic spaces $\mathbf{H}\bar{H}^{n-1}$, $\mathbf{H}\bar{S}_l^{n-1}$ and $\mathbf{H}\bar{H}_l^{n-1}$ defined just as spaces $\mathbf{C}\bar{H}^{n-1}$, $\mathbf{C}\bar{S}_l^{n-1}$ and $\mathbf{C}\bar{H}_l^{n-1}$. The names and notations of pseudo-elliptic and pseudo-hyperbolic spaces are used by Wolf [1984]. The geometry of all these spaces except $\mathbf{H}\bar{S}y^n$ were described in author's book [1955]. The spaces $\mathbf{H}\bar{S}y^n$ were described by Rumyanceva [1963a] (let us note that the spaces S_l^n and H_{l-1}^n have the same absolute but the curvature of the first and second spaces is positive and negative respectively).

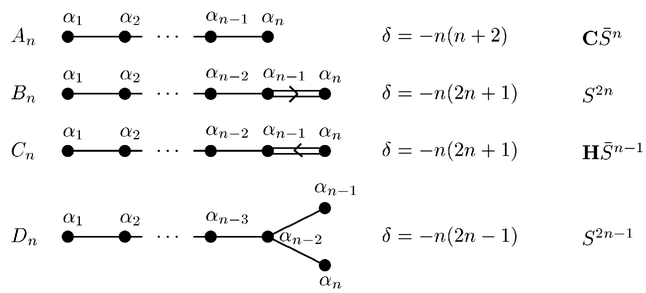


Fig. 1. Dynkin diagrams for classical groups A_n, B_n, C_n, D_n

The complex, compact and split real simple Lie groups are characterized by *Dynkin diagrams* (Fig. 1) whose dots represent the simple roots of these groups (all these roots for compact real groups are purely imaginary; for split groups all these roots are real). The non-compact and non-split simple Lie groups are characterized by *Satake diagrams* (Fig. 2), whose white dots represent the real simple roots of these groups, while the black dots represent their purely imaginary simple roots, and the white dots which are joined by double arrows represent their conjugate imaginary simple roots.

2. Isomorphisms of the classical groups

In the papers mentioned above Killing and Cartan found also the isomorphisms between complex and real simple groups $A_1 = B_1 = C_1$, $D_2 = A_1 \times A_1$, $B_2 = C_2$ and $A_3 = D_3$ and all their geometric interpretations (Cartan [1915] called the geometries with isomorphic groups *equivalent geometries*).

Let us note that the compact real group C_1 is the group of automorphisms of the skew field $\mathbf{H}$ of quaternions. The Dynkin and Satake diagrams of the isomorphic groups are similar (Fig. 3 and 4). The isomorphisms $A_1 = B_1$ and $B_2 = C_2$ are connected with isometries of lines $\mathbf{C}\bar{S}^1$ and $\mathbf{H}\bar{S}^1$ to the spheres of the Euclidean spaces R^3 and R^5 respectively.

Let us note also that if we replace in the definition of the space $\mathbf{C}\bar{S}^n$ the field $\mathbf{C}$ by the algebra $\mathbf{C}'$ of split complex numbers $a + be$, where a, b are real numbers, $e^2 = +1$ (this algebra is isomorphic to the direct sum $\mathbf{R} \oplus \mathbf{R}$ of two fields $\mathbf{R}$ of real numbers), then we obtain the space $\mathbf{C}'\bar{S}^n$ whose group of motions is isomorphic to the group of collineations of P^n (the space $\mathbf{C}'\bar{S}^n$ admits interpretation as manifold of pairs point+hyperplane of P^n.

If in the definitions of the spaces $\mathbf{H}\bar{P}^n$, $\mathbf{H}\bar{S}^n$ and $\mathbf{H}\bar{S}y^n$ we replace the field $\mathbf{H}$ by the algebra $\mathbf{H}'$ of *split quaternions* $a + bi + ce + df$, where a, b, c, d are real numbers, $i^2 = -1$, $e^2 = +1$, $ie = -ei = f$ (this algebra is isomorphic to the algebra $\mathbf{R}_2$ of real 2-matrices), then we obtain the spaces $\mathbf{H}'\bar{P}^n$, $\mathbf{H}'\bar{S}^n$ and $\mathbf{H}'\bar{S}y^n$ whose

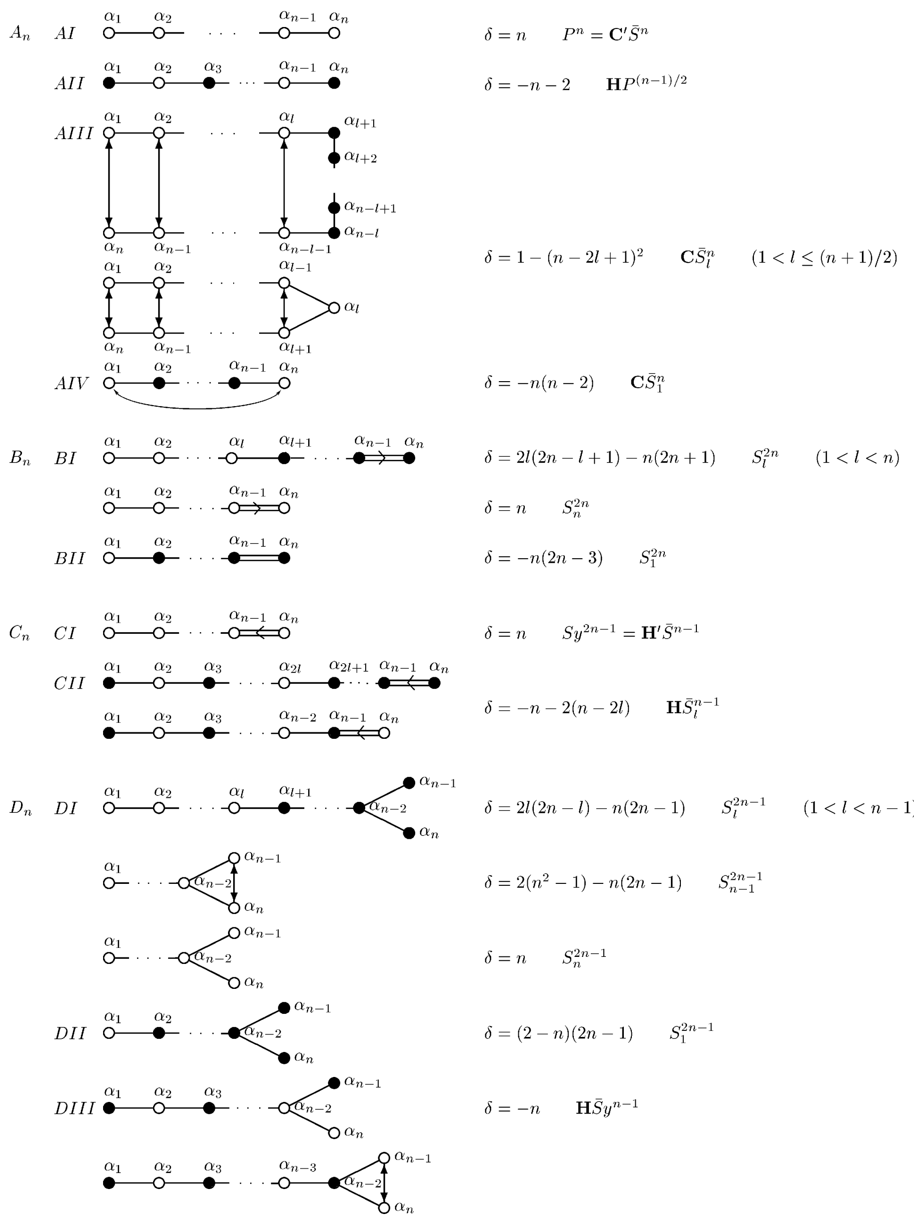


Fig. 2. Satake diagrams for classical groups A_n, B_n, C_n, D_n

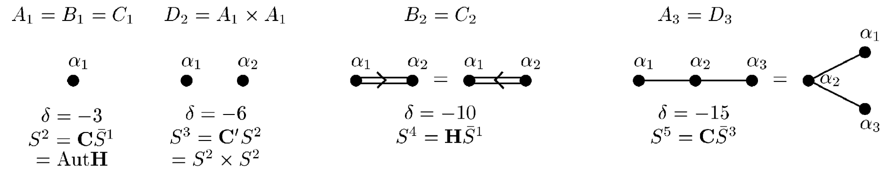


Fig. 3. Dynkin diagrams for isomorphic classical groups $A_1 = B_1 = C_1$, $D_2 = A_1 \times A_1$, $B_2 = C_2$, $A_3 = D_3$

fundamental groups are isomorphic to the groups of collineations of P^{2n+1}, to the groups of symplectic transformations of Sy^{2n+1} and to the gruop of motions of S^{2n+1} respectively (these split quaternion spaces admit interpretations as manifolds of lines of real spaces).

