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ON CERTAIN DIVISOR FUNCTION IN SHORT INTERVALS

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Abstract. An asymptotic formula for the integral of the summatory function of $f(n)$ in short intervals is obtained, where $f(n)$ represents a certain divisor function. In particular $f(n)$ can be the characteristic function of the set of k-free or k-full numbers.

1. Introduction. Let $f(n)$ be the divisor function defined by

$$f(n) = \sum_{d^k|n} g\left(\frac{n}{d^k}\right), \tag{1.1}$$

where $k \ge 1$ is a fixed integer and

$$\sum_{n\le x} |g(n)| \ll x^b \qquad (0 \le b < 1/k). \tag{1.2}$$

Here $u(x) \ll v(x)$ (same as $u(x) = O(v(x))$) means that $|u(x)| \le Cv(x)$ for $C > 0$, $v(x) > 0$ and $x \ge x_0 > 0$. Divisor functions of this type often occur in multiplicative number theory (see Section 3 for some examples). Let $\mathcal{L}$ denote the class of slowly varying functions $L(x)$ which are increasing for $x \ge x_0 > 0$ and satisfy

$$\lim_{x\to\infty} L(x) = +\infty. \tag{1.3}$$

By a slowly varying function we mean a function $L(x)$ which is positive and continuous for $x \ge x_0$, and satisfies

$$\lim_{x\to\infty} \frac{L(cx)}{L(x)} = 1 \tag{1.4}$$

for any $c > 0$ (see Bingham et al. [**1**] and Seneta [**9**] for an extensive account on slowly varying functions). It is known that

$$L(x) \ll x^\varepsilon \tag{1.5}$$

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for any given $\varepsilon > 0$, and functions from $\mathcal{L}$ (such as $\log x, e^{\sqrt{\log x}}$ etc.) often occur in analytic number theory. Note that we can choose our $L(x)$ to tend to infinity arbitrarily slowly, since $\log L(x)$ is slowly varying if $L(x)$ is and (1.3) holds. In particular, we can take $L(x) = \log_r x$, the r-fold natural logarithm of x.

The aim of this paper is to study $f(n)$ in short intervals $(x, x+h]$, where

$$h = h(x) = x^{(k-1)/k} u(x), \ \ 0 < u(x) \ll x^{\theta} \ \ (0 \le \theta < 1/k), \tag{1.6}$$

and $u(x)$ is increasing for $x \ge x_0$. The interval $(x, x+h]$ is short in the sense that $h = o(x)$ as $x \to \infty$ and the form of h given by (1.6) will become transparent later from the proof (see (2.2)). Naturally it is desirable to have $u(x)$ in (1.6) as small as possible. From (1.1) and (1.2) it follows by an elementary argument that

$$\sum_{n \le x} f(n) = \left(\sum_{m=1}^{\infty} g(m) m^{-1/k} \right) x^{1/k} + R(x), \ \ R(x) \ll x^b. \tag{1.7}$$

Hence for h given by (1.6) it follows that, as $x \to \infty$,

$$\sum_{x < n \le x+h} f(n) = \left(\frac{C}{k} + o(1) \right) u(x) + R(x+h) - R(x), \ \ C = \sum_{m=1}^{\infty} g(m) m^{-1/k}. \tag{1.8}$$

The problem is to determine for which $u(x)$ one has in (1.8)

$$R(x+h) - R(x) = o(u(x)) \qquad (x \to \infty), \tag{1.9}$$

so that the sum in (1.8) is asymptotic to $Ck^{-1}u(x)$, which is the quantity that one expects to obtain from (1.7). In general this is a difficult problem, and even in the classical cases when $f(n)$ represents the characteristic function of squarefree or squarefull numbers, it has not yet been solved satisfactorily (for latest results see Liu [**8**] and Shiu [**10**], respectively). Thus we shall consider here the less difficult problem of the integral of $R(x+h) - R(x)$. In that case we can obtain a very satisfactory solution, in the sense that we get the "expected" quantity when we take in (1.6) $u(x) = L(x)$, $L(x) \in \mathcal{L}$. This is in a certain sense optimal, since $L(x)$ may tend to infinity arbitrarily slowly. The result is the following

THEOREM. *If $f(n)$ satisfies* (1.1) *and* (1.2), *$h = x^{(k-1)/k} L(x)$ with $L(x) \in \mathcal{L}$, then as $X \to \infty$ we have*

$$\int_X^{2X} \left(\sum_{x < n \le x+h} f(n) \right) dx = \left(\frac{C}{k} + o(1) \right) X L(X), \tag{1.10}$$

where C is given by (1.8).

