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ENLARGEMENT OF THE CLASS OF GEOMETRICALLY INFINITELY DIVISIBLE RANDOM VARIABLES

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Abstract. The class of negative binomial infinitely divisible random variables is introduced in the following way: Random variable Y is called *negative binomial infinitely divisible* if there exist i.i.d. random variables $X_p^{(1)}, X_p^{(2)}, \dots$, $p \in (0,1)$, independent of Y and $\nu_p^{(r)}$ and such that

$$Y \stackrel{\mathrm{d}}{=} \lim_{p\to 0} \sum_{j=1}^{\nu_p^{(r)}} X_p^{(j)},$$

where $\nu_p^{(r)}$ has negative binomial law.

The representation of characteristic functions from the class of negative binomial infinitely divisible random variables is given and also some related properties discussed. When $r = 1$ the above class reduces to the well known class of geometrically infinitely divisible random variables.

Klebanov et al. [**4**] introduced the notion of geometric infinite divisibility in the following way: Let $X_p^{(1)}, X_p^{(2)}, \dots$ be i.i.d. random variables and suppose ν_p has a geometric distribution

$$p_n = P(\nu_p = n) = pq^{n-1}, n = 1, 2, \dots (p + q = 1, p > 0) \tag{1}$$

Put:

$$Y \stackrel{\mathrm{d}}{=} \sum_{j=1}^{\nu_p} X_p^{(j)} \tag{2}$$

Definition 1. Random variable Y is called *geometrically infinitely divisible* if, for every $p \in (0,1)$, there exist random variables $X_p^{(1)}, X_p^{(2)}, \dots$ such that Y could be presented as the random sum (2), where Y, ν_p and $X_p^{(j)}$, $j = 1, 2, \dots$ are independent.

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It was proved by Klebanov et al. [**4**] that f is the characteristic function of geometrically infinitely divisible distribution if and only if it is of the form

$$f(t)=\frac{1}{1-\ln\psi(t)}, \tag{3}$$

where ψ is some infinitely divisible characteristic function,

$$\ln\psi(t)=ita+\int\limits_{-\infty}^{+\infty}\left[e^{itx}-1-\frac{itx}{1+x^2}\right]\frac{1+x^2}{x^2}d\theta(x)$$

where $a\in R^1$, θ - nondecreasing bounded function, $\theta(-\infty)=0$. At origin the integrand is defined by continuity and equals $-t^2/2$.

Kruglov and Korolev presented results concerning geometric random sums in chapter 8 of [**5**]. We propose here some natural enlargements of the class of geometrically infinitely divisible random variables.

Suppose random variable ν_p is distributed according to the Pascal distribution

$$p_n=P(\nu_p=n)=\binom{-r}{n-r}p^r(-q)^{n-r}=\binom{n-1}{r-1}p^rq^{n-r},\ n\geq r,\ p>0,\ p+q=1, \tag{4}$$

and put

$$Y\stackrel{\rm d}{=}\sum_{n=r}^{\nu_p}X_p^{(n)} \tag{5}$$

Definition 2. Random variable Y is called *Pascal infinitely divisible* if, for every $p\in(0,1)$, there exist random variables $X_p^{(1)},X_p^{(2)},\dots$ such that Y could be presented as the random sum (5), where ν_p has Pascal distribution (4) and Y, ν_p and $X_p^{(j)}$, $j=1,2,\dots$ are independent.

THEOREM 1. *The characteristic function f is Pascal infinitely divisible if and only if it is of the form*

$$f(t)=(1-\ln\psi(t))^{-r} \tag{6}$$

for some $r\in N$, where ψ is an infinitely divisible characteristic function.

We see that, when $r=1$, the class of Pascal infinitely divisible characteristic functions reduces to the class of geometrically infinitely divisible characteristic functions.

