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EXISTENCE OF SOLUTIONS FOR INTEGRODIFFERENTIAL INCLUSIONS IN BANACH SPACES

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Abstract. We examine nonlinear integrodifferential inclusions defined in a separable Banach space. Using a compactness type hypothesis involving the ball measure of noncompactness, we establish two existence results. One involving convex-valued orientor fields and the other nonconvex valued ones.

1. Introduction. In this paper, we prove two existence theorems for integrodifferential inclusions in a separable Banach space. The first existence theorem concerns convex-valued orientor fields, while the second deals with nonconvex-valued ones. Our "convex" result extends the works of Davy [**3**, Theorem 4.2], Mukshinov [**8**] and Papageorgiou [**15**, Theorems 3.2 and 3.5]. All these works treated differential inclusions with no Volterra operator present. Similarly, our "nonconvex" result extends the work of Kisielewicz [**5**]. Furthermore, the results of the present paper extend to multivalued integrodifferential systems, the recent work of the author [**14**] on Volterra integral inclusions.

2. Preliminaries. The purpose of this section is to briefly review some basic facts about the measurability and continuity properties of multifunctions (set valued functions) that we will need in the sequel.

Let (Ω, Σ) be a measurable space and X a separable Banach space. Throughout this paper we will be using the following notations:

$$P_{f(c)}(X) = \{A \subseteq X\colon \text{ nonempty, closed, (convex)}\}$$

and $P_{(w)k(c)}(X) = \{A \subseteq X\colon \text{ nonempty, (weakly-) compact, (convex)}\}$.

A multifunction $F\colon \Omega \to P_f(X)$ is said to be measurable, if for every $x \in X$ the $\mathbf{R}_+$-valued function $\omega \to d(x, F(\omega)) = \inf\{\|x - z\|\colon z \in F(\omega)\}$ is measurable. In fact this is equivalent to saying that for every $U \subseteq X$ open, $F^-(U) = \{\omega \in \Omega\colon F(\omega) \cap U \neq \emptyset\}$ or that there exists a sequence $\{f_n\}_{n\geq 1}$ of measurable

AMS Subject Classification (1980): Primary 34G05, 45G05 functions $f_n: \Omega \to X$ such that $F(\omega) = \overline{\{f_n(\omega)\}}_{n \geq 1}$ for all $\omega \in \Omega$. A multifunction $F: \Omega \to 2^X \setminus \{\emptyset\}$ is said to be weakly (or scalarly) measurable, if for every $x^* \in X^*$ its support function $\omega \to \sigma(x^*, F(\omega)) = \sup\{(x^*, x): x \in F(\omega)\}$ is measurable. It is clear that for $P_f(X)$-valued multifunctions measurability implies weak measurability (just observe that for every $x^* \in X^*$, $\sigma(x^*, F(\omega)) = \sup_{n \geq 1}(x^*, f_n(\omega))$, where $f_n: \Omega \to X$ are measurable functions such that $F(\omega) = \overline{\{f_n(\omega)\}}_{n \geq 1}$ for all $\omega \in \Omega$). The converse is true if there is a σ-finite measure $\mu(\cdot)$ defined on Σ, Σ is μ-complete and $F(\cdot)$ is $P_{wkc}(X)$-valued. For a multifunction $F: \Omega \to 2^X \setminus \{\emptyset\}$, the graph of $F(\cdot)$ is defined by $\mathrm{Gr}F = \{(\omega, x) \in \Omega \times X: x \in F(\omega)\}$. We say that $F(\cdot)$ is graph measurable, if $\mathrm{Gr}F \in \Sigma \times B(X)$, with $B(X)$ being the Borel σ-field of X. For $P_f(X)$-valued multifunctions measurability implies graph measurability. Indeed let $f_n: \Omega \to X$, $n \geq 1$ be a sequence of measurable maps such that $F(\omega) = \overline{\{f_n(\omega)\}}_{n \geq 1}$ for all $\omega \in \Omega$ and note that $\mathrm{Gr}F = \{(\omega, x) \in \Omega \times X: d(x, F(\omega)) = 0\}$. But $d(x, F(\omega)) = \inf_{n \geq 1} \|x - f_n(\omega)\|$ and for each $n \geq 1$, $(\omega, x) \to \|x - f_n(\omega)\|$ is measurable in ω, continuous in x (i.e. a Caratheodory function), hence $(\omega, x) \to \|x - f_n(\omega)\|$ is jointly measurable $\Rightarrow (\omega, x) \to d(x, F(\omega)) = \inf_{n \geq 1} \|x - f_n(\omega)\|$ is jointly measurable $\Rightarrow \mathrm{Gr}F \in \Sigma \times B(X)$. Again the converse is true if there is a σ-finite measure $\mu(\cdot)$ defined on Σ and Σ is μ-complete. For more details we refer to Wagner [**18**].

