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A PROOF OF BÁRÁNY'S THEOREM

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Abstract. We give a new proof of the following theorem of I. Bárány and L. Lowasz: Let $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_{d+1}$ be finite nonempty families of convex sets from R^d and suppose that for any choice $C_1 \in \mathcal{S}_1, \ldots, C_{d+1} \in \mathcal{S}_{d+1}$ the intersection $\cap C_i$ is not empty. Then for some $i = 1, \ldots, d+1$ all the sets in family S_i have a common point.

1. Introduction. The well-known Helly's theorem (finite form) says that given a finite family $\mathcal{F} = \{K_1, \ldots, K_r\}$ of convex sets in R^d, with $r \geq d+1$, such that every $(d+1)$ members of the families have nonempty intersection, then $\bigcap_{i=1}^r K_i \neq \emptyset$. In this paper we prove a generalization of Helly's theorem due to Bárány [**2**].

Let $[m] = \{1, \ldots, m\}$. $C \subseteq \mathcal{P}([m])$ is abstract simplicial complex if $s' \subseteq s \Rightarrow s' \in C$, for every $s \in C$. Element $s \in C$ is called n-dimenzional simplex if $\|s\| = n$.

To each finite family $\mathcal{F}$ of sets we associate a finite abstract simplicial complex $\mathcal{N}(\mathcal{F}) = \{s \subseteq [m] \mid \bigcap_{i\in s} K_i \neq \emptyset\}$, the nerv of $\mathcal{F}$. It contains full information about all intersections in the family $\mathcal{F}$. The complexes which are nervs of finite families of convex sets in R^d are called d-representable complexes.

Let C denote an arbitrary finite simplicial complex. For any simplex s, by $\operatorname{cost}(s, C)$ we denote the subcomplex of C containing all members of C which do not contain s. Simplex s is free (in C) if there is only one maximal simplex s' such that $s \subseteq s'$. Let s be free simplex of dimension less then d. The operation of transforming complex C to the complex $D = \operatorname{cost}(s, C)$, denoted by $C_d\searrow D$, will be called elementary d-collapsing. A finite chain $C_d\searrow C_{1\,d}\searrow \cdots {}_d\searrow D$ of elementary d-collapsings is called a d-collapsing and also denoted by $C_d\searrow D$. Finally, C is d-collapsible if C d-collapses to the empty complex $\emptyset$.

Wegner, [**1**], proved an important result which shows that d-collapsibility is a fundamental property of nerves of finite families of convex setys:

THEOREM 1.1. [**1**, p. 319] *If $C(\neq \emptyset)$ is d-representable, then C contains a free simplex s of dimension less then d such that* $\operatorname{cost}(s, C)$ *is again d-representable.*

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Theorem 1.2. [**1**, p. 319] *Every d-representable complex is d-collapsible.*

We will use that in the next section, in the proof of our main theorem 1.1. Importance of Wegner's theorem is nicely demonstrated by G. Kalai who successfully used d-collapsibility as well as specially defined new type of homology groups to establish the Eckhoff's conjecture and other results about nerves of convex sets (see [**4**], [**5**]).

Though the idea of using Wegner's theorem in the context of Bárány's theorem was noticed by him and others who applied this result in other context, yet we could not find such a proof published anywhere. Also, there is a growing interest in various generalizations of Bárány's theorem (see [**3**] and the conjecture at the end of this paper), so it is certainly important to analize and list all avilable proving techniques.

2. The main result. This section contains the proof of multiplied Helly's theorem, the main result of this paper:

Theorem 2.1. *Let $\mathcal{S}_1, \dots, \mathcal{S}_{d+1}$ be finite nonempty families of convex sets from R^d and for any choice $C_1 \in \mathcal{S}_1, \dots, C_{d+1} \in \mathcal{S}_{d+1}$ the intersection $\bigcap_{i=1}^{d+1} C_i$ is not empty. Then for some $i = 1, \dots, d+1$ all the sets in family $\mathcal{S}_i$ have a point in common.*

Proof. Let C denote the nerv of union $\mathcal{S}_1 \cup \mathcal{S}_2 \cup \ldots \cup \mathcal{S}_{d+1}$ and let $s = (A, B, C)$ denote the simplex s generated by nonempty intersection of the sets $A, B, C \in \bigcup_{i=1}^{d+1} \mathcal{S}_i$ is d-representable simplicial complex. According to Theorem 1.1 there exists a free simplex $s \in C$ such that $\dim(s) < d$ and $\operatorname{cost}(s, C)$ is d-representable. Depending on whether all the sets that define simplex $s \in C$ belong to different families $\mathcal{S}_1, \dots, \mathcal{S}_{d+1}$ or not, there are two options:

(1) Let $s = (A_1, \dots, A_k)$, $k \leq d$ and sets $A_1, \dots, A_k$ belong to different families S_i. Suppose that $(A_1, \dots, A_k) \in \mathcal{S}_1 \times \mathcal{S}_2 \times \ldots \times \mathcal{S}_k$, $k \leq d$. By assumption of the theorem, for each $B \in \mathcal{S}_{d+1}$ $(k+1)$-tuple $(A_1, \dots, A_k, B)$ is an element of C. Since $s = (A_1, \dots, A_k)$ is a free simplex, $(k+m)$-tuple $(A_1, \dots, A_k, B_1, B_2, \dots, B_m)$, where $m = \|\mathcal{S}_{d+1}\|$ and $\mathcal{S}_{d+1} = \{B_1, \dots, B_m\}$, is also an element of C. Therefore, $\mathcal{S}_{d+1}$ has nonvoid intersection.

(2) Let $s = (A_1, \dots, A_k)$, $k \leq d$, and $i, j \leq k$ be such that the sets A_i and A_j are members of the same family $\mathcal{S}_m$, $m = 1, \dots, d+1$. Then any $(d+1)$-tuple $(B_1, B_2, \dots, B_{d+1}) \in \mathcal{S}_1 \times \mathcal{S}_2 \times \ldots \mathcal{S}_{d+1}$ is an element of the simplicial complex $\operatorname{cost}(s, C)$. Hence, $\operatorname{cost}(s, C)$ is the nerv of the union of $(d+1)$ families of convex sets which satisfy the assumption of the theorem.

According to (2), necessary condition for given complex C to be converted to the void complex $\emptyset$ by finite sequence of elementary d-collapsings is appearance of a simplex of type (1) in some step of d-collapsing process. In that case, we have already proved in (1) that there exist family $\mathcal{S}_1$, $1 \leq i \leq d+1$, with nonempty intersection.

By Theorem 1.2 and above conclusions the assertion of our theorem follows. $\square$

As remarked by Bárány, Helly's theorem follows from Theorem 2.1 by putting $\mathcal{S}_1 = \mathcal{S}_2 = \ldots = \mathcal{S}_{d+1}$.

3. A Generalisation of Carathéodory's theorem. A Carathéodory's theorem says that, given a set $V \subseteq R^d$ and a point $a \in \operatorname{conv}(V)$, there exist a subset $A \subseteq V$ such that $\|A\| \leq n+1$ and $a \in \operatorname{conv}(V)$. It is well known that Carathéodory's and Helly's theorem imply each other, so it is not surprising that generalised Helly's theorem yields appropriate multiplied version of Carathéodory's theorem. This is exactly the approach of Bárány who deduced the generalization of Carathéodory's theorem. For completness and for the readers convenience, we include a simple duality statement which alows us to prove multiplied Carathéodory's from multiplied Helly's theorem.

Lemma 3.1. *Let $S = \{(a_i, -\alpha_i) \mid i = 1, \ldots, s,\ a_i \in R^{d-1},\ \alpha_i \in R\} \subseteq R^d$ be an arbitrary sistem of vectors in R^d and $h_i = \{x \in R^{d-1} \mid \langle a_i, x\rangle \geq \alpha_i\}$ $i = 1, \ldots, s$, be an appropriate family of closed halfspaces in R^{d-1}. Then*

$$(0,-1) \in \operatorname{pos}(S) \text{ iff } \bigcap_{i=1}^{s} h_i = \emptyset, \qquad 0 = (0, \ldots, 0) \in R^{d-1}$$

Proof. Suppose that $(0,-1) \in \operatorname{pos}(S)$ and $\bigcap_{i=1}^{s} h_i$ is nonempty. Then there exists $\lambda_1, \ldots, \lambda_s \geq 0$ and $x \in R^{d-1}$ such that:

$$(0,-1) = \lambda_1(a_1, -\lambda_1) + \ldots + \lambda_s(a_s, -\lambda_s) \tag{1}$$

$$\langle a_i, x\rangle \geq \alpha_i, \text{ i.e. } \langle (a_i, -\alpha_i), (x, 1)\rangle \geq 0, \quad i = 1, \ldots, s \tag{2}$$

The inequality $-1 \geq 0$ can be deduced directly from (1) and (2).