The proof (1.10) will be given in Section 2, while some applications are to be found in Section 3. It is possible to generalize the above Theorem in several ways. For example, instead of (1.1) one may consider

$$f(n) = \sum_{(d_1 \dots d_r)^k m = n} g(m),$$

where $k, r \geq 1$ are given integers and g satisfies (1.2). This would entail the appearance of $\Delta_r(x)$, the error term in the general Dirichlet divisor problem (see Ch. 13 of [**5**]). In particular Δ_r would appear instead of Ψ in (2.5), which would make the ensuing estimation much more complicated. Also k in (1.1) does not have to be an integer, but only $k > 0$ may be required, as long as (1.2) holds, and instead of (1.1) we may suppose that

$$f(n) = \sum_{d^k|n} h(d) g\left(\frac{n}{d^k}\right),$$

where the summatory function of $h(n)$ has a "nice" shape. However, for the sake of simplicity, I found it expedient to prove the result in the form given by (1.10).

2. Proof of the Theorem. Let first $X \leq x \leq 2X$, $H = H(x) = x^{(k-1)/k} L(X)$ so that L does not depend on x. Then

$$\begin{aligned}\sum_{x<n\leq x+H} f(n) &= \sum_{x<d^k m\leq x+H,\ m\leq L(X)} g(m) + O\left(\sum_{x<d^k m\leq x+H,\ m>L(X)} |g(m)|\right) \\ &= S_1 + O(S_2),\end{aligned}$$

say. By using (1.2) we have

$$\begin{aligned}S_1 &= \sum_{m\leq L(X)} g(m) \sum_{(x/m)^{1/k}<d\leq((x+H)/m)^{1/k}} 1 \\ &= \left((x+H)^{1/k} - x^{1/k}\right) \sum_{m\leq L(X)} g(m) m^{-1/k} + O\left(\sum_{m\leq L(X)} |g(m)|\right) \\ &= (1/k + o(1))\, L(X) \left(\sum_{m=1}^{\infty} g(m) m^{-1/k} + \sum_{m>L(X)} g(m) m^{-1/k}\right) + O(L^b(X)) \\ &= C/k + o(1)) L(X),\end{aligned} \tag{2.1}$$

where C is given by (1.8), since $0 \leq b < 1/k$ and

$$\begin{aligned}(x+H)^{1/k} - x^{1/k} &= x^{1/k}\left(H/kx + O((H/x)^2)\right) \\ &= L(X)(1/k + O(L(X)X^{-1/k})).\end{aligned} \tag{2.2}$$

Also

$$\begin{aligned}S_2 &= \sum_{L(X)<m\leq x+H} |g(m)| \sum_{(x/m)^{1/k}<d\leq((x+H)/m)^{1/k}} 1 \\ &= \left((x+H)^{1/k} - x^{1/k}\right) \sum_{L(X)<m\leq x+H} |g(m)| m^{-1/k} \\ &\quad + \sum_{L(X)<m\leq x+H} |g(m)| \left\{\Psi\left(\left(\frac{x}{m}\right)^{1/k}\right) - \Psi\left(\left(\frac{x+H}{m}\right)^{1/k}\right)\right\}\end{aligned} \tag{2.3}$$

$$= o(L(X)) + \sum_{L(X)<M\le x+H} |g(m)| \left\{ \Psi\left(\left(\frac{x}{m}\right)^{1/k}\right) - \Psi\left(\left(\frac{x+H}{m}\right)^{1/k}\right)\right\},$$

where $\Psi(t) = t - [t] - 1/2$. Hence from (2.1) and (2.3) it follows that

$$\int_X^{2X} \left(\sum_{x<n\le x+H} f(n)\right) dx = \left(\frac{C}{k} + o(1)\right) XL(X) + O\left(\sum\right), \tag{2.4}$$

where

$$\sum := \int_X^{2X} \sum_{L(X)<m\le x+H} |g(m)| \left\{ \Psi\left(\left(\frac{x}{m}\right)^{1/k}\right) - \Psi\left(\left(\frac{x+H}{m}\right)^{1/k}\right)\right\} dx. \tag{2.5}$$