Now suppose that (Ω, Σ, μ) is a finite measure space and $F: \Omega \to 2^X \setminus \{\emptyset\}$ a multifunction. By S_F^1 we will denote the set of integrable selectors of $F(\cdot)$; i.e. $S_F^1 = \{f \in L^1(X): f(\omega) \in F(\omega) \mu\text{–a.e.}\}$. This set may be empty. For a graph measurable multifunction, it is nonempty if and only if $\omega \to \inf\{\|z\|: z \in F(\omega)\} \in L^1_+$. This is the case if $\omega \to |F(\omega)| = \sup\{\|z\|: z \in F(\omega)\} \in L^1_+$ and such a multifunction is called "integrably bounded". For a graph measurable multifunction S_F^1 is closed in $L^1(X)$ if and only if $F(\cdot)$ is $P_f(X)$-valued, and is convex if and only if $F(\cdot)$ is convex valued. Also the set S_F^1 is decomposable, in the sense that if $f_1, f_2 \in S_F^1$ and $A \in \Sigma$, $f = \chi_A f_1 + \chi_{A^c} f_2 \in S_F^1$. For further details we refer to [**12**] and [**13**]. Using the set S_F^1, we can define a set valued integral for $F(\cdot)$ by setting

$$\int_\Omega f(\omega) d\mu(\omega) = \left\{ \int_\Omega f(\omega) d\mu(\omega): f \in S_F^1 \right\}.$$

The vector valued integrals involved in this definition are understood in the sense of Bochner. A detailed study of this set valued integral can be found in Kandilakis-Papageorgiou [**4**].

Next let Y, Z be Hausdorff topological spaces and $F: Y \to 2^Z \setminus \{\emptyset\}$. We say that $F(\cdot)$ is upper semicontinuous (u.s.c.) (resp. lower semicontinuos (l.s.c.)), if for every $U \subseteq Z$ open, the set $F^+(U) = \{y \in Y: F(y) \subseteq U\}$ (resp. $F^-(U) = \{y \in Y: F(y) \cap U \neq \emptyset\}$) is open in Y. If $F(\cdot)$ is both u.s.c. and l.s.c., then we say that $F(\cdot)$ is continuous. In fact continuity is equivalent to saying that $F: Y \to 2^Z \setminus \{\emptyset\}$ is continuous from Y into $2^Z \setminus \{\emptyset\}$ the latter equipped with the Vietoris topology. If Z is a metric space, on $P_f(Z)$ we can define a generalized metric, known in the literature as the Hausdorff metric, by setting $h(A, B) = \max[\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)]$. The metric space $(P_f(Z), h)$ is complete if Z is complete. A multifunction $F: Y \to P_f(Z)$ is said to be Hausdorff continuous

(h-continuous), if it is continuous from Y into $(P_f(Z), h)$. Since on $P_k(Z)$ the Vietoris and Hausdorff topologies coincide (see Klein–Thompson [**6**, Corollary 4.2.3, p. 41]), a $P_k(Z)$-valued multifunction is continuous if and only if it is h-continuous.

Let X be a Banach space and $\mathcal{B}$ its family of bounded sets. Then the Hausdorff (ball)-measure of noncompactness $\beta: \mathcal{B} \to \mathbf{R}_+$ is defined by

$$\beta(B) = \inf\{r > 0: B \text{ can be covered by finitely many balls of radius } r\}.$$

A comprehensive introduction to the subject of measures of noncompactness can be found in Banas–Goebel [**1**].