Proof. Put $F(x)=P(Y<x)$ and $F_p(x)=P(X_p<x)$ and denote by f and f_p characteristic functions of F and F_p, respectively. Then from (5) we have

$$F(x)=\sum_{n=r}^{\infty}\binom{-r}{n-r}p^r(-q)^{n-r}F_p^{*n}(x)$$

and

$$f(t)=\left[\frac{pf_p(t)}{1-(1-p)f_p(t)}\right]^r. \tag{7}$$

Let us prove that f is never zero. There exists an interval $(-\varepsilon,\varepsilon)$, $\varepsilon > 0$, where both f and f_p are different from zero. For $t \in (-\varepsilon,\varepsilon)$, (7) could be written in the form

$$f(t) = \left[1 + \frac{1}{p}\left[\frac{1}{f_p(t)} - 1\right]\right]^{-r}, \quad t \in (-\varepsilon,\varepsilon).$$

When $p \to 0$, there must be $f_p(t) \to 1$, for $t \in (-\varepsilon,\varepsilon)$. Using the well known inequality for characteristic functions

$$1- \mid f(2t) \mid^2 \leq 4(1- \mid f(t) \mid^2),$$

we get that $f_p(t) \to 1$ for $t \in (-2\varepsilon, 2\varepsilon)$. Proceeding with the same procedure we conclude that $f_p(t) \to 1$, when $p \to 0$, for every $t \in R$ and therefore f is never zero. Since $f(t) \neq 0$ for all $t \in R$, roots and logarithms of f are uniquely defined in the usual way using the principal branch of $ln f(t)$.

If we divide by p both nominator and denominator of the right-hand side of (7) and let $p \to 0$, then we get

$$\lim_{p\to 0} \exp\left[\frac{1}{p}\left[f_p(t) - 1\right]\right] = \exp\left[1 - \frac{1}{f(t)^{1/r}}\right]. \tag{8}$$

We have that the right-hand side of (8) is continuous at zero and, since f_p is a characteristic function, the left-hand side of (8) is the limit of the characteristic functions of compound Poisson distributions. Therefore that limit is an infinitely divisible characteristic function (which we shall denote by ψ). It follows that (6) is valid.

Let us now prove the converse - that the characteristic function of the form (6)

$$f(t) = \left[\frac{1}{1 - \ln\psi(t)}\right]^{r}, \quad r \in N,$$

where ψ is characteristic function of an infinitely divisible distribution, could be represented as (7), for every $p \in (0,1)$. Put $f_p(t) = [1 - \ln\psi^p(t)]^{-1}$ and (7) follows immediately.

THEOREM 2. *Pascal infinitely divisible characteristic functions are infinitely divisible.*

Proof. Suppose X is infinitely divisible random variable with the characteristic function ψ, and A is probability distribution of some positive random variable. It is well known (Feller [**1**]) that power mixtures

$$\phi(t) = \int_0^{+\infty} (\psi(t))^u dA(u),$$

with infinitely divisible mixing distribution A, are infinitely divisible. Indeed if we denote by α the characteristic function of A, $\alpha(t) = (\alpha_n(t))^n$ for each n, then

$$\phi(t) = \alpha(-i\ln\psi(t)) = (\alpha_n(-i\ln\psi(t)))^n = (\phi_n(t))^n.$$

If we take A to be Gamma distribution

$$A(x) = \int_0^x \frac{u^{\lambda-1}e^{-u}}{\Gamma(\lambda)}du, \ \ \lambda > 0, \ \ x > 0,$$

which has the following infinitely divisible characteristic function

$$\alpha(t) = (1 - it)^{-\lambda}, \ \ \lambda > 0,$$

then

$$\phi(t) = \alpha(-i\ln\psi(t)) = (1 - \ln\psi(t))^{-\lambda}, \ \ \lambda > 0 \tag{9}$$

and we have that all the functions of the form (9) are infinitely divisible characteristic functions.

We proved in fact more than we stated, namely that for all $\lambda > 0$, the function ϕ is infinitely divisible, not only for $\lambda \in N$.