Now if t is not an integer one has the Fourier expansion

$$\Psi(t) = -\sum_{n=1}^{\infty} \frac{\sin(2\pi n t)}{\pi n},$$

where the series is boundedly convergent and thus can be integrated termwise. Therefore (2.5) gives

$$\sum = \frac{1}{\pi} \sum_{L(X)<m\le 2X+H} |g(m)|$$

$$\times \sum_{n=1}^{\infty} \frac{1}{n} \int_{X_1}^{2X} \left\{ \sin\left(2\pi n \left(\frac{x+H}{m}\right)^{1/k}\right) - \sin\left(2\pi n \left(\frac{x}{m}\right)^{1/k}\right)\right\} dx$$

with $X_1 = \max(X, m-H)$. For those n which satisfy $n > X$ we write the sines as exponentials and estimate the integrals by the first derivative test (Lemma 2.1 of [**5**]) to obtain that the total contribution of such n is

$$\ll X^{(k-1)/k} \sum_{m\le 2X+H} |g(m)| m^{1/k} \sum_{n>X} n^{-2} \ll X^b.$$

For $n \le X$ write

$$\int_{X_1}^{2X} \left\{ \sin\left(2\pi n \left(\frac{x+H}{m}\right)^{1/k}\right) - \sin\left(2\pi n \left(\frac{x}{m}\right)^{1/k}\right)\right\} dx = 2I_1 + 2I_2,$$

say, where

$$I_1 = \int_{X_1}^{2X} (\sin A - A)\cos B dx, \quad I_2 = \int_{X_1}^{2X} A\cos B dx,$$

$$A = A(x) = \pi n \left(\left(\frac{x+H}{m}\right)^{1/k} - \left(\frac{x}{m}\right)^{1/k}\right) \ll n m^{-1/k} L(X),$$

$$B = B(x) = \pi n \left(\left(\frac{x+H}{m}\right)^{1/k} + \left(\frac{x}{m}\right)^{1/k}\right).$$

For $X \le x \le 2X$ we have

$$\begin{aligned} A'(x) &= \frac{\pi}{k} n m^{-1/k} x^{(1-k)/k} \left\{ \left(1+\frac{H}{x}\right)^{(1-k)/k} (1+H') - 1 \right\} \\ &= \pi n m^{-1/k} x^{-(k+1)/k} L^2(X) \left(\frac{k-1}{2k^3} + O(X^{-1/k} L(X)) \right) > 0 \end{aligned}$$

if $k > 1$, and if $k = 1$ then A does not depend on x so (2.6) will follow more simply. Consequently $(\sin A - A)' = A'(\cos A - 1) \le 0$. Hence we can apply the second mean value theorem for integrals to I_1 and I_2. For some C_1 satisfying $X_1 \le C_1 \le 2X$ we have

$$\begin{aligned} I_1 = (\sin A(2X) - A(2X)) \int_{C_1}^{2X} \cos B(x) dx &\ll n m^{-1/k} L(X) n^{-1} m^{1/k} X^{(k-1)/k} \\ &= L(X) X^{(k-1)/k}, \end{aligned}$$

and for some C_2 satisfying $X_1 \le C_2 \le 2X$

$$I_2 = A(2X) \int_{C_2}^{2X} \cos B(x) dx \ll L(X) X^{(k-1)/k}.$$

Here we used $|\sin x| \le |x|$ and the first derivative test for the cosine integrals. This gives

$$\begin{aligned} \sum &\ll X^b + \sum_{m \le 2X+H} |g(m)| \sum_{n \le X} n^{-1} L(X) X^{(k-1)/k} \\ &\ll X^b + X^{(k-1)/k} X^b L(X) \log X = o(XL(X)) \end{aligned} \tag{2.6}$$

since $b < 1/k$. To complete the proof of the Theorem let

$$\begin{aligned} \chi = \chi(x) = x^{(k-1)/k} L(2X), \quad h = h(x) &= x^{(k-1)/k} L(x), \\ H = H(x) &= x^{(k-1)/k} L(X). \end{aligned}$$