Finally, if $\{A_n\}_{n\geq 1} \subseteq 2^X \setminus \{\emptyset\}$, we set

$$w\text{-}\overline{\lim} A_n = \{x \in X: x = w\text{-}\lim x_{n_k}, x_{n_k} \in A_{n_k}, n_1 < n_2 < \ldots < n_k < \ldots\}$$

3. Existence results. Let $T = [0, r]$ and X be a separable Banach space. Let $K: \Delta = \{(t, s): 0 \leq s \leq t \leq r\} \to \mathcal{L}(X)$ be a strongly continuous kernel (i.e. it is continuous from Δ into $\mathcal{L}(X) = \{$bounded liner operators from X into itself$\}$ equipped with the strong operator topology) and let $V: C(T, X) \to C(T, X)$ be the Volterra integral operator corresponding to the kernel $K(t, s)$; i.e. $V(x)(t) = \int_0^t K(t, s)x(s)\, ds$. We consider the following integrodifferential inclusion:

$$\dot{x} \in F(t, x(t), V(x)(t)) \text{ a.e.,} \qquad x(0) = x_0 \tag{$*$}$$

By a solution of $(*)$, we understand a function $x(\cdot) \in C(T, X)$ such that

$$x(t) = x_0 + \int_0^t f(s) ds$$

for all $t \in T$ and with $f \in S^1_{F(\cdot, x(\cdot), V(x)(\cdot))}$. Note that such a function is almost everywhere differentiable and $\dot{x} = f(t) \in F(t, x(t), V(x)(t))$ a.e.

We will start with a "convex" result. For this we will need the following hypothesis on the orientor field $F(t, x, y)$.

$H(F)_1$ $F: T \times X \times X \to P_{fc}(X)$ is multifunction such that

(1) $(t, x, y) \to F(t, x, y)$ is weakly measurable,

(2) $(x, y) \to F(t, x, y)$ is u.s.c. from $X \times X$ into X_w, where X_w denotes the Banach space X equipped with the weak topology,

(3) $\beta(F(t, B_1, B_2)) \leq k(t)[\beta(B_1) + \beta(B_2)]$, for all $B_1, B_2 \subseteq X$ nonempty bounded and with $k(\cdot) \in L^1_+$,

(4) $|F(t, x, y)| = \sup\{\|z\|: z \in F(t, x, y)\} = a(t) + b(t)(\|x\| + \|y\|)$ a.e. with $a(\cdot), b(\cdot) \in L^1_+$.

In the proof of the "convex" existence result, we will need the following lemma, which in fact is of independent interest.

Lemma 3.1 *If (Ω, Σ, μ) is a σ-finite, complete measure space, X is a separable Banach space, $F: \Omega \to 2^X \setminus \{\emptyset\}$ is a graph measurable multifunction and*

$u:\Omega \times X \to \mathbf{R}$ is a measurable function, then $\omega \to m(\omega) = \sup\{u(\omega,x): x \in F(\omega)\}$ is measurable.

Proof. We need to show that for every $\theta \in \mathbf{R}$, the level set $\{\omega \in \Omega: m(\omega) > \theta\} \in \Sigma$. Note that $m(\omega) > \theta$ if and only if there exists $x \in F(\omega)$ such that $u(\omega,x) > \theta$. So $\{\omega \in \Omega: m(\omega) > \theta\} = \mathrm{proj}_\Omega\{(\omega,x) \in \mathrm{Gr}F: u(\omega,x) > \theta\}$.

But since by hypothesis $F(\cdot)$ is graph measurable and $u(\cdot,\cdot)$ is measurable, we have $\{(\omega,x) \in \mathrm{Gr}F: u(\omega,x) > \theta\} \in \Sigma \times B(X)$. Then from von Neumann's projection theorem (see Saint-Beuve [**16**, Theorem 4]), we get that $\mathrm{proj}_\Omega\{(\omega,x) \in \mathrm{Gr}F: u(\omega,x) > \theta\} \in \Sigma$. So $m(\cdot)$ is indeed measurable as claimed by the lemma.

Now we are ready to state and prove our first existence theorem:

THEOREM 3.2 *If the hypothesis $H(F)_1$ holds, then the problem $(*)$ admits a solution.*

Proof. We will start by deriving an a priori bound for the solutions of $(*)$. So let $x(\cdot) \in C(T,X)$ be such a solution. Then by definiton we have:

$$x(t) = x_0 + \int_0^t f(s)\,ds$$

for all $t \in T$ and with $f \in L^1(X), f(t) \in F(t, x(t), V(x)(t))$ a.e. Hence

$$\begin{aligned}
\|x(t)\| &\le \|x_0\| + \int_0^t \|f(s)\| \\
&\le \|x_0\| + \int_0^t \left(a(s) + b(s) \left(\|x(s)\| + \left\| \int_0^s K(s,\tau)x(\tau)\,d\tau \right\| \right) \right) ds \\
&\le \|x_0\| + \int_0^t \left(a(s) + b(s)\|x(s)\| + b(s) \int_0^s M_1 \|x(\tau)\|\,d\tau \right) ds
\end{aligned}$$

where $\|K(t,s)\|_{\mathcal{L}} \le M_1$ for all $(t,s) \in \Delta$. Invoking Pachpatte's inequality (see Theorem 1 in [**9**]), we get that there exists $M_2 > 0$ such that for all $t \in T$ $\|x(t)\| \le M_2$. Then $V(x)(t) \le \int_0^t M_1\|x(s)\|ds \le M_1M_2b = M_3$