Let $X_1, X_2, \dots$ be i.i.d. with probability distribution F, independent of the variable ν_p with Pascal distribution

$$p_n = P(\nu_p = n) = \binom{-r}{n-r} p^r(-q)^{n-r} = \binom{n-1}{r-1} p^r q^{n-r}, \ n \geq r, \ p > 0, \ p+q = 1.$$

If for some choice of constants $A(p) > 0$ and $B(p)$ the distribution of Pascal random sums

$$S_{\nu(p)} = \frac{1}{A(p)} \sum_{k=r}^{\nu(p)} (X_k - B(p))$$

converges weakly as $p \to 0$ to probability distribution G, we say that F *is in the domain of Pascal attraction of* G.

THEOREM 3. *In order that probability distribution G has a nonempty domain of Pascal attraction, it is necessary and sufficient that its characteristic function g could be represented in the following way:*

$$g(t) = \left[\frac{1}{1 - ln\psi(t)}\right]^r, \tag{10}$$

where r is positive integer and ψ is a stable characteristic function

$$\ln\psi(t) = i\gamma t - c \mid t \mid^{\alpha} \left\{1 + i\beta \frac{\mid t \mid}{t} \omega(t, \alpha)\right\}, \tag{11}$$

where $\gamma \in R$, $-1 \leq \beta \leq 1$, $0 < \alpha \leq 2$, $c \geq 0$ and

$$\omega(t, \alpha) = \begin{cases} \mathrm{tg}(\pi\alpha/2), & \text{for } \alpha \neq 1 \\ 2\pi^{-1}\ln.|t|, & \text{for } \alpha = 1 \end{cases}$$

The random variable whose characteristic function is defined by (10) and (11) we call *Pascal stable*. Analogous notions and statements for geometric sums were studied by Freyer [**2**] and Kruglov and Korolev [**5**].

Proof. We exclude the degenerate case when ψ from (11) equals $e^{it\gamma}$. The characteristic function of $S_{\nu(p)}$ is:

$$\left[\frac{pf(t/A(p))\exp(-itB(p)/A(p))}{1-(1-p)f(t/A(p))\exp(-itB(p)/A(p))}\right]^r.$$

From Theorem 1 it follows that the limiting distribution G is Pascal infinitely divisible, and from (8) we have:

$$\lim_{p\to 0}\left[p^{-1}\left[f(t/A(p))\exp(-itB(p)/A(p))-1\right]\right]=\ln\psi(t),$$

where ψ is an infinitely divisible characteristic function. That limit also holds for $p=1/n$. Put $B_n=B(1/n)$, $A_n=A(1/n)$, and $C_n=B_n/A_n$. We have

$$\lim_{n\to\infty} n\left[f(t/A_n)\exp(-itC_n)-1\right]=\ln\psi(t),$$

which is equivalent to

$$\left[f(t/A_n)\exp(-itC_n)\right]^n \underset{n\to\infty}{\to} \psi(t)$$

(Feller [**1**, ch.XVII, Th.1]). Above we have the sequence of characteristic functions of linearly normed sums of i.i.d. random variables which tend to a limit and (see Zolotarev [**8**, p.13]), the only characteristic function which can appear as a weak limit is the characteristic function of the stable distribution function. So we have that ψ has the form (11).

Conversely, in order to show that each function g of the form (10) (with (11)) has a nonempty domain of Pascal attraction, it is enough to take F with the characteristic function ψ (11), $A(p)=p^{-1/\alpha}$ and, when $\alpha\neq 1$, $B(p)=\gamma p^{-1/\alpha}(p^{1/\alpha-1}-1)$, when $\alpha=1$, then take $B(p)=2/\pi^{-1}c\beta p^{-1}\ln p$. Using the fact that for such ψ, A and B we have:

$$\psi(t/A(p))\exp(-itB(p)/A(p))=\psi^p(t)$$

it follows easily that the characteristic function of $S_{\nu(p)}$ tends, as $p\to 0$, to the characteristic function g.