Recalling that $L(x)$ is increasing for $x \ge x_0$ it follows that

$$I_H \le I_h \le I_\chi, \tag{2.7}$$

where

$$I_F := \int_X^{2X} \left(\sum_{x < n \le x+F} f(n) \right) dx \qquad (F = H,\ h \text{ or } \chi).$$

But from (2.4) and (2.6) we have

$$I_H = (C/k + o(1))\, X L(X) \qquad (X \to \infty),$$

and similarly it folows that

$$I_\chi = (C/k + o(1)) X L(2X) \qquad (X \to \infty).$$

Since $L(x)$ is slowly varying (1.4) gives

$$L(2X) = (1+o(1))L(X) \qquad (X \to \infty),$$

and we obtain (1.10) from (2.7). It is the weak asymptotic formula (2.8) that accounts for $o(1)$ in (1.10), and hypotheses on $L(x)$ that would give a sharpening of (2.8) would lead to a sharpening of (1.10).

3. Applications. Some applications of the Theorem will be given now, although many more can be found. Let $k \ge 2$ be a fixed integer. A natural number n is k-free (resp. k-full) if $n = 1$ or if all the exponents in the canonical decomposition of n are $\le k-1$ (resp. $\ge k$). Let $Q(k)$ denote the set of k-free and $G(k)$ the set of k-full numbers. Then

$$f(n) = \sum_{d^k|n} \mu(d) = \begin{cases} 1 & n \in Q(k) \\ 0 & n \notin Q(k) \end{cases} \tag{3.1}$$

is the characteristic function of $Q(k)$. One has (see (14.24) of [**5**])

$$F_k(x) := \sum_{n \le x} f(n) = \sum_{n \le x, n \in Q(k)} 1 = \frac{x}{\zeta(k)} + R_k(x) \tag{3.2}$$

with

$$R_k(x) \ll x \exp\left(-C(k)\log^{3/5} x (\log\log x)^{-1/5}\right) \qquad (C(k) > 0). \tag{3.3}$$

Rewriting (3.1) as

$$f(n) = \sum_{d|n} g\left(\frac{n}{d}\right), \quad g(m) = \begin{cases} \mu(m) & m = l^k, \\ 0 & m \ne l^k, \end{cases}$$

it is seen that (1.1) and (1.2) hold with $k = 1$ and $b = 1/k$, where in the latter case k refers to $Q(k)$. Hence applying the Theorem we obtain

Corollary 1. *If $F_k(x)$ is given by* (3.2) *and $L(x) \in \mathcal{L}$, then as $X \to \infty$*

$$\int_X^{2X} (F_k(x+L(x)) - F_k(x))dx = \left(\frac{1}{\zeta(k)} + o(1)\right) XL(X). \tag{3.4}$$

If $n \in G(k)$, then n can be written uniquely as

$$n = a_1^k a_2^{k+1} \dots a_k^{2k-1}, \quad \mu^2(a_2 \dots a_k) = 1.$$

In this case $f(n)$ represents the characteristic function of $G(k)$, and it has the form

$$f(n) = \sum_{d^k|n} g\left(\frac{n}{d^k}\right), \quad g(n) = \sum_{a_2^{k+1} \dots a_k^{2k-1} = n} \mu^2(a_2 \dots a_k). \tag{3.5}$$

If $A_k(x)$ denotes the number of k-full integers not exceeding x,then we may write

$$A_k(x) = \sum_{n\le x, n\in G(k)} 1 = \gamma_{0,k}x^{1/k} + \gamma_{1,k}x^{1/(k+1)} + \ldots + \gamma_{k-1,k}x^{1/(2k-1)} + \delta_k(x) \quad (3.6)$$

where $\delta_k(x)$ may be considered as the error term in the asymptotic formula for $A_k(x)$. For estimates of $\delta_k(x)$ the reader is referred to Ch. 14 of [**5**]. One has $\gamma_{0,2} = \zeta(3/2)/\zeta(3)$, and in general

$$\gamma_{0,k} = \sum_{m=1}^{\infty} g(m)m^{-1/k} = \sum_{a_2=1}^{\infty} \ldots \sum_{a_k=1}^{\infty} \mu^2(a_2\ldots a_k)\left(a_2^{k+1}\ldots a_k^{2k-1}\right)^{-1/k}. \quad (3.7)$$