Define $\hat{F}: T \times X \times X \to P_{fc}(X)$ by $\hat{F}(t,x,y) = F(t, p_{M_2}(x), p_{M_3}(y))$, where $p_{M_2}(\cdot), p_{M_3}(\cdot): X \to X$ are the M_2 and M_3 radial retractions respectively. Recalling that $p_{M_2}(\cdot)$, $p_{M_3}(\cdot)$ are Lipschitz continuous and using the hypothesis $H(F)_1(1)$, we see that $(t,x,y) \to \hat{F}(t,x,y)$ is weakly measurable, while from the hypothesis $H(F)_1(2)$ and Theorem 7.3.11, of Klein–Thompson [**6**], we have that $(x,y) \to \hat{F}(t,x,y)$ is u.s.c. from $X \times X$ into X_w. Also if $B_1, B_2 \subseteq X$ are nonempty, bounded sets, then we have

$$\beta(\hat{F}(t,B_1,B_2)) = \beta(F(t, p_{M_2}(B_1), p_{M_3}(B_2)) \le k(t)[\beta(p_{M_2}(B_1) + \beta(p_{M_3}(B_2))].$$

But note that

$$p_{M_2}(B_1) \subseteq \overline{\mathrm{conv}}(B_1 \cup \{0\} \Rightarrow \beta(p_{M_2}(B_1)) \le \beta(\overline{\mathrm{conv}}(B_1 \cup \{0\})) = \beta(B_1)$$

and similarly we get that $\beta(p_{M_3}(B_2)) \le \beta(B_2)$. Therefore we have that

$$\beta(\hat{F}(t, B_1, B_2)) \le k(t)[\beta(B_1) + \beta(B_2)] \text{ a.e.}$$

Finally, observe that $|\hat{F}(t,x,y)| \le a(t) + b(t)(M_2 + M_3) = \varphi(t)$ a.e. with $\varphi(\cdot) \in L^1_+$. Let

$$W = \{y \in C(T,X) : y(t) = x_0 + \int_0^t g(s)ds, \;\; t \in T, \;\; \|g(t)\| \le \varphi(t) \quad \text{a.e.}\}$$

Clearly this is a nonempty, bounded, equicontinuous and closed subset of $C(T,X)$.

Next let $R: W \to 2^W$ be defined by

$$R(x) = \{y \in W : y(t) = x_0 + \int_0^t f(s)ds,$$
$$t \in T, \;\; f \in L^1(X), \;\; f(t) \in \hat{F}(t, x(t), V(x)(t)) \text{ a.e.}\}$$