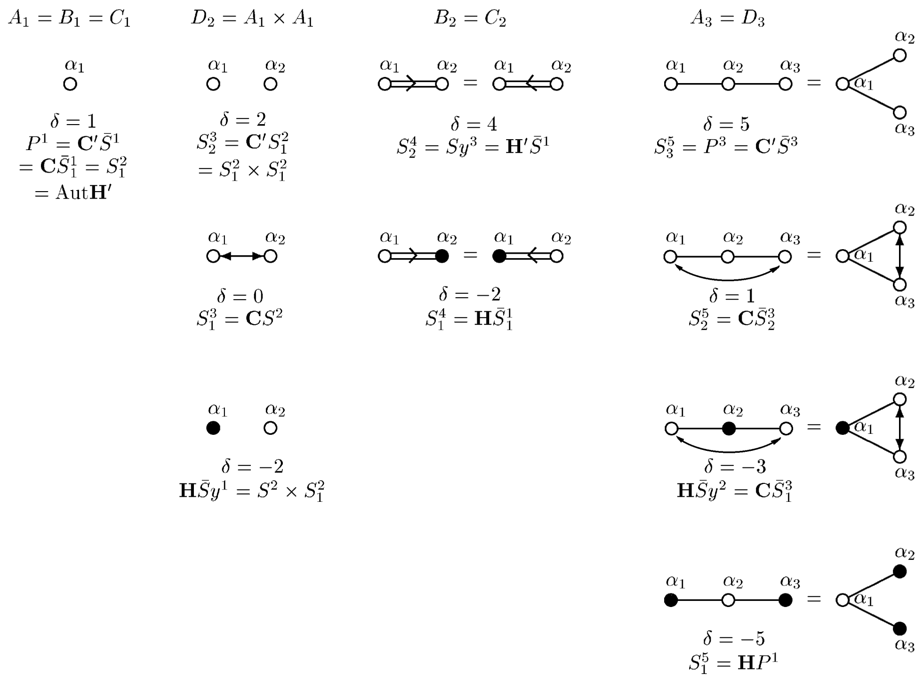


Fig. 4. Satake diagrams for isomorphic classical groups $A_1 = B_1 = C_1$, $D_2 = A_1 \times A_1$, $B_2 = C_2$, $A_3 = D_3$

If in the definitions of the spaces $\mathbf{C}\bar{S}^n$ and $\mathbf{H}\bar{S}^n$ we replace the fields $\mathbf{C}$ and $\mathbf{H}$ by the tensor products $\mathbf{C} \otimes \mathbf{C}$, $\mathbf{C} \otimes \mathbf{H}$ and $\mathbf{H} \otimes \mathbf{H}$, then we obtain the spaces $(\mathbf{C} \otimes \mathbf{C})\tilde{\bar{S}}^n$, $(\mathbf{C} \otimes \mathbf{H})\tilde{\bar{S}}^n$ and $(\mathbf{H} \otimes \mathbf{H})\tilde{\bar{S}}^n$ investigated by Abbasov [1962, 1963] and Rumyanceva [1963b]. The equations of absolutes of the elliptic spaces over tensor products of fields have the form $\sum_i \tilde{\bar{x}}^i x^i = 0$, where the involution $x \to \tilde{\bar{x}}$ consists of the involutions $x \to \bar{x}$ and $x \to \tilde{x}$ in both tensor factors. Abbasov and

Rumyanceva proved that the groups of motions of these spaces are isomorphic to the direct products of two groups of motions of $\mathbf{C}\bar{S}^n$ and to the groups of motions of $\mathbf{C}\bar{S}^{2n+1}$ and S^{4n+3} respectively, and that these spaces admit interpretations as manifolds of pairs of points of two $\mathbf{C}\bar{S}^n$, of lines of $\mathbf{C}\bar{S}^{2n+1}$ and of 3-planes of S^{4n+3} respectively.

3. Octave planes

As was shown by Killing [1888-1890] and Cartan [1894], besides the infinite series A_n, B_n, C_n, D_n there are 5 classes of exceptional complex simple Lie groups G_2, F_4, E_6, E_7 and E_8. Cartan [1914] found all the real simple Lie groups of these classes and showed that the compact real group G_2 is the group of automorphisms of the alternative skew field $\mathbf{O}$ of octaves (Cayley numbers). The geometric meaning of the remaining exceptional Lie groups was for a long time a riddle for the matematicians.

Borel [1950] and Freudenthal [1951] proved that the compact real group F_4 is the group of motions of the octave Hermitian elliptic plane $\mathbf{O}\bar{S}^2$ and that one of the non-compact real groups E_5 is the group of collineations of the octave projective plane $\mathbf{O}\bar{P}^2$ investigated by Moufang [1939] and Hirsch [1949]. Borel defined the planes $\mathbf{O}\bar{P}^2$ and OS^2 by means of topological methods, Freudenthal by means of algebraic methods. Tits [1956] proved that one of the non-compact real groups F_4 is the group of motions of octave Hermitian hyperbolic plane $\mathbf{O}\bar{H}^2$.

In the cases of the spaces P^n, $\mathbf{C}P^n$, $\mathbf{C}'P^n$, $\mathbf{H}P^n$ and $\mathbf{H}'P^n$ over the associative fields and algebras the points of these spaces are characterized by coordinates x^i defined up to the multiciplation $x^i \to x^i\lambda$, where λ is non-zero element of the fields $\mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$, and algebras $\mathbf{C}'$ and $\mathbf{H}'$ and in the two last cases the multiplier λ must not be a zero divisor. But in the case of the plane $\mathbf{O}P^2$ and the analogous plane $\mathbf{O}'P^2$ over the alternative algebra $\mathbf{O}'$ of *split octaves* (having the common complex form with field $\mathbf{O}$) this is impossible since $(x^i\lambda)\mu \neq x^i(\lambda\mu)$. Therefore Freudenthal used for definition of the plane $\mathbf{O}P^2$ the *Jordan algebra* $\mathbf{J}_3$ of octave 3-matrices (x^{ij}) for which $x^{ij} = \bar{x}^{ji}$ and $x^{ij}x^{jk} = x^{jj}x^{ik}$. All elements of these matrices are from an associative subfield of the alternative field $\mathbf{O}$, since all a^{ii} are real. These associative subfields are isomorphic to the fields $\mathbf{H}$, $\mathbf{C}$ or $\mathbf{R}$. Therefore the points of the plane $\mathbf{O}P^2$ can be defined by three coordinates x^0, x^1, x^2 from an associative subfield of $\mathbf{O}$ connected with elements x^{ij} of 3-matrices (x^{ij}) of $\mathbf{J}_3$ by relations $x^{ij} = \bar{x}^i x^j$. The coordinates of points of the plane $\mathbf{O}'P^2$ are defined analogously. The absolutes of the planes $\mathbf{O}\bar{S}^2$ and $\mathbf{O}'\bar{S}^2$ are Hermitian conics with equations $\bar{x}^0x^0 + \bar{x}^1x^1 + \bar{x}^2x^2 = 0$, the absolute of the plane $\mathbf{O}\bar{H}^2$ is a Hermitian conic with equation $-\bar{x}^0x^0 + \bar{x}^1x^1 + \bar{x}^2x^2 = 0$.

The collineations in the spaces over associative fields and algebras have the form $'x^i = \sum_j a_i^j f(x^j)$, where a_i^j and x_j are arbitrary elements of a field or of an algebra and $x \to f(x)$ is an automorphism of this field or algebra. In the spaces over non-associative fields and algebras the collineations have the analogous form $'x^i = \sum_j \bar{\bar{a}}_j^i f(x^j)$ where the elements x^j are from an associative subfield or subalgebra. Here the elements a_i^j are arbitrary elements of a field or of an algebra, $x \to f(x)$ is an automorphism of this field or algebra and the operation $x \to \bar{\bar{x}}$ is a "projection" of the element x onto the associative subfield or subalgebra. This "projection" for the field $\mathbf{O}$ and its subfield $\mathbf{H}$ is defined as follows: in the field $\mathbf{O}$ there is an automorphism $x \to \hat{x}$ preserving the subfield $\mathbf{H}$, for instance for $a = a_0 + a_1 i + a_2 j + a_3 k + a_4 l + a_5 p + a_6 q + a_7 r$ we have $a \to \hat{a} = a_0 + a_1 i + a_2 j + a_3 k - a_4 l - a_5 p - a_6 q - a_7 r$ and $\bar{\bar{x}} = (x + \hat{x})/2$. In other cases this "projection" is defined analogously. The motions of the planes $\mathbf{O}\bar{S}^2$, $\mathbf{O}\bar{H}^2$ and $\mathbf{O}'\bar{S}^2$ are the collineations of the planes $\mathbf{O}P^2$ and $\mathbf{O}'P^2$ preserving the absolutes. Therefore the Lie algebras of the groups of collineations of planes $\mathbf{O}P^2$ and $\mathbf{O}'P^2$ are direct sums of the sets of 3-matrices (a_{ij}) of the algebras $\mathbf{O}_3$ or $\mathbf{O}'_3$ over the field $\mathbf{O}$ or over the algebra $\mathbf{O}'$ satisfying one condition $a_{00} + a_{11} + a_{22} = 0$ and of Lie algebras of the groups of automorphisms of the field $\mathbf{O}$ or of the algebra $\mathbf{O}'$; these groups of automorphisms are compact and split groups G_2 respectively. Analogously, the groups of motions of elliptic planes $\mathbf{O}\bar{S}^2$ and $\mathbf{O}'\bar{S}^2$ are the direct sums of the sets of 3-matrices (a_{ij}) of algebras $\mathbf{O}_3$ or $\mathbf{O}'_3$ satisfying the conditions $a_{ji} = -\bar{a}_{ji}$ and $a_{00} + a_{11} + a_{22} = 0$ and of Lie algebras of groups of automorphisms of the field $\mathbf{O}$ or of the algebra $\mathbf{O}'$, and the Lie algebra of the group of motions of the plane $\mathbf{O}\bar{H}^2$ is the direct sum of the sets of 3-matrices (a_{ij}) of the algebra $\mathbf{O}_3$ satisfying the conditions $a_{ij} = -\bar{a}_{ji}\varepsilon_i\varepsilon_j$ $(\varepsilon_0 = -1, \varepsilon_1 = \varepsilon_2 = 1)$ and $-a_{00} + a_{11} + a_{22} = 0$ and of Lie algebras of the groups of automorphisms of the field $\mathbf{O}$. Therefore in the case of the groups F_4 the dimension of these Lie algebras is $8+8+8+7+7+14 = 52$, and in the case of the groups E_6 the dimension of these Lie algebras is $8 \cdot 8 + 14 = 78$. These numbers (52 and 78) coincide with the dimensions of the groups F_4 and E_6.