Suppose now that random variable ν_p is distributed according to negative binomial distribution

$$p_n=P(\nu_p=n)=\binom{-r}{n}p^r(-q)^n,\ \ n=0,1,2,\dots,\ p>0,\ p+q=1,\ r\in R^+. \tag{12}$$

Definition 3. Random variable Y (with characteristic function f) is called *negative binomial infinitely divisible* if, for every $p\in(0,1)$, there exist random variables X_p (with characteristic functions f_p) such that f could be represented as

$$f(t)=(f_p(t))^r\sum_{n=0}^{+\infty}p_n(f_p(t))^n,\ \ r>0. \tag{13}$$

When $r\in N$, then (13) reduces to (5).

The following theorem can be proved in the same way as Theorem 1 concerning Pascal infinitely divisible characteristic functions.

THEOREM 4. *The characteristic function f is negative binomial infinitely divisible if and only if it is of the form*

$$f(t) = (1 - \ln\psi(t))^{-r}, \quad r \in R^+,$$

where ψ is an infinitely divisible characteristic function.

Although in the definition of negative binomial infinitely divisible characteristic function f, the functional equation by which f is defined seems slightly artificial because of the factor $(f_p(t))^r$, it can be shown that functions of that class appear, as limiting, in the Transfer theorem for random sums. We quote Gnedenko and Fahim [**3**]:

TRANSFER THEOREM. *Let $\xi_{n1}, \xi_{n2}, \dots$ be i.i.d. for every $n \in N$, $F_n(x) = P(\xi_{nk} < x)$, f_n – characteristic function of F_n. Let $\{k_n\}$ be a sequence of positive integers and let $\{\nu_n\}$ be a sequence of positive integer valued random variables, independent of ξ_{nk}. If*

$$\text{(A)} \qquad P\left\{\sum_{k=1}^{k_n} \xi_{nk} < x\right\} \to F(x) \ \text{ and } \ \text{(B)} \qquad P\left\{\frac{\nu_n}{k_n} < x\right\} \to A(x)$$

as $n \to \infty$, where F and A are distribution functions, then

$$\text{(C)} \qquad P\left\{\sum_{k=1}^{\nu_n} \xi_{nk} < x\right\} \to G(x).$$

The distribution G is determined by its characteristic function g

$$g(t) = \int_0^{+\infty} [f(t)]^z \, dA(z)$$

where f is the characteristic function of F.

From the classical theory of summation it follows that f is infinitely divisible characteristic function. We shall take ν_n to be negative binomial with parameter $1/n$:

$$p_k = P(\nu_n = k) = \binom{-r}{k} (1/n)^r (-1 + 1/n)^k, \quad k = 0, 1, 2, \dots, \quad r > 0.$$

Let us show that the condition (B) is fulfilled with $k_n = n$. If we denote by $a_n(t)$ the Laplace transform of the variable ν_n, we have

$$a_n(t) = \left[\frac{1/n}{1 - (1 - 1/n)e^{-t}}\right]^r.$$

The Laplace transform of ν_n/n is

$$a_n(t/n) = \left[\frac{1/n}{1-(1-1/n)e^{-t/n}}\right]^r = \left[\frac{1}{n-ne^{-t/n}+e^{-t/n}}\right]^r =$$
$$= \left[\frac{1}{e^{-t/n}+n(1-e^{-t/n})}\right]^r \underset{n\to\infty}{\to} \left[\frac{1}{1+t}\right]^r, \quad r>0.$$

So when we apply the Transfer theorem with the sequence of random indexes defined as above, we obtain as limiting the random variable with negative binomial infinitely divisible characteristic function.

We see that the relationship existing between geometric, negative binomial and their continuous analogues exponential and Gamma distributions are also present in the simple enlargements of the class of geometrically infinitely divisible distributions which are introduced here. There also exists independent interest in studying power mixtures with negative binomial mixing distribution. For instance, Willmot [**6**],[**7**] investigated the tail behavior of such mixtures in connection with some problems of risk and insurance. Hence we are interested to investigate the relationship between mixtures with negative binomial mixing distribution and the class of negative binomial infinitely divisible distributions introduced in the formal way by Definition 3.