First we will show that $R(\cdot)$ has nonempty values. Fix $x(\cdot) \in C(T,X)$ and $x^* \in X^*$ and let $\theta_1 : T \to T \times X \times X$ be defined by $\theta_1(t) = (t, x(t), V(x)(t))$. Clearly $\theta_1(\cdot)$ is measurable. Also let $\theta_2 : T \times X \times X \times X \to \mathbf{R}$ be defined by $\theta_2(t,x,y) = \sigma(x^*, \hat{F}(t,x,y))$. Because of hypothesis $H(F)_1(1)$, $\theta_2(\cdot,\cdot,\cdot)$ is measurable. Then $\theta_2 \circ \theta_1 : T \to \mathbf{R}$ defined by $(\theta_2 \circ \theta_1)(t) = \sigma(x^*, \hat{F}(t, x(t), V(x)(t))$ is measurable $\Rightarrow t \to \hat{F}(t, x(t), V(x)(t))$ is measurable for the Lebesgue σ-field on T (see Section 2). So from Aumann's selection theorem (see Theorem 5.10 of Wagner [**18**]) and since $|\hat{F}(t,x,y)| \le \varphi(t)$ a.e. with $\varphi(\cdot) \in L^1_+$, we see that $s^1_{\hat{F}(\cdot, x(\cdot), V(x)(\cdot))} \ne \emptyset$, which of course implies that $R(x) \ne \emptyset$ for all $x(\cdot) \in C(T,X)$. Clearly $R(x)$ is convex and we will now show that it is closed. Indeed let $\{y_n\}_{n \ge 1} \subseteq R(x)$ and assume that $y_n \to y$ in $C(T,X)$. Then by definition $y_n(t) = x_0 + \int_0^1 f_n(s)ds$, for all $t \in T$ and with $f_n \in S^1_{\hat{F}(\cdot, x(\cdot), V(x)(\cdot))}$, $n \ge 1$. Let $G(t) = \overline{\text{conv}}\{f_n(t)\}_{n \ge 1}$. Clearly $G(\cdot)$ is measurable and $G(t) \subseteq \hat{F}(t, x(t), V(t)) \in P_{kc}(X)$ (since $\beta(\hat{F}(t, B_1, B_2) \le k(t)[\beta(B_1) + \beta(B_2)]$ a.e.; just take $B_1 = \{x(t)\}$ and $B_2 = \{V(x)(t)\}$, note that $\beta(B_1) = \beta(B_2) = 0$ and so obtain that $\beta(\hat{F}(t, B_1, B_2)) = 0 \Rightarrow \hat{F}(t, x(t), V(x)(t)) \in P_{kc}(X)$. Hence $G(t) \in P_{kc}(X)$, $t \in T$, and $|G(t)| \le |\hat{F}(t, x(t), V(x)(t))| \le \varphi(t)$ a.e. So invoking proposition 3.1 of [**10**], we deduce that $S^1_G \in P_{wkc}(L^1(X))$ and so by the Eberlein–Smulian theorem and by passing to a subsequence if necessary, we may assume that $f_n \xrightarrow{w} f$ in $L^1(X)$. Since $S^1_{\hat{F}(\cdot, x(\cdot), V(x)(\cdot))} \in P_{wkc}(X)$ (see Proposition 3.1 in [**10**]), we get that $f \in S^1_{\hat{F}(\cdot, x(\cdot), V(x)(\cdot))}$. Then $y(t) = x_0 + \int_0^t f(s)ds$ for all $t \in T$ and with $f \in S^1_{\hat{F}(\cdot, x(\cdot), V(x)(\cdot))}$. So $R(x) \in P_{fc}(C(T,X))$.

Next let $B \subseteq W$ be nonempty and closed. In what follows, we set $B(t) = \{x(t) : x(\cdot) \in B\}$. We have

$$R(B)(t) =$$
$$\left\{x_0 + \int_0^t f(s)ds : f \in L^1(X), f(s) \in \hat{F}(s, x(s), V(x)(s)) \text{ a.e.}, x \in B\right\}$$

Note that $\{\hat{F}(s,x(s),V(x)(s)) : x \in B\} \subseteq \hat{F}(s,B(s),\overline{V(B)(s)})$ for all $s \in T$. Also for every $x^* \in X^*$, we have

$$\sigma(x^*,\hat{F}(s,B(s),\overline{V(B)(s)})) = \sigma\left(x^*, \bigcup_{\substack{x\in B(s)\\ y\in\overline{V(B)(s)}}} \hat{F}(s,x,y)\right)$$
$$= \sup[\sigma(x^*,\hat{F}(s,x,y)) : (x,y) \in B(s) \times \overline{V(B)(s)}].$$

Observe that $s \to B(s)$ is measurable, since if $\{x_n\}_{n\geq 1} \subseteq B$ is dense in B, then from the continuity of the evaluation map, we have that $B(s) = \overline{\{x_n(s)\}}_{n\geq 1}$, establishing the measurability of $B(\cdot)$. Similarly, using theorem 3.1 of Kandilakis–Papageorgiou [**4**], we have that

$$\overline{V(B)(s)} = \overline{\left\{\int_0^s K(s,\tau)x_n(\tau)d\tau\right\}}_{n\geq 1} \Rightarrow s \to \overline{V(B)(s)} \text{ is measurable.}$$

Since $(s,x,y) \to \sigma(x^*,\hat{F}(s,x,y)$ is measurable (follows from hypothesis $H(F)(1)$, from Lemma 3.1, we deduce that

$$\begin{aligned} &s \to \sup[\sigma(x^*,F(s,x,y)) : x \in B(s), y \in \overline{V(B)(s)}] \text{ is measurable}\\ &\qquad \Rightarrow s \to \sigma(x^*,\hat{F}(s,B(s),\overline{V(B)(s)}) \text{ is measurable}\\ &\qquad \Rightarrow s \to \overline{\text{conv}}F(s,B(s),\overline{V(B)(s)}) = H(s) \text{ is measurable for the Lebesgue}\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \sigma\text{-field on } T \text{ (see Section 2).} \end{aligned}$$