Let us note that the plane $\mathbf{O}\bar{S}^2$ is *compact symmetric Riemannian space* V^{16} which is the irreducible symmetric space FII according to Cartan [1926–1927]. The lines of this plane are isometric to spheres of R^9. The plane $\mathbf{O}\bar{H}^2$ is divided by its absolute into two domains, one of which is *non-compact Riemannian symmetric space* V^{16}, and the other is *pseudo-Riemannian symmetric space* V_1^{16}. The plane $\mathbf{O}'\bar{S}^2$ is divided by its absolute into two domains which are the pseudo-Riemannian symmetric spaces V_8^{16}. All these symmetric spaces are spaces of rank 1. The isotropy groups of these spaces (the groups of rotations around their points) are locally isomorphic to the groups of motions of lines $\mathbf{O}\bar{S}^1$, $\mathbf{O}\bar{H}^1$ and $\mathbf{O}'\bar{S}^1$, that is to the groups of rotations of R^9, R_1^9, or R_4^9 or to the groups of motions of the spaces S^8, H^8 and S_4^8.

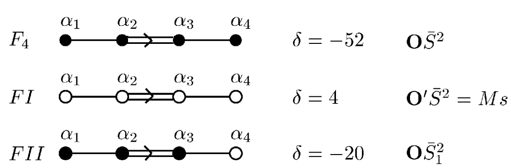


Fig. 5. Dynkin and Satake diagrams for exceptional groups F_4

Since all symmetric spaces with groups of motions (isometries) isomorphic to the fundamental groups of certain spaces can be interpreted as manifolds of *symmetry figures* of this space, the symmetry figures of the plane $\mathbf{O}\bar{S}^2$ are *points with their polar lines* forming an interpretation of the symmetric Riemannian 16-space FII, and *normal quaternion plane chains* (isometric to the plane $\mathbf{H}\bar{S}^2$) forming an interpretation of the symmetric 28-space FI. The symmetry figures of the plane $\mathbf{O}P^2$ are pairs point+line forming an interpretation of the symmetric 32-space EIII, *Hermitian conics* forming an interpretation of the symmetric 26-space EIV and more complicated geometric figures interpreting the symmetric 42-space EI and the 40-space EII (we shall consider these figures in the next section). Fig. 5 represents Dynkin and Satake diagrams for real simple groups F_4.

4. Planes over tensor products

In our paper [1954] it was proved that the group of collineations of the plane $\mathbf{O}P^2$ is isomorphic to the group of motions of the Hermitian elliptic plane $(\mathbf{C}'\otimes\mathbf{O})\tilde{\bar{S}}^q$ admitting the interpretation as the manifold of pairs point+line of the plane $\mathbf{O}P^2$ (analogous to the interpretation of the space $\mathbf{C}'\bar{S}^n$ by the manifold of pairs point+hyperplane of P^n) and that the compact group E_6 is isomorphic to the group of motions of the Hermitian elliptic plane $(\mathbf{C}\otimes\mathbf{O})\tilde{\bar{S}}^q$.

In our paper [1956] this result was generalized to the compact groups E_7 and E_8 and was proved that these groups are isomorphic to the groups of motions of the planes $(\mathbf{H}\otimes\mathbf{O})\tilde{\bar{S}}^q$ and $(\mathbf{O}\otimes\mathbf{O})\tilde{\bar{S}}^q$. These planes and their groups of motions are defined analogously as for the plane $\mathbf{O}\bar{S}^2$ and its group of motions, but in these cases the equations of the absolutes have the form $\tilde{\bar{x}}^0x^0+\tilde{\bar{x}}^1x^1+\tilde{\bar{x}}^2x^2=0$, where the involution $x\to\tilde{\bar{x}}$ as in the cases of tensor products of associative algebras, consists of the involutions $x\to\bar{x}$ and $x\to\tilde{x}$ in both tensor factors. The Lie algebras of these groups are the direct sums of the sets of 3-matrices (a_{ij}) of the algebras $(\mathbf{C}\otimes\mathbf{O})_3$, $(\mathbf{H}\otimes\mathbf{O})_3$ and $(\mathbf{O}\otimes\mathbf{O})_3$ satisfying the conditions $a_{ij}=-\tilde{\bar{a}}_{ji}$ and $a_{00}+a_{11}+a_{22}=0$ and of the Lie algebras of groups of automorphisms of the corresponding algebras. These groups of automorphisms are direct sums of the Lie algebras of groups of automorphisms of tensor factors of these tensor products. Therefore in the case of the compact group E_6 the dimension of this Lie algebra is $16+16+16+8+8+14=78$, in the case of the compact group E_7 the dimension of this Lie algebra is $32+32+32+10+10+14+3=133$, and in the case of the compact group E_8 the dimension of this Lie algebra is $64+64+64+14+14+14+14=248$. All these numbers (78, 133 and 248) coincide with the dimension of the groups E_6, E_7 and E_8.

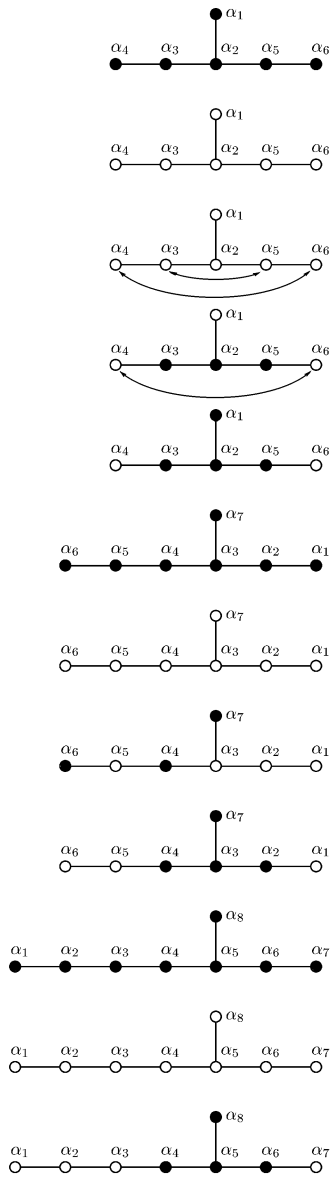
The lines of these planes admit the interpretations as the *Grassmann manifolds* $G_{9,1}$, $G_{11,3}$ and $G_{15,7}$ of lines of the elliptic space S^9, of 3-planes of S^{11} and of 7-planes of S^{15} respectively. The polar manifolds of all these lines, 3-planes are 7-planes, thus the Hermitian elliptic line over the tensor product of 2^p- and 2^q-dimensional fields admit an interpretation as the Grassmann manifold $G_{p+q-1,p-1}$ of $(p-1)$- or $(q-1)$-planes of S^{p+q-1} (this rule is also valid for $p,q=1,2$). The isotropy groups of points of these elliptic planes are locally isomorphic respectively to direct products of the groups of motions of S^9, S^{11} and S^{15} and the groups $|\alpha| = 1$ of the corresponding fields (in the cases of the fields $\mathbf{C}$ and $\mathbf{H}$ these groups are the groups of rotation of a circle and of a 3-sphere; in the case of $\mathbf{O}$ the set $|\alpha| = 1$ is not a group, but it is a loop).

The non-compact simple Lie groups admit analogous geometric interpretations as the planes obtained from Hermitian elliptic planes over tensor products $\mathbf{C} \otimes \mathbf{O}$, $\mathbf{H} \otimes \mathbf{O}$ and $\mathbf{O} \otimes \mathbf{O}$ by replacing the absolutes of elliptic planes by the Hermitian conics $-\tilde{\bar{x}}^0 x^0 + \tilde{\bar{x}}^1 x^1 + \tilde{\bar{x}}^2 x^2 = 0$ which are the absolutes of hyperbolic planes, or by replacing one or both fields $\mathbf{C}$, $\mathbf{H}$ and $\mathbf{O}$ by the coresponding split algebra $\mathbf{C}'$, $\mathbf{H}'$ and $\mathbf{O}'$.