It should be pointed out that Definitions 1 and 2 allow the random index ν_p only to have Pascal distribution shifted by the value of the parameter $r \in N$ ($r=1$ for geometric distribution). The representation theorem (see [**4**]) for characteristic functions of geometrically infinitely divisible distributions will not be true if instead of (1), we take

$$p_n = P(\nu_p = n) = pq^n, \;\; n=0,1,2,\dots \;\; (p+q=1, \;\; p>0)$$

in the Definition 1, because in the "only if" part of that theorem it should be proved that, for every $p\in(0,1)$, a characteristic function $f_p(t)$ exists, such that for every $p\in(0,1)$,

$$\frac{p}{1-(1-p)f_p(t)} = \frac{1}{1-\ln\psi(t)} \tag{14}$$

holds. But this is impossible because from (14) it follows that, for every $p\in(0,1)$,

$$f_p(t) = 1 + p/(1-p)\ln\psi(t).$$

Obviously the above $f_p(t)$ is not a characteristic function because it does not satisfy the condition $\mid f_p(t) \mid \leq 1$, for every $p\in(0,1)$.

Now we shall give another definition of negative binomial infinitely divisible random variables, without the previously mentioned disadvantages.

Let $\nu_p^{(r)}$ be a random variable having negative binomial law (12).

Definition 4. Random variable Y is called *negative binomial infinitely divisible* if there exist i.i.d. random variables $X_p^{(1)}, X_p^{(2)}, \dots,$ $p\in(0,1)$, independent of

Y and $\nu_p^{(r)}$ and such that

$$Y \stackrel{\mathrm{d}}{=} \lim_{p\to 0} \sum_{j=1}^{\nu_p^{(r)}} X_p^{(j)}.$$

In terms of probability distributions and characteristic functions the preceding equation becomes (F, f are probability distribution and characteristic function of Y, and F_p, f_p are probability distribution and characteristic function of X_p, respectively):

$$F(x) = \lim_{p\to 0} \sum_{n=0}^{\infty} \binom{-r}{n} p^r (-q)^n F_p^{*n}(x)$$

and

$$f(t) = \lim_{p\to 0} \left[\frac{p}{1-(1-p)f_p(t)}\right]^r.$$

We shall see from the following theorem that Definitions 3 and 4 determine the same class of characteristic functions. When $r \in N$ this class reduces to (6), and when $r = 1$ to (3).

THEOREM 5. *The characteristic function f is negative binomial infinitely divisible if and only if it is of the form*

$$f(t) = (1 - \ln\psi(t))^{-r}, \quad r \in R^+$$

where ψ is an infinitely divisible characteristic function.

Proof. The "if" part is identical as in the Theorem 1, and in order to prove the "only if" part we have to prove that, for every infinitely divisible characteristic function ψ, there exist characteristic functions f_p, $p \in (0,1)$, such that

$$\left[\frac{1}{1-\ln\psi(t)}\right]^r = \lim_{p\to 0} \left[\frac{p}{1-(1-p)f_p(t)}\right]^r.$$

Put $f_p(t) = \psi^p(t)$, then we have

$$\lim_{p\to 0} \left[\frac{p}{1-(1-p)\psi^p(t)}\right]^r = \lim_{p\to 0} \left[\frac{1}{\psi^p(t) - 1/p(\psi^p(t)-1)}\right]^r.$$

When $p \to 0$ we have $\psi^p(t) \to 1$ and also $|1-\psi^p(t)| < 1$. Then for p small enough and from Taylor expansion of the function $\log(1-z)$ we conclude that

$$\begin{aligned}\log\psi(t) = p^{-1}\log\psi^p(t) = p^{-1}\log[1-(1-\psi^p(t))] = \\ = -p^{-1}[1-\psi^p(t)] - p^{-1}[1-\psi^p(t)]^2 - \ldots\end{aligned}$$

When $p \to 0$, $\log\psi(t) = \lim_{p\to 0} p^{-1}[\psi^p(t) - 1]$. The proof is completed.

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