Thus there exist $h_n : T \to X$, $n \geq 1$, Lebesgue measurable functions such that for all $t \in T$, $H(t) = \overline{\{h_n(t)\}}_{n\geq 1}$. So invoking Proposition 1.6 of Mönch [**7**] (see also Lemma 2.2 of Kisielewicz [**5**]), we have

$$\begin{aligned} \beta(R(B)(t)) &\leq \beta\left[\int_0^t \{h_n(s)\}_{n\geq 1} ds\right] \leq \int_0^t \beta(\{h_n(s)\}_{n\geq 1} ds\\ &= \int_0^t \beta(H(s))ds = \int_0^t \beta(\hat{F}(s,B(s),\overline{V(B)(s)})ds\\ &\leq \int_0^t k(s)[\beta(B(s)) + \beta(\overline{V(B)(s)}]ds\\ &= \int_0^t k(s)[\beta(B(s)) + \beta(V(B)(s))]ds. \end{aligned}$$

From the definition of the Volterra integral operator $V(\cdot)$, we have

$$\begin{aligned}
\beta(\overline{V(B)(s)}) &= \beta(V(B)(s)) = \beta\left[\int_0^s K(s,t)B(\tau)d\tau\right] \\
&= \beta\left[\int_o^s K(s,\tau)x_n(\tau)d\tau : n \geq 1\right] \\
&\leq \int_0^s \beta(K(s,\tau)x_n(\tau): n \geq 1)d\tau \qquad \text{(see Mönch [}\mathbf{7}\text{])} \\
&\leq \int_0^s M_1\beta(\{x_n(\tau)\}_{n\geq 1})d\tau = \int_0^s M_1\beta(B(\tau))d\tau \\
\Rightarrow \quad \int_0^t \beta(V(B)(s))ds &\leq \int_0^t \int_0^s M_1\beta(B(\tau))d\tau ds \leq M_1 r \int_0^t \beta(B(\tau))d\tau.
\end{aligned}$$

So we have:

$$\beta(R(B(t)) \leq \int_0^t k(s)[\beta(B(s)) + M_1 r\beta(B(s))]ds.$$

Let $\psi(B) = \sup_{t\in T}\left[\exp\left(-\lambda\int_0^t k(s)ds\right)\beta(B(t))\right]$, $\lambda > 0$. Using the properties of $\beta(\cdot)$ and the fact that $W \subseteq C(T,H)$ is equicontinuous, we can easily check that $\psi(\cdot)$ is a sublinear measure of noncompactness, in the sense of Banas–Goebel [**1**]. We have

$$\begin{aligned}
\beta(R(B)(t)) &\leq \int_0^t k(s)(1+M_1 r)\exp\left(-\lambda\int_0^t k(\tau)d\tau\right) \times \\
&\qquad\qquad \exp\left(\lambda\int_0^t k(\tau)d\tau\right)\beta(B(s))ds \\
&\leq \int_0^t k(s)(1+M_1 r)\psi(B)\exp\left(\lambda\int_0^s k(\tau)d\tau\right)ds \\
&= \frac{(1+M_1 r)\psi(B)}{\lambda}\int_0^t d\left(\exp\left(\lambda\int_0^t k(\tau)d\tau\right)\right) \\
\Rightarrow \quad & \beta(R(B)(t)\exp\left(-r\int_0^t k(s)ds\right) \leq \left(\frac{1+M_1 r}{\lambda}\right)\psi(B), \quad t \in T \\
\Rightarrow \quad & \psi(R(B)) \leq \left(\frac{1+M_1 r}{\lambda}\right)\psi(B).
\end{aligned}$$

So if we choose $\lambda > (1+M_1 r)$, we have that $R(\cdot)$ is ψ-contraction.