Let us note that the planes $(\mathbf{C} \otimes \mathbf{O})\tilde{\bar{S}}^2$, $(\mathbf{H} \otimes \mathbf{O})\tilde{\bar{S}}^2$ and $(\mathbf{O} \otimes \mathbf{O})S^2$ form the interpretations of compact symmetric Riemannian 32-space EIII, 64-space EVI and 128-space EVIII defined in Cartan [1926-1927] and the corresponding Hermitian hyperbolic planes and Hermitian planes over another tensor products form the interpretations of symmetric Riemannian or pseudo-Riemannian spaces of the same dimensions. The ranks of the symmetric 32-, 64-, and 128-spaces EIII, EVI and EVIII are equal to the ranks of symmetric 16-, 32- and 64-spaces interpreted as lines of the corresponding elliptic planes over tensor products and as the corresponding Grassmann manifolds $G_{9,1}$, $G_{11,3}$, and $G_{15,7}$; these ranks are 2, 4 and 8 respectively.

The symmetry figures corresponding to the compact symmetric Riemannian spaces EIII, EVI and EVIII are *points and their polar lines* of the planes $(\mathbf{C}\otimes\mathbf{O})\tilde{\bar{S}}^2$, $(\mathbf{H} \otimes \mathbf{O})\tilde{\bar{S}}^2$ and $(\mathbf{O} \otimes \mathbf{O})\tilde{\bar{S}}^2$ respectively. The symmetry figures corresponding to the compact symmetric 40-space EII and 26-space EIV are *normal biquaternion plane chains* isometric to $(\mathbf{C} \otimes \mathbf{H})\tilde{\bar{S}}^2$ and *normal octave plain chains* isometric to $\mathbf{O}\bar{S}^2$ respectively. The symmetry figures corresponding to the compact symmetric 54-space EVII are normal plane chains isometric to $(\mathbf{C} \otimes \mathbf{O})\tilde{\bar{S}}^2$ (the isotropy group of the manifold of normal plane chains of this plane isometric to $(\mathbf{H} \otimes \mathbf{H})\tilde{\bar{S}}^2$ is isomorphic to the isotropy group of this plane). The symmetry figures corresponding to the compact symmetric 112-space EIX are normal plane chains isometric to $(\mathbf{H} \otimes \mathbf{O})\tilde{\bar{S}}^2$.

The symmetries of plane chains are determined by involutive automorphisms of one tensor factor of the tensor product. The symmetry figures with symmetries determined by involutive automorphisms in both factors of the tensor product are *normal bichains*. These figures correspond to the compact symmetric 42-space EI and 70-space EV (the isotropy group of the manifold of normal bichains of $(\mathbf{O} \otimes \mathbf{O})\tilde{\bar{S}}^2$ is isomorphic to the isotropy group of this plane). The normal bichains are isometric to the split octave bi-Hermitian plane $\mathbf{O}'\tilde{\bar{S}}^2$ determind by N. M. Zablotskikh [1969] and to analogous bi-Hermitian planes over the algebra with basis elements $1, i, j, I, Jl, Jp, Jq, Jr, Kl, Kp, Kq, Kr$, where $1, i, \ldots, r$ are the basis elements of the algebra $\mathbf{O}$ and $1, I, J, K$ are the basis elements of the



E_6	$\delta = -78$	$(\mathbf{C}\otimes\mathbf{O})\tilde{\tilde{S}}^2$
EI	$\delta = 6$	$(\mathbf{C}'\otimes\mathbf{O}')\tilde{\tilde{S}}^2 = \mathbf{O}'P^2 = \mathbf{C}'\bar{M}s$
EII	$\delta = 2$	$(\mathbf{C}\otimes\mathbf{O}')\tilde{\tilde{S}}^2 = \mathbf{C}\bar{M}s$
$EIII$	$\delta = -14$	$(\mathbf{C}\otimes\mathbf{O})\tilde{\tilde{S}}^2_1$
EIV	$\delta = -26$	$(\mathbf{C}'\otimes\mathbf{O})\tilde{\tilde{S}}^2 = \mathbf{O}P^2$
E_7	$\delta = -133$	$(\mathbf{H}\otimes\mathbf{O})\tilde{\tilde{S}}^2$
EV	$\delta = 7$	$(\mathbf{H}'\otimes\mathbf{O}')\tilde{\tilde{S}}^2 = \mathbf{O}'\bar{S}y^5 = \mathbf{H}'\bar{M}s$
EVI	$\delta = -5$	$(\mathbf{H}\otimes\mathbf{O}')\tilde{\tilde{S}}^2 = (\mathbf{H}\otimes\mathbf{O})\tilde{\tilde{S}}^2_1 = \mathbf{H}\bar{M}s$
$EVII$	$\delta = -25$	$(\mathbf{H}'\otimes\mathbf{O})\tilde{\tilde{S}}^2 = \mathbf{O}\bar{S}y^5$
E_8	$\delta = -248$	$(\mathbf{O}\otimes\mathbf{O})\tilde{\tilde{S}}^2$
$EVIII$	$\delta = 8$	$(\mathbf{O}'\otimes\mathbf{O}')\tilde{\tilde{S}}^2 = (\mathbf{O}\otimes\mathbf{O})\tilde{\tilde{S}}^2_1 = \mathbf{O}'\bar{M}s$
EIX	$\delta = -24$	$(\mathbf{O}\otimes\mathbf{O}')\tilde{\tilde{S}}^2 = \mathbf{O}\bar{M}s$

Fig. 6. Dynkin and Satake diagrams for exceptional groups E_6, E_7, E_8

algebra $\mathbf{H}$ commuting with $1, i, \ldots, r$, and the algebra with basis elements $1, i, j, I, J, K, Ll, Lp, Lq, Lr, Pl, Pp, Pq, Pr, Ql, Qp, Qq\ Qr, Rl, Rp, Rq, Rr$, where $1, i, \ldots, r$ and $1, I, \ldots, R$ are the basis elements of two commuting algebras $\mathbf{O}$; the subalgebras of these algebras with basis elements $1, i, j, k, Jl, Jp, Jq, Jr, \ldots$, and $1, i, j, k$, Rl, Rp, Rq, Rr are isomorphic to the algebra $\mathbf{O}'$. The fundamental groups of these bi-Hermitian planes are isomorphic to the groups of motions of these spaces $\mathbf{H}\bar{\bar{S}}^3$, $(\mathbf{C} \otimes \mathbf{H})\bar{\bar{S}}^3$, and $(\mathbf{H} \otimes \mathbf{H})\bar{\bar{S}}^3$, respectively and there are the bijective correspondence between points of the absolutes of these spaces and bi-Hermitian planes and between lines of these spaces and points of the bi-Hermitian planes. These interpretations of the manifolds of lines of Hermitian spaces on the bi-Hermitian planes are analogous to the Kotelnikov–Study interpretation of the manifold of lines of the space S^3 in the form of plane $\mathbf{C}'S^2$. Fig. 6 represents the Dynkin and Satake diagrams of real simple Lie groups E_6, E_7 and E_8.

5. Symplectic and metasymplectic geometries

Freudenthal [1954-1963] defined the *octave Hermitian symplectic 5-space* $\mathbf{O}\bar{S}y^5$ and proved that its fundamental group is a non-compact group E_7. In the same paper he defined also four *metasymplectic geometries* and proved that their fundamental groups are non-compact groups of all 4 classes F_4, E_5, E_7 and E_8.

Since a projective space over a non-associative field or algebra can exist only if its dimension is ≤ 2, the definition of $\mathbf{O}\bar{S}y^5$ as the space $\mathbf{O}\bar{P}^5$ with restricted group of collineations is impossible; therefore Freudenthal defined the space $\mathbf{O}\bar{S}y^5$ only as the set of 2-planes $\mathbf{O}\bar{P}^2$ analogous to isotropic 2-planes P^2 and $\mathbf{H}P^2$ of the spaces Sy^5 and $\mathbf{H}\bar{S}y^5$.

If in the definition of the space $\mathbf{H}\bar{S}y^5$ we replace the field $\mathbf{H}$ by the field $\mathbf{C}$, then we obtain the complex Hermitian symplectic spaces $\mathbf{C}\bar{S}y^n$ coinciding with space $\mathbf{C}\bar{S}^n$ or $\mathbf{C}\bar{S}^n_l$ (the general form of the absolute $\sum_i \bar{x}^i i x^i = 0$ is $\sum_{ij} \bar{x}^j a_{ij} x^i = 0$, where $a_{ij} = -\bar{a}_{ji}$, but in $\mathbf{C}P^n$ this equation is the equation of an Hermitian quadric); the isotropic points, lines and planes of the complex Hermitian symplectic spaces coincide with points and rectilinear and planar generators of the Hermitian quadric and the symplectic transformations of the spaces $\mathbf{C}\bar{S}y^n$ coincide with the motions of $\mathbf{C}\bar{S}^n$ and $\mathbf{C}\bar{S}^n_l$.

Freudenthal [1954-1963] defined also the "magic square" now called *the Freudenthal magic square.* This square consists of 16 groups

$$\begin{array}{cccc}
B_1 & A_2 & C_3 & F_4 \\
A_2 & A_2 \times A_2 & A_5 & E_6 \\
C_3 & A_5 & D_6 & E_7 \\
F_4 & E_6 & E_7 & E_8
\end{array}$$

The groups of the first row of this square are the groups of motions of the elliptic planes S^2, $\mathbf{C}\bar{S}^2$, $\mathbf{H}\bar{S}^2$, $\mathbf{O}\bar{S}^2$. The groups of the second row are the qroups of collineations of the projective planes P^2, $\mathbf{C}P^2$, $\mathbf{H}P^2$, $\mathbf{O}P^2$. The groups of the third row are the groups of symplectic transformations of the symplectic 5-spaces Sy^5, $\mathbf{C}\bar{S}y^5$, $\mathbf{H}\bar{S}y^5$, $\mathbf{O}\bar{S}y^5$, where $\mathbf{C}\bar{S}y^5$ is the space $\mathbf{C}\bar{S}^5_l$ with real 2-planer generators, that is $\mathbf{C}\bar{S}_3^5$. Therefore Freudenthal called the geometries of the fourth row *metasymplectic geometries*. Let us denote these geometries by Ms, $\mathbf{C}\bar{M}s$, $\mathbf{H}\bar{M}s$, $\mathbf{O}\bar{M}s$.