From (3.5) it follows that

$$\sum_{n\le x}|g(n)| \le \sum_{a_2^{k+1}\ldots a_k^{2k-1}\le x} 1 \ll x^{1/(k+1)}.$$

Thus we have $b = 1/(k+1)$ in (1.2) and (1.10) gives

Corollary 2. *If $A_k(x)$ is given by* (3.6), $\gamma_{0,k}$ *by* (3.7) *and* $h = x^{(k-1)/k}L(x)$ *with* $L(x)\in\mathcal{L}$, *then as* $X\to\infty$

$$\int_X^{2X} (A_k(x+h) - A_k(x))dx = \left(\frac{1}{k}\gamma_{0,k} + o(1)\right) XL(X). \quad (3.8)$$

As our third example consider $a(n)$, the multiplicative function which denotes the number of nonisomorphic abelian groups with n elements. For $\operatorname{Re} s > 1$ one has (see Ch. 1 and Ch. 14 of [**5**])

$$\sum_{n=1}^{\infty} a(n)n^{-s} = \zeta(s)\zeta(2s)\zeta(3s)\ldots, \quad (3.9)$$

while

$$A(x) := \sum_{n\le x} a(n) = C_1x + C_2x^{1/2} + C_3x^{1/3} + \delta(x), \delta(x) \ll x^{\alpha+\varepsilon}, \quad (3.10)$$

where the currently best value $\alpha = 40/159 = 0.25172\ldots$ is due to Liu [**7**], and

$$C_j = \prod_{r=1, r\ne j}^{\infty} \zeta\left(\frac{r}{j}\right). \quad (3.11)$$

From (3.9) one has

$$a(n) = \sum_{d|n} g\left(\frac{n}{d}\right), \quad \sum_{m\le x}|g(m)| = \sum_{m\le x} g(m) \ll x^{1/2},$$

hence (1.1) and (1.2) hold with $f(n) = a(n)$ and $k = 1$, $b = 1/2$. Thus (1.10) gives

COROLLARY 3. *If $A(x)$ is given by* (3.10), C_1 *by* (3.11) *and* $L(x) \in \mathcal{L}$, *then as* $X \to \infty$

$$\int_X^{2X} (A(x + L(x)) - A(x))dx = (C_1 + o(1))XL(X). \tag{3.12}$$

Known pointwise estimates for the functions $F_k(x)$, $A_k(x)$ and $A(x)$ are not nearly as good as (3.4), (3.8) and (3.12), respectively. For k-free numbers in short intervals the reader should see Filaseta's papers [**2**], [**3**], for squarefull and cubefull numbers the already mentioned papers [**8**] and [**10**], and for $a(n)$ the author's paper [**6**].

In concluding it may be mentioned that the Theorem makes it possible to obtain useful arithmetic information in certain cases. As an example consider k-free integers, and let $A = A(x, X)$ denote the set of x from $[X, 2X]$ such that $(x, x + L(x)]$ contains a k-free number, where $L(x) \in \mathcal{L}$. If $F_k(x)$ is given by (3.2), then $x \in A$ is equivalent to $F_k(x + L(x)) - F_k(x) > 0$. If $m(A)$ is the measure of A, then from (3.4) and its proof it follows that

$$\begin{aligned} \left(\frac{1}{\zeta(k)} + o(1)\right) XL(X) &= \int_A (F_k(x + L(x)) - F_k(x))\,dx \\ &\ll m(A) \sup_{x\in[X,2X]} (F_k(x + L(x)) - F_k(x)) \end{aligned}$$

$$\ll m(A)\left\{L(X) + \sup_{x\in[X,2X]}\left|\sum_{L(x)<n\le(x+L(x))_{1/k}} \left(\Psi\left(\frac{x + L(x)}{n^k}\right) - \Psi\left(\frac{x}{n^k}\right)\right)\right|\right\}.$$

Trivial estimation gives at once $m(A) \gg X^{(k-1)/k}L(X)$. Use of Lemma 3 of Halberstam and Roth [**4**] leads to the better bound $m(A) \gg X^{(2k-2)/(2k-1)-\varepsilon}$ for any given $\varepsilon > 0$, and the recent methods of Filaseta [**2**] would lead to some further improvement.

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