Next we will show that the multifunction $R(\cdot)$ has a closed graph. To this end, let $[x_n, y_n] \in \mathrm{Gr}R$, $n \geq 1$, $[x_n, y_n] \to [x, y]$ in $C(T,X)\times C(T,X)$. By definition for every $n \geq 1$, we have $y_n(t) = x_0 + \int_0^t f_n(s)ds$ for all $t \in T$ and with $f_n \in L^1(X)$, $f_n(t) \in F(t, x_n(t), V(x_n)(t)$ a.e. But $x_n \to x$ in $C(T,X)$ and so $V(x_n) \to V(x)$ in

$C(T,X)$. Since $\hat{F}(t,\cdot,\cdot)$ is u.s.c. from $X \times X$ into X_ω, using Theorem 7.4.2, p. 90 of Klein–Thompson [**6**], we have

$$\overline{\text{conv}} \bigcup_{n\geq 1} \hat{F}(t, x_n(t), V(x_n)(t)) = G(t) \in P_{wkc}(X)$$

and clearly $t \to G(t)$ is a measurable multifunction, with $|G(t)| = \sup\{\|z\|: z \in G(t)\} \leq \varphi(t)$ a.e. So invoking once again Proposition 3.1 of [**10**], we get that $S^1_G \in P_{wkc}(L^1(X))$. Since $\{f_n\}_{n\geq 1} \subseteq S^1_G$, from the Eberlein–Smulian theorem and by passing to a subsequence if necessary, we may assume that $f_n \xrightarrow{w} f$ in $L^1(X)$. Using Theorem 3.1 of [**11**] we have

$$\begin{aligned} f(t) \in \overline{\text{conv}}\, w\text{-}\overline{\lim}\{f_n(t)\}_{n\geq 1} &\subseteq \overline{\text{conv}}\, w\text{-}\overline{\lim}\hat{F}(t, x_n(t), V(x_n)(t)) \\ &\subseteq \hat{F}(t, x(t), V(x)(t)) \text{ a.e.}, \end{aligned}$$

the last inclusion following from the fact that $\hat{F}(t,\cdot,\cdot)$ is u.s.c. from $X \times X$ into X_w, is convex-valued and $x_n \xrightarrow{s} x$, $V(x_n) \xrightarrow{s} V(x)$ in $C(T,X)$. Hence

$$y(t) = x_0 + \int_0^t f(s)ds$$

for all $t \in T$ and with $f \in S^1_{\hat{F}(\cdot,x(\cdot),V(x)(\cdot))}$. Thus $[x,y] \in \text{Gr}R$; i.e., $R(\cdot)$ has a closed graph in $W \times W$.

Apply Theorem 4.1 of Tarafdar–Vyborny [**17**], to get $x \in R(x)$. Then as in the beginning of the proof, using the definition of $\hat{F}(t,x,y)$ and Pachpatte's inequality (Theorem 1, in [**9**]), we get that $\|x(t)\| \leq M_2$, $\|V(x)(t)\| \leq M_3$ for all $t \in T$ and so $\hat{F}(t, x(t), V(x)(t)) = F(t, x(t), V(x)(t))$. Hence $x(\cdot) \in C(T,X)$ is the desired solution of $(*)$.

Now we will prove the "nonconvex" analog of Theorem 3.2. For this we will need the following hypothesis on the orientor field $F(t,x,y)$.

$H(F)_2$ $F: T \times X \times X \to P_f(X)$ is a multifunction such that

(1) $(t,x,y) \to P_f(X)$ is graph measurable,

(2) $(x,y) \to F(t,x,y)$ is l.s.c.,

(3) $\beta(F(t,B_1,B_2)) \leq k(t)[\beta(B_1)+\beta(B_2)]$ a.e. for all $B_1, B_2 \subseteq X$ nonempty, bounded and with $k(\cdot) \in L^1_+$,

(4) $|F(t,x,y)| = \sup\{\|z\|: z \in F(t,x,y)\} \leq a(t) + b(t)(\|x\| + \|y\|)$ a.e., with $a(\cdot), b(\cdot) \in L^1_+$.

THEOREM 4.3 *If hypothesis $H(F)_2$ holds, then $(*)$ admits a solution.*

Proof. As in the beginning of the proof of Theorem 3.2, we can get that for every solution $x(\cdot) \in C(T,X)$ of $(*)$, we have $\|x(t)\| \leq M_2$ and $\|V(x)(t)\| \leq M_3$ for all $t \in T$. As before, define $\hat{F}(t,x,y) = F(t, p_{M_2}(x), p_{M_3}(y))$. Again we can easily check that $\hat{F}(\cdot,\cdot,\cdot)$ is graph measurable, $\hat{F}(t,\cdot,\cdot)$ is l.s.c., $\beta(\hat{F}(t,B_1,B_2)) \leq$

$k(t)[\beta(B_1) + \beta(B_2)]$ a.e. and $|\hat{F}(t,x,y) = \sup\{\|z\|: x \in \hat{F}(t,x,y)\} \leq \varphi(t)$ a.e. with $\varphi(\cdot) \in L^1_+$. Set

$$W = \{y \in C(T,X): y(t) = x_0 + \int_0^t f(s)ds,\, t \in T,\, \|f(t)\| \leq \varphi(t) \text{ a.e.}\}.$$

Then W is a closed, convex, bounded and equicontinuous subset in $C(T,X)$. Let $R: W \to 2^{L^1(X)}$ be defined by $R(x) = S^1_{\hat{F}(\cdot,x(\cdot),V(x)(\cdot))}$.