Freudenthal considered these geometries as geometries of the sets of *symplecta* which are sets of isotropic 2-planes of the spaces of the third row, these 2-planes are planes of the second row, and they contain the projective lines and points.

The compact groups of the same classes as the groups of Freudenthal magic sguare are the groups of motions of the Hermitian elliptic planes

$$\begin{array}{cccc} S^2 & \mathbf{C}\bar{S}^2 & \mathbf{H}\bar{S}^2 & \mathbf{O}\bar{S}^2 \\ \mathbf{C}\bar{S}^2 & (\mathbf{C}\otimes\mathbf{C})\bar{\bar{S}}^2 & (\mathbf{C}\otimes\mathbf{H})\bar{\bar{S}}^2 & (\mathbf{C}\otimes\mathbf{O})\bar{\bar{S}}^2 \\ \mathbf{H}\bar{S}^2 & (\mathbf{H}\otimes\mathbf{C})\bar{\bar{S}}^2 & (\mathbf{H}\otimes\mathbf{H})\bar{\bar{S}}^2 & (\mathbf{H}\otimes\mathbf{O})\bar{\bar{S}}^2 \\ \mathbf{O}\bar{S}^2 & (\mathbf{O}\otimes\mathbf{C})\bar{\bar{S}}^2 & (\mathbf{O}\otimes\mathbf{H})\bar{\bar{S}}^2 & (\mathbf{O}\otimes\mathbf{O})\bar{\bar{S}}^2 \end{array}$$

The split groups of the same classes are the groups of motions of the planes obtained from compact groups by substitution of all fields by corresponding split algebras. In the groups considered by Freudenthal himself only the first tensor factors of tensor products are replaced by corresponding split algebras. If in the square of planes with the compact groups of motions we replace all planes by the lines over the same algebras, then we obtain the square of Hermitian elliptic lines with the groups of motions isomorphic to the groups of motions of the real elliptic spaces forming the square

$$\begin{array}{cccc} S^1 & S^2 & S^4 & S^8 \\ S^2 & S^3 & S^5 & S^9 \\ S^4 & S^5 & S^7 & S^{11} \\ S^8 & S^9 & S^{11} & S^{15} \end{array}$$

6. Parabolic figures

Tits [1956] defined an important class of geometric figures called by him *fundamental elements*. These figures are the cases of *parabolic figures* – figures whose isotropy groups are parabolic subgroups of the fundamental group of the space. Tits [1957] called the manifolds of such figures R-*spaces*, Wolf [1969] called them *flag manifolds*. These manifolds can be called, by analogy with symmetric spaces, *parabolic spaces*, (see our paper with Zamakhovsky and Timoshenko [1990]). Each kind of Tits' fundamental elements corresponds to one simple root of the fundamental group of the space and to one dot on the Dynkin or Satake diagram of this group; the real and imaginary figures correspond to white and black dots of the Satake diagrams respectively, the imaginary conjugated figures correspond to white dots of these diagrams which are joined by double arrows, general parabolic figures correspond to a set of simple roots or dots of the Dynkin or Satake diagrams. Let us call the parabolic figures corresponding to simple roots α_i or to sets of simple roots $\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_k}$ α_i-*figures* and $(\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_k})$-*figures* respectively.

The α_i-figures of the spaces with split classical fundamental groups are as follows. The α_1-figures of the space P^n (see Fig. 1) are points of this space, the

α_2-figures are lines, the α_i-figures are $(i-1)$-planes, the (α_1, α_n)-figures are pairs of incident points and hyperplanes of P^n or points of the absolute of $\mathbf{C}'\bar{S}^n$, the $(\alpha_i, \alpha_{n-i+1})$-figures are pairs of incident $(i-1)$-planes and $(n-i)$-planes of P^n or $(i-1)$-planar generators of the absolute of $\mathbf{C}'\bar{S}^n$, the general parabolic figures of P^n are flags consisting of incident planes of different dimension (hence the term "flag manifold"). The α_1-figures of S^{2n}_n are the points of its absolute, the α_i-figures are the $(i-1)$-planar generators of the absolute. The α_1-figures of S^{2n-1}_y are points of this space, the α_i-figures are isotropic $(i-1)$-planes. The α_i-figures of S^{2n-1}_n are points of its absolute, the α_i-figures $(i = 2, 3, \ldots, n-2)$ are $(i-1)$-planar generators of the absolute, the α_{n-1}- and α_n-figures are $(n-1)$-planar generators of the first and the second families of the absolute ($(n-2)$-planar generators are (α_{n-1}, α_n)-figures).

The symplecta, 2-planes, lines and points of metasymplectic geometries Ms, $\mathbf{C}\bar{M}s$, $\mathbf{H}\bar{M}s$ and $\mathbf{O}\bar{M}s$ are also parabolic figures. The symplecta of these geometries are α_i-figures of the geometries (see Fig. 5 and 6), the 2-planes are α_2-figures of these geometries, the lines are α_3-figures of Ms, $\mathbf{H}\bar{M}s$ and $\mathbf{O}\bar{M}s$ and (α_3, α_5)-figures of $\mathbf{C}\bar{M}s$, the points are α_4-figures of Ms, (α_4, α_5)-figures of $\mathbf{C}\bar{M}s$, α_5-figures of $\mathbf{H}\bar{M}s$ and α_7-figures of $\mathbf{O}\bar{M}s$. All these parabolic figures are real.

The same parabolic figures can be defined on all Hermitian planes over $\mathbf{O}$, $\mathbf{O}'$ and their tensor products by $\mathbf{C}$, $\mathbf{C}'$, $\mathbf{H}$, $\mathbf{H}'$, $\mathbf{O}$ and $\mathbf{O}'$. The points of metasymplectic geometries are points of absolutes of corresponding planes (see our paper with Stepashko [1983]). Symplecta, 2-planes, lines and other parabolic figures of these geometries are sets of points of absolutes of these planes.

The Hermitian planes with split groups of motions are the planes $\mathbf{O}'\bar{S}^2$, $(\mathbf{C}'\otimes \mathbf{O}')\tilde{\bar{S}}^2$, $(\mathbf{H}'\otimes\mathbf{O}')\tilde{\bar{S}}^2$ and $(\mathbf{O}'\otimes\mathbf{O}')\tilde{\bar{S}}^2$. The geometries of these groups can be considered also as matasymplectic geometries Ms, $\mathbf{C}'\bar{M}s$, $\mathbf{H}'\bar{M}s$ and $\mathbf{O}'\bar{M}s$; the second and third of these geometries can also be considered as geometries of the plane $\mathbf{O}'P^2$ and of the space $\mathbf{O}'\bar{S}y^5$. All parabolic figures of these geometries are real.

The dimension of α_3-, α_2- and α_i-figures of $\mathbf{O}'\bar{S}^2$ are 1, 2 and 5. The dimensions of manifolds of α_1-, α_2-, α_3- and α_4-figures of this plane are 15, 20, 20 and 15.

α_4- and α_6-figures of the plane $\mathbf{O}'P^2$ are its points and lines, the dimensions of α_i-figures $(i = 1, 2, 3, 5, 6)$ of $\mathbf{O}'P^2$ are 9, 4, 1, 4 and 8. The dimensions of manifolds of α_i-figures of this plane $(i = 1, 2, \ldots, 6)$ are 21, 29, 25, 16, 25 and 16.

α_1-, α_7- and α_6-figures of the space $\mathbf{O}'\bar{S}y^5$ are its isotropic points, lines and 2-planes, the dimensions of two last parabolic figures of this space are 8 and 16. The dimensions of α_i-figures of $\mathbf{H}'\bar{M}s$ $(i = 1, 2, 3, 4, 5, 6, 7)$ are 33, 8, 4, 2, 10 and 11. The dimensions of manifolds of α_i-figures of this geometry $(i = 1, 2, \ldots, 7)$ are 33, 47, 53, 51, 42, 27 and 42.

The dimensions of α_i-figures of $\mathbf{O}'\bar{M}s$ $(i = 1, 2, 3, 4, 5, 6, 8)$ are 33, 16, 8, 4, 2, 1 and 7. The dimensions of manifolds of α_i-figures of $\mathbf{O}'\bar{M}s$ $(i = 1, 2, \ldots, 8)$ are 57, 83, 97, 104, 106, 98, 78 and 92.

7. The geometries of the groups D_4 and G_2

Among classical simple Lie group D_4 plays a special role since its Dynkin diagram (Fig. 7 represents the Dynkin and Satake diagrams of real simple groups D_4) has trilateral symmetry. The compact real group D_4 is the group of motions of the elliptic 7-space S^7, the split real group D_4 is the group of motions of the pseudo-elliptic space S^7_4, the other real groups D_4 are the groups of motions of the hyperbolic space H^7 and the pseudo-elliptic spaces S^7_2 and S^7_3 and the group of symplectic transformations of the space $\mathbf{H}\bar{S}y^3$.

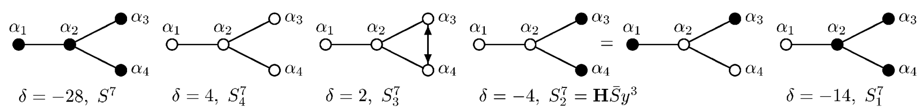


Fig. 7. Dynkin and Satake diagrams for classical group D_4

The symmetry figures of the space S^7 are points with their polar hyperplanes, lines with their polar 5-planes, 2-planes with their polar 4-planes, pairs of polar 3-planes, and *paratactic congruences* of lines, that is sets of lines joining the conjugate imaginary points of conjugate imaginary 3-planar generators of the absolute of this space belonging to one family of these generators. The space S^7 is the compact symmetric Riemannian space DII. The manifolds of lines and planes of S^7 form interpretations of compact symmetric Riemannian spaces DI, the manifold of paratactic congruences forms an interpretation of compact symmetric Riemannian space DIII.

α_i-figures of the space S^7_4 are the points of its absolute, α_2-figures of this space are the rectilinear generators of the absolute, α_3- and α_4-figures of this space are the 3-planar generators of the absolute belonging to two families of these generators.

Triliteral symmetry of the Dynkin diagram of this group defines the *principle of triality* of spaces S^7 and S^7_4 discovered by Cartan [1925]. According to this principle the points of absolutes of these spaces correspond to its 3-planar generators of both families and the lines joining the conjugate imaginary points of the absolute of S^7 and arbitrary points of the absolute of S^7_4 correspond to paratactic congruences (defined by 3-planar generators belonging to one family); the corresponding figures have isomorphic stationary subgroups.

This isomorphism defines an isomorphism of the fundamental groups of the spaces S^7_2 and $\mathbf{H}\bar{S}y^3$ and an interpretation of one of these spaces in the other.

The principle of triality of the space S^7 and S^7_4 is connected with algebras $\mathbf{O}$ and $\mathbf{O}'$: in these algebras one can intrduce the metrics of the spaces R^8 and R^8_4: the distance between α and β is $|\beta - \alpha|$. The hyperspheres $|\alpha| = 1$ of these spaces with identified antipodal points are models of the spaces S^7 and S^7_4; the absolutes of these spaces are imaged by bioctaves (elements of the tensor product $\mathbf{C} \otimes \mathbf{O}$) or split octaves of zero moduls. The 3-planar generators of these absolutes are defined by equations $\alpha\xi = 0$ and $\xi\alpha = 0$; therefore these generators are also defined by bioctaves or split octaves of zero moduls. If α and β define two 3-planar generators of different families, then their product $\alpha\beta$ defines the unique point of their intersection; if this product is equal to zero, then the intersection of these 3-planar generators is a 2-planar generator (see Weiss [1938]).

Trilateral symmetry of the Dynkin diagram of the group D_4 defines a 3-involutive automorphisms σ of this group ($\sigma^3 = 1$). The elements invariant under this automorphism form the group G_2 which is a subgroup of the group D_4 and of its subgroup B_3. The real simple groups G_2 are two groups: the compact group is the group of automorphisms of the field $\mathbf{O}$ and the split group is the group of automorphisms of the algebra $\mathbf{O}'$. If in the algebras $\mathbf{O}$ and $\mathbf{O}'$ we introduce the metrics of the spaces R^8 and R^8_4 as above, then the groups of automorphisms of these algebras are subgroups of groups of rotations of these spaces, which are the groups D_4, preserving the real axes ($\alpha = \bar{\alpha}$) of these spaces and 7-planes $\alpha = -\bar{\alpha}$ perpendicular to these axes, which are the spaces R^7 and R^7_3. Therefore the compact and split groups G_2 are subgroups of rotations of these 7-spaces or of the groups B_3. The groups G_2 are also subgroups of the groups of motions of the elliptic space S^6 and pseudo-elliptic space S^6_3, admitting models as 6-sphere of the spaces R^7 and R^7_3 with identified antipodal points. The groups of automorphisms of algebras $\mathbf{O}$ and $\mathbf{O}'$ are transitive on the space S^6 and on each of domains of S^6_3 defined by its absolute. The space S^6 and S^6_3, whose fundamental groups are these subgroups, were studied by Adamushko [1969] and were called by her *G-elliptic space* Sg^6 and G-*pseudo-elliptic space* Sg^6_3. Fig. 8 represents the Dynkin and Satake diagrams of real groups G_2 and the groups B_3 and D_4 containing these groups.

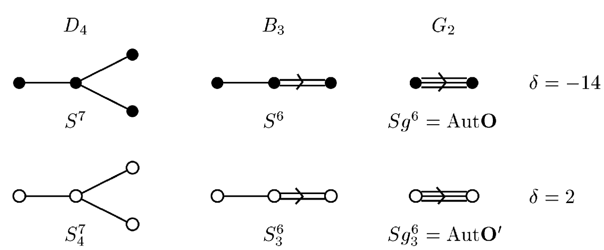


Fig. 8. Dynkin and Satake diagrams for exceptional groups G_2 and for classical groups D_4 and B_3 containing these groups

The symmetry figures of the space Sg^6 are special 2-planes obtained from the intersections of the 6-sphere which is the intersection of the hyperplane $\alpha = -\bar{\alpha}$ and the 7-sphere $|\alpha| = 1$ with associative subfields of the field $\mathbf{O}$ isomorphic to the field $\mathbf{H}$. The manifold of these 2-planes forms an interpretation of the compact symmetric 8-space G.

The α_i-figures of the space Sg^6_3 are *special rectilinear generators* of the absolute of this space which are the intersections of this absolute with 2-planes defined by associative subalgebras of the algebra $\mathbf{O}'$ isomorphic to the algebras $\mathbf{H}^0$ and $\mathbf{H}'^0$ of *semiquaternions* and *split semiquaternions* $a + bi + c\varepsilon + d\eta$ and $a + be + c\varepsilon + d\theta$ where a, b, c, d are real numbers, $i^2 = -1$, $e^2 = -1$, $\varepsilon^2 = 0$ $i\varepsilon = -\varepsilon i = \eta$, $e\varepsilon = -\varepsilon e = \theta$. The α_2-figures of this space are *points of the absolute* together with *special* 2*-planar generators* of this absolute defined by associative subalgebras of the algebra $\mathbf{O}'$ isomorphic to the algebra $\mathbf{H}^{00}$ of *quarter-quaternions* $a + b\varepsilon + c\eta + d\omega$, where a, b, c, d are real numbers, $\varepsilon^2 = \eta^2 = 0$, $\varepsilon\eta = -\eta\varepsilon = \omega$, through these points. The special generators of the absolute of Sg_3^6 were studied by Schellekens [1962].

8. Linear representations of the groups

The theory of linear representations of simple Lie groups was created by Cartan [1913]. He proved that each linear representation of these groups is defined by the dominant weight which is rational linear combination of simple roots α_i. If the simple roots of a simple Lie group are α_i $(i = 1, 2, ..., n)$ and if $\pi_i = 2\alpha_i/(\alpha_i, \alpha_i)$, where (α, β) is the inner product in the Cartan subalgebra with the metric of R^n induced in this subalgebra by Killing–Cartan metric in the Lie algebra of the simple Lie group, then the inner products $a_{ij} = (\alpha_i, \pi_j)$ are integers forming the *Cartan matrix* $A = (a_{ij})$ which is equivalent to the Dynkin diagram of the simple Lie group. If the vectors π^i form the basis reciprocal to the basis π_i $((\pi_i, \pi^j) = \delta_i^j)$, then the simple roots α_i and vectors π^i are connected by the relations $\alpha_i = a_{ij}\pi^j$, $\pi^i = A^{ij}\alpha_j$, where $(A^{ij}) = A^{-1}$ is the matrix inverse to the Cartan matrix A. Cartan proved that all linear representations are reduced to the fundamental representations, the number of which is equal to the rank of the group, and that the dominant weights of the fundamental representations of simple Lie groups are the vectors π_i mentioned above; the vectors π^i are called *fundamental weights.* The linear combinations of simple roots α_i with integer coefficients form the *root lattice*; the linear combinations of fundamental weights π^i with integer coefficients form the *weight lattice*. Both these lattices are additive groups and first of them is a subgroup of the second one and the quotient qroup of these lattices is a finite group, whose order, the *index of connectedness* of simple Lie group, is equal to the determinant of the Cartan matrix. All weights of linear representations are vectors of the weight lattice. Each fundamental representation corresponds to a fundamental weight π^i and to the corresponding simple root α_i; therefore it corresponds to the corresponding Tits' fundamental element and this representation can be considered as a linear transformation of coordinates of this element. Let us denote the fundamental representation corresponding to the weight π^i and to the simple root α_i by φ_i.