Note that for every $x \in W$, the map $\eta_x: T \times T \to T \times X \times X \times X$ defined by $\eta_x(t,v) = (t, x(t), V(x)(t), v)$ is measurable. Then $\mathrm{Gr}\hat{F}(\cdot,x(\cdot),V(x)(\cdot)) = \eta_x^{-1}(\mathrm{Gr}\hat{F}) \in B(T) \times B(X)$ since $\hat{F}(\cdot,\cdot,\cdot)$ is graph measurable. So

$$t \to \hat{F}(t,x(t),V(x)(t))$$

is measurable for the Lebesgue σ-field on T. Thus $S^1_{\hat{F}(\cdot,x(\cdot),V(x)(\cdot))} \neq \emptyset$ and in fact it belongs in $P_f(L^1(X))$ and is decomposable (see Section 2). Hence $R: W \to P_f(L^1(X))$ and from Theorem 4.1 of [**11**], we know that it is l.s.c. So we can apply Theorem 3 of Bressan–Colombo [**2**] and get $v: W \to L^1(X)$ continuous map such that $v(x) \in R(x)$ for all $x \in W$. Then define $u: W \to W$ by

$$u(x)(t) = x_0 + \int_0^t v(x)(s)ds, \;\; t \in T.$$

Let $B \subseteq W$ be nonempty and closed. We have $\beta(u(B)(t)) \leq \beta\left[\int_0^t v(B)(s)ds\right]$.

Let $\{x_n\}_{n\geq 1}$ be dense in B. Then $\overline{v(B)(s)} = \overline{\{v(x_n)(s)\}}_{n\geq 1}$. So using Proposition 1.6 of Mönch [**7**], we get

$$\begin{aligned}\beta(u(B)(t)) &\leq \beta\left[\int_0^t v(x_n)(s)ds: n \geq 1\right] \leq \int_0^t \beta(v(x_n)(s): n \geq 1)ds \\ &\leq \int_0^t \beta(\hat{F}(s,B(s),V(B)(s)))ds \leq \int_0^t k(s)[\beta(B(s)) + \beta(V(B)(s))]ds \\ &\leq \int_0^t k(s)[\beta(B(s)) + M_1 r\beta((B)(s))]ds.\end{aligned}$$

As in the proof of Theorem 3.2, set $\psi(B) = \sup_{t\in T}\left[\exp\left(-\lambda\int_0^t k(s)ds\right)\right]$, which is a sublinear measure of noncompactness on the nonempty subsets of W. Then, as before, we get $\psi(u(B)) \leq (1 + rM_1)\lambda^{-1}\psi(B)$

If we choose $\lambda > (1 + rM_1)$, we get that $u(\cdot)$ is a ψ-contraction. Clearly $u(\cdot)$ is continuous, since $v(\cdot)$ is. Hence Theorem 4.1 of Tarafdar–Vyborny [**17**], tells us that $u(\cdot)$ has a fixed point; i.e. there exists $x \in W$ such that $x = u(x)$. Using the definition of $\hat{F}(t,x,y)$ and Pachpatte's inequality, we can easily check that $\|x(t)\| \leq M_2$ and $\|V(x)\| \leq M_3$ for all $t \in T$. So $\hat{F}(t,x(t),V(x)(t)) = F(t,x(t),V(x)(t))$. Therefore $x(\cdot) \in C(T,X)$ solves $(*)$.

Remark. If $F: T \times X \times X \to P_f(X)$ is a multifunction such that $(x,y) \to F(t,x,y))$ is h-continuous, then $(x,y) \to \mathrm{bd}F(t,x,y)$ is h-continuous too (here $\mathrm{bd}F(t,x,y)$ denotes the boundary of $F(t,x,y))$. Thus if $F(\cdot,\cdot,\cdot)$ satisfies also $H(F)_2(1)$ (3) and (4), we deduce that the integrodifferential inclusion $\dot{x} \in \mathrm{bd}F(t,x(t),V(x)(t))$, $x(0) = x_0$ has a solution. Such results are useful in control theory, in connection with the maximum principle.

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