The fundamental representation φ_1 of the group of collineations of P^n, is the representation of this group by linear transformations of the linear space L^{n+1}, whose vector coordinates are equal to projective coordinates of points of P^n. The representation φ_n of this group is the representation by linear transformations of L^{n+1}, whose vector coordinates are equal to tangential coordinates of hyperplanes of P^n. The representations φ_k of this group $(k = 2, 3, \dots, n-1)$ are the representations by linear transformations of L^n, where $N = C_{n+1}^k$, whose vector coordinates are equal to *Grassmannian coordinates* $p^{i_1 i_2 \dots i_k} = x_1^{[i_1} x_2^{i_2} \dots x_k^{i_k]}$ (the brackets [] denote alternation of indices) of $(k-1)$-planes, for $k = 2$ Plückerian coordinates of lines of P^n.

The fundamental representations φ_1 of the groups B_n and D_n, which are the groups of motions of S^{2n} and S^{2n-1}, are the representations of these groups by linear transformations L^{2n+1} and L^{2n}, whose vector coordinates are equal to projective coordinates of points of S^{2n} and S^{2n-1}. The representations φ_k of these groups ($k = 2, 3, \dots, n-1$ for the groups B_n, $k = 2, 3, \dots, n-2$ for the groups D_n) are representations by linear transformations in the same spaces L^N as the representation φ_k of the group A_n. The fundamental representations φ_n of the group B_n and φ_{n-1} and φ_n of the group D_n are *spinor representations* of these groups by linear transformations of L^N, where $N = 2^n$ for the group B_n and $N = 2^{n-1}$ for the group D_n, whose coordinates are equal to coordinates of $(n-1)$-planar generators of absolutes of S^{2n}. In the last case the representations φ_{n-1} and φ_n correspond to different families of these generators (see Cartan [1938]). Other representations φ_k of these groups can also be considered as transformations of coordinates of points and rectilinear and $(k-1)$-planar generators of absolutes of S^{2n} and S^{2n-1}.

The fundamental representation φ_1 of the group C_n of symplectic transformations of Sy^{2n-1} are analogous to the representations φ_1 of the groups A_n, B_n and D_n, but the representations analogous to the representations φ_k of these groups ($1 < k < n$ for the groups A_n and B_n, $1 < k < n-1$ for the groups D_n) are reducible and the representations φ_k of the groups C_n are their irreducible parts.

The important linear representations of Lie groups are their *adjoint groups* – the linear groups in their Lie algebras defined by their inner automorphisms $x \to axa^{-1}$. The dimensions of the spaces of these representations are equal to dimensions of the groups. In the cases of groups B_n and D_n these representations are representations φ_2. In the case of group A_n this representation is the Kronecker product $\varphi_1 \otimes \varphi_n$ of representations φ_1 and φ_n. In the case of group C_n this representation is the representation φ_1^2 in the space of symmetric tensors $z^{ij} = z^{ji}$.

The fundamental representations φ_1 and φ_2 of the group G_2 have the dimensions of their representation spaces (14 and 7). The first of them is the representation by the adjoint group, the second one is the representation by rotations of space R^7 or R^7_3.

The fundamental representations φ_1 of the groups F_4, E_6, E_7 and E_8 are representations by adjoint groups. These representations correspond to symplecta of the metasymplectic geometries. The dimensions of the spaces of these representations are equal to the dimensions of the groups 52, 78, 133 and 248.

The dominant weights of adjoint representations are $\pi^1 + \pi^n$ for the group A_n, π^2 for the groups B_n and D_n, $2\pi^1$ for the group C_n and π^1 for all exceptional simple Lie group. Let us note that the dots of the Dynkin diagrams corresponding to non-zero coefficients at π^i in these expressions of dominant weight of the adjoint representations of these groups are those dots of these diagrams with which are joined to the dots of the extended Dynkin diagram corresponding to the minimal root $-\mu$ (see Wolf's book [1984, p. 269]).

The linear representations of the exceptional simple Lie groups with minimal dimension of the spaces of representation are in the case of the groups F_4 the representations φ_4 corresponding to points of absolutes of Hermitian planes (in this case this dimension is 26); in the case of groups E_6 the representations φ_4 and φ_6 corresponding to points and lines of projective planes (this dimension is 27); in the case of groups E_7 the representations φ_6 corresponding to isotropic 2-planes of symplectic 5-spaces (this dimension is 56), in the case of groups E_8 they are the representations φ_1 mentioned above. The role of vectors of the representations φ_4 and φ_6 of the groups E_6 is played by the 3-matrices $X = (x^{ij})$ $(x^{ij} = \bar{x}^{ji})$ representing the points and lines of projective planes. The role of these vectors of the representation φ_4 is played by the same 3-matrices satisfying the conditions $x^{00} + x^{11} + x^{22} = 0$ in the cases of elliptic planes and $-x^{00} + x^{11} + x^{22} = 0$ in the case of hyperbolic plane equivalent to equations of absolutes of these planes. The role of these vectors of the representation φ_6 is played by matrices representing isotropic 2-planes of symplectic 5-spaces.

The dimensions of the spaces of the representations φ_2 and φ_3 of the groups F_4 are 1274 and 273; the dimensions of the spaces of the representations φ_2, φ_3 and φ_5 of the groups E_6 are 2925, 351 and 351; the dimensions of the spaces of the representations φ_i $(i = 2, 3, 4, 5$ and $7)$ of the groups E_7 are 8645, 365750, 27664, 1539 and 912; the dimensions of the spaces of the representations φ_i $(i = 2, 3, \dots, 8)$ of the groups E_8 are 30380, 2450240, 146325270, 6899072464, 6696000, 3875 and 147250. Let us note that the representations φ_4 of the groups F_4, $\varphi_4 \otimes \varphi_6$ of the groups E_6, φ_5 of the groups E_7 and φ_7 of the groups E_8 are representations by transformations of coordinates of points of corresponding metasymplectic geometries and of points of absolutes of corresponding Hermitian planes.

9. Local absolutes of symmetric spaces

If a Riemannian symmetric space V^n is a space of constant curvature, then its isotropy group is the group O_n of the rotations of the space R^n tangent to V^n in this point. If a Riemannian symmetric space V^n is not a space of constant curvature, then its isotropy group is a subgroup of the group O_n. If this subgroup preserves a cone in the tangent space R^n and this cone cuts a surface from the hyperplane at infinity of this space R^n, which is the elliptic space S^{n-1}, then we call this surface *local absolute* of the symmetric space V^n. Analogous local absolutes are defined in pseudo-Riemannian symmetric spaces V^n_1. If a symmetric space V^{rn} or V^{rn}_L is interpreted as n-space over an r-dimensional algebra $\mathbf{A}$, then the tangent rn-spaces R^{rn} and R^{rn}_L of the spaces V^{rn} and V^{rn}_L can be considered as affine n-spaces $\mathbf{A}E^n$ over this algebra and the points of local absolutes are points at infinity of straight lines of the tangent spaces whose director vectors correspond to *singular vectors* of the space $\mathbf{A}E^n$, that is non-zero vectors $\mathbf{a}$ with zero products by zero divisors $(\mathbf{a}\lambda = \mathbf{0})$. In the general case local absolutes are subsets of the sets of points at infinity of lines with singular director vectors. In this case the complete set of these points in infinity can be called *local superabsolute* of the symmetric space. The local superabsolutes are real only if the algebra $\mathbf{A}$ has zero divisors, and are imaginary if this algebra is a field.

The local absolutes and superabsolutes were considered in the papers of the author with Kostrikina and others [1990], with Burceva [1990] and with Masagutova [1991]. In the first and the third of these papers it was proved that the local absolute of the space V_n^{2n} isometric to $\mathbf{C}'\bar{S}^n$ consists of two $(n-1)$-planar generators of the absolute quadric of S_n^{2n-1}, and that the local absolute of the space V^{2n} isometric to $\mathbf{C}\bar{S}^n$ consists of two conjugate imaginary $(n-1)$-planar generators of the imaginary absolute quadric of S^{2n-1}. In these papers it was also proved that the local absolute of the space V_{2n}^{4n} isometric to $\mathbf{H}'\bar{S}^n$ is a *segrean* $\sum_{i,2n-i}$ on the absolute of S_{2n}^{4n-1} (segrean $\sum_{m,n}$ is an algebraic $(m+n)$-surface of order $C_m^{m+n} = C_n^{m+n}$ in projective $(mn+m+n)$-space defined by parametric equations $z^{i\alpha} = x^i y^\alpha$ or equations $z^{i\alpha}z^{j\beta} - z^{i\beta}z^{j\alpha} = 0$, and that the local absolute of the space V^{4n} isometric to $\mathbf{H}\bar{S}^n$ is the imaginary segrean $\sum_{i,2n-i}$ on the absolute of S^{4n-1} defined by equations differing from the equations of the real segrean $\sum_{i,2n-i}$ by replacing of the forms $z_1z_2 - z_3z_4$ by sums of four squares of coordinates.

In the second case the real lines joining imaginary conjugate points of the $(n-1)$-planes form the *paratactic congruence* of S^{2n-1} isometric to $\mathbf{C}\bar{S}^{n-1}$ and the isotropy group of the symmetric space V^{2n} is isomorphic to the direct product of the groups of motions of $\mathbf{C}\bar{S}^{n-1}$ and S^1; therefore the metric of S^{2n-1} induces on $(n-1)$-planar generators of its absolute the metric of $\mathbf{C}\bar{S}^{n-1}$. These imaginary $(n-1)$-planes form the *focal surface* of the paratactic congruence of lines. The lines of this congruence are cut from the hyperplane at infinity of the tangent space R^{2n} of the space V^{2n} by tangent 2-planes to the 2-spheres of V^{2n} which are isometric to lines of $\mathbf{C}\bar{S}^n$.

In the fourth case the metric of S^{4n-1} induces on the rectilinear generators of the imaginary segrean $\sum_{i,2n-i}$ the metric of $\mathbf{C}\bar{S}^1$ and on the $(2n-1)$-planar generators of this segrean metric of $\mathbf{C}\bar{S}^{2n-1}$. The tangent hyperplane to the absolute in each point of this $(2n-1)$-plane cut from this $(2n-1)$-plane a $(2n-2)$-plane defined by a null-system; therefore in $\mathbf{C}\bar{S}^{2n-1}$ is defined a paratactic congruence of lines which is isomorphic to $\mathbf{H}\bar{S}^{n-1}$, and the isotropy group of the space V^{4n} is isomorphic to the direct product of the groups of motions of $\mathbf{H}\bar{S}^{n-1}$ and $\mathbf{C}\bar{S}^i$. This imaginary segrean is the focal surface of the paratactic congruence of 3-planes which are cut from the hyperplane in infinity of the tangent space R^{4n} of the space V^{4n} by tangent 4-planes to the 4-spheres of V^{4n} which are isometric to lines of $\mathbf{H}\bar{S}^n$.

In the same papers it was also proved that the local absolute of the space $V^{(m+1)(n-m)}$ which can be interpreted as a *Grassmann manifold* $G_{m,n}$ of m-planes of S^n and also as a $\frac{n+1}{m+1}$-space over the algebra $\mathbf{R}_{m+1}$ of real $(m+1)$-matrices (the dimension of this space can be integer and fractional; about spaces of fractional dimension, see Chakhtauri [1971]) is a segrean $\sum_{m,n-m-1}$ in $S^{(m+1)(n-m)-1}$. The space $V^{(m+1)(n-m)}$ is isometric to a grassmannian $\Gamma_{n,m}$ - an algebraic

$(m+1)(n-m)$-surface in the space S^N, where $N = C_{n+1}^{m+1} - 1$, with equations $p^{i_0 i_1 \dots i_{m-1}[i_m} p^{i_{m+1} \dots i_1 \dots i_{2m-1}]} = 0$ (in these equations $p^{i_0 i_1 \dots i_m} = x^{[i_0} x^{i_1} \dots x^{i_m]}$ are Grassmannian coordinates of m-planes). This symmetric space has also local superabsolutes consisting of points of m-planes intersecting segreans $\sum_{m,n-m-1}$ in $m+1$ points.

Compact real simple Lie groups with the Killing-Cartan metric defined by the metric tensor $a_{ij} = -C_{ik}^h C_{jh}^k$, are also symmetric Riemannian spaces. The local absolute of the space $V^{n(n-1)/2}$, which is the group O_n with the Killing-Cartan metric, is a grassmannian $\Gamma_{n-1,1}$.

Since the symmetric space V^8 with fundamental compact group G_2 can be interpreted as a manifold of special 2-planes of the space Sg^6 and since through each line of this space passes only one special 2-plane space, the space V^8 can be interpreted as quotient space $G_{6,1}/G_{2,1} = V^{10}/S^2$. Therefore the local absolute of V^8 can be obtained from the segrean $\sum_{1,4}$ by corresponding factorization and consists of a line with fixed points in S^7 and of the quadric with 2-planar generators (which is the grassmannian $\Gamma_{3,1}$) in the polar 5-plane of this line. The isotropy group of V^8 is locally isomorphic to the group O_4.

In the paper of the author and Burceva [1991] the *lipschitzian* Ω_n was defined as an algebraic $n(n-1)/2$ surface in the projective $(2^{n-1}-1)$-space defined by equations

$$
\begin{aligned}
x^0 x^{i_1 i_2 i_3 i_4} &= 3!! x^{i_1 [i_2} x^{i_3 i_4]}, \\
x^0 x^{i_1 i_2 i_3 i_4 i_5 i_6} &= 5!! x^{i_1 [i_2} x^{i_3 i_4 i_5 i_6]}, \\
&\dots \\
x^0 x^{i_1 i_2 \dots i_{2k}} &= (2k-1)!! x^{i_1 [i_2} x^{i_3 i_4 \dots i_2 k]}, \\
&\dots
\end{aligned}
$$

In these equations $(2k-1)!!$ is the product of all odd integers from 1 to $2k-1$ and the brackets [] denote alternation of indices. The intersection of the cone with these equations in an Euclidean 2^{n-1}-space with the sphere $\mathbf{x}^2 = 1$ of this space was considered by Lipshitz [1886]: if this 2^{n-1}-space is a *Clifford algebra* with the basis 1, e_i $(i = 1, 2, \dots, n-1)$ and $e_{i_1 i_2 \dots i_k} = e_{i_1} e_{i_2} \dots e_{i_k}$ where $e_i^2 = -1$ and $e_i e_j = -e_j e_i$ and if the coordinate at 1 is denoted by x^0 and to each coordinate with odd number of indices the index n is added, then these equations define the subgroup of the group of invertible elements of Clifford algebra which is *spinor group* $\mathbf{S}\bar{O}_n$ covering the group SO_n. Therefore the lipshitcian Ω_n is homeomorphic to the group SO_n.

In author's paper with Burceva [1990] it was proved that the local absolute of the symmetric space V^{16} isometric to the plane $\mathbf{O}\bar{S}^2$ is an imaginary lipschitzian Ω_5 which is the imaginary 10-surface in S^{15}; this imaginary lipschitzian is the focal surface of the paratactic congruence of 7-planes of S^{15} which are cut from the hyperplane in infinity of the tangent space R^{16} to the symmetric space V^{16} by tangent 8-planes to the 8-spheres of V^{16} which are isometric to lines of $\mathbf{O}\bar{S}^2$. The isotropy group of the space V^{16} is represented by the fundamental representation φ_4 of the group B_4. The group B_4 is the group of motions of S^8 and the fundamental representation φ_4 of this group is defined by 3-planar generators of the absolute of this space. Cartan [1938] proved that the manifold of planar generators of maximal dimension of quadric in P^{2n-2} and each of two famillies of planar generators of maximal dimension of quadric in P^{2n-1} are homeomorphic to the group SO_n. Therefore the manifold of 3-planar generators of the absolute of S^8 is homeomorphic to the complex group SO_5, and therefore to the imaginary lipschitzian Ω_5, and the isotropy group of V^{16} represents transformations of the manifold of 3-planar generators of the absolute of S^8 under the fundamental linear transformation φ_4.

Analogous are the proofs that the local absolutes of the symmetric spaces V^{32}, V^{64} and V^{126} which are isometric to elliptic Hermitian 2-planes over tensor product $\mathbf{O}\otimes\mathbf{C}$, $\mathbf{O}\otimes\mathbf{H}$ and $\mathbf{O}\otimes\mathbf{O}$, are located on the imaginary absolutes of the elliptic spaces S^{31}, S^{63} and S^{127} and are respectively a couple of conjugate imaginary lipschitzians Ω_5 in two conjugate imaginary 15-planar generators of the absolute of S^{31}, an imaginary segrean $\sum_{1,31}$ in 31-planar generators of which are given imaginary lipszhitzians Ω_6, and an imaginary lipschitzian Ω_8.

The isotropy groups of these symmetric spaces are, respectively, represented by direct product of the fundamental representation φ_5 of the group D_5 and of the group of motions of the line S^1 joining conjugate imaginary points of imaginary segreans Ω_5, by direct product of the fundamental representation φ_6 of the group D_6 and of the group of motions of the line $\mathbf{C}\bar{S}^1$, and by the fundamental representation φ_8 of the group D_8.

The lipschitzians mentioned above are homeomorphic, respectively, to the families of planar generators of maximal dimension of absolutes of S^9, S^{11} and S^{15}, whose groups of motions are isomorphic to the groups of motions of elliptic Hermitian lines over corresponding tensor products. The local superabsolutes of these symmetric spaces V^{32}, V^{64} and V^{128} are focal surfaces of the congruences, respectively, of 15-planes of S^{31}, of 31-planes of S^{63} and of 63-planes of S^{127}, which are cut from hyperplanes at infinity of the tangent spaces R^{32}, R^{64} and R^{128} to the symmetric spaces by tangent 16-, 32- and 64-planes to the surfaces of these symmetric spaces defined by lines of corresponding elliptic Hermitian 2-planes over tensor products. The intersection of local superabsolutes of these symmetric spaces with the planes of congruences mentioned above are isometric, respectively, to the sets of lines, 3-planes and 7-planes intersecting in 2, 4 and 8 points segreans $\sum_{1,7}$, $\sum_{3,7}$ and $\sum_{7,7}$ which are local superabsolutes of the Grassmann manifolds $G_{9,1}$, $G_{11,3}$ and $G_{15,7}$. These local superabsolutes are real.

Since the families of 4-planar generators of the absolute of S^9, of 5-planar generators of the absolute of S^{11}, and of 7-planar generators of the absolute of S^{15} are homeomorphic, respectively, to the lipschitzians Ω_5, Ω_6 and Ω_8, the isotropy groups of the spaces V^{31}, V^{63} and V^{127} represents transformations of these families of planar generators under corresponding fundamental linear representations φ_5, φ_6 and φ_8 of the groups D_5, D_5 and D_8.

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