Now we show that $\bigcap_{i=i}^{s} h_i \neq \emptyset$ imply $(0,-1) \in \operatorname{pos}(S)$. Suppose, to the contrary, that $(0,-1) \notin \operatorname{pos}(S)$. The set $\operatorname{pos}(S)$ is closed and convex, so there exists a hyperplane $\{x \in R^d \mid \langle (c, \beta), x\rangle = 0\}$, $c \in R^{d-1}$, $\beta \in R$, which separates the cone $\operatorname{pos}(S)$ and the point $(0,-1)$, i.e.

$$\langle (c, \beta), (0,-1)\rangle < 0 \tag{3}$$

$$\langle (c, \beta), (a_i, -\alpha_i)\rangle \geq 0, \quad i = 1, \ldots, s \tag{4}$$

(3) implies that $\beta > 0$ and we may suppose that $\beta = 1$. Using inequalities (4) we get: $\langle a_i, c\rangle \geq \alpha_i$, $i = 1, \ldots, s$. Thus, $c \in \bigcap_{i=1}^{s} h_i$ and this intersection is nonempty. $\square$

Theorem 3.2. (Multiplied cone version) *Suppose $V_1, \ldots, V_d \subseteq R^d$ and $a \in \operatorname{pos}(V_i)$ for $i = 1, \ldots, d$. Then for each i there exist elements $v_i \in V_i$, such that $a \in \operatorname{pos}\{v_1, \ldots, v_d\}$.*

Proof. Suppose, and we may do so without loss of generality, that $a = (0, 0, \ldots, 0, -1) \in R^d$. To abbreviate notation we will write $a = (0,-1)$. To each vector $(c,-\beta)=(c_1,\dots,c_{d-1},-\beta)\in R^d$, let us associate closed halfspace $H_{c,\beta}=\{x\in R^{d-1}\mid\langle c,x\rangle\geq\beta\}=\{x\in R^{d-1}\mid\langle(c,-\beta),(x,1)\rangle\geq 0\}$. Thus, to given sistems of vectors $V_1,\dots,V_d$ we associate families $\mathcal{S}_1,\dots,\mathcal{S}_d$ of closed halfspaces in R^{d-1}. Let $(0,-1)\in\operatorname{pos}(V_i)$, $i=1,\dots,d$. By the previous Lemma, each of $\mathcal{S}_i$, $i=\overline{1,d}$, has nonempty intersection. By Theorem 2.1 there exist $H_1\in\mathcal{S}_1,\dots,H_d\in\mathcal{S}_d$ such that $H_1\cap H_2\cap\dots\cap H_d=\emptyset$. Each of the spaces $H_1,\dots,H_d$ has form H_{s_i,β_i} for some vector $v_i=(s_i,-\beta_i)\in V_i$, $i=1,\dots,d$. Lemma 3.1 implies that $a=(0,-1)\in\operatorname{pos}(v_1,\dots,v_d)$. □

THEOREM 3.3. (Multiplied Carathéodory's theorem) *If* $V_1,\dots,V_{d+1}\subseteq R^d$ *and* $a\in\cap_{i=1}^{d+1}\operatorname{conv}(V_i)$, *then there exist* $v_i\in V_i$, $i=\overline{1,d+1}$ *such that* $a\in\operatorname{conv}\{v_1,\dots,v_{d+1}\}$.

Proof. By Caratheodory's theorem we can suppose that each V_i is finite. Let us define map $\tilde{}: R^d\to R^{d+1}$ by $\tilde{x}=(x,1)$. Assume that $V_1,\dots,V_{d+1}\subseteq R^d$ and $a\in\operatorname{conv}(V_i)$, $i=1,\dots,d+1$. It is easy to check that $\tilde{a}\in\operatorname{pos}(\tilde{V}_i)$. Theorem 3.2. implies that there exist vectors $\tilde{v}_i\in\tilde{V}_i$ such that $\tilde{a}\in\operatorname{pos}(\tilde{v}_1,\dots\tilde{v}_{d+1})$, i.e. $(0,-1)=\lambda_1(v_1,1)+\dots+\lambda_{d+1}(v_{d+1},1)=(\lambda_1v_1+\dots+\lambda_{d+1}v_{d+1},\lambda_1+\dots+\lambda_{d+1})$, $\lambda_i\geq 0$, $i=1,\dots,d+1$. Therefore, $a=\lambda_1v_1+\dots+\lambda_{d+1}v_{d+1}$ and $\sum_{i=1}^{d+1}\lambda_i=1$, $\lambda_i\geq 0$, $i=\overline{1,d+1}$, i.e. $a\in\operatorname{conv}(v_1,\dots v_{d+1})$. □

4. A conjecture.

There are several conjectures in [**3**] about finite families of vectors in R^d, which are similar in spirit with Theorem 3.3. Here is one conjecture and one of the theorems proved there.

THEOREM. *If simplexes* $\mathcal{S}_1,\mathcal{S}_2,\dots,\mathcal{S}_5$ *in* 2*-dimensional space* R^2 *have a point in common and all vertices in* $\mathcal{S}_i$, $i=1,\dots,5$, *are coloured by one of three different colours, then there exists a simplex* $\operatorname{conv}\{x_1,x_2,x_3\}$ *containing this point, such that* x_1,x_2,x_3 *are vertices of different simplexes* $\mathcal{S}_1,\mathcal{S}_2,\dots,\mathcal{S}_5$ *and* x_i, $i=1,2,3$, *are coloured by different colours.*

Conjecture: Let $\{K_{i,j}\mid 1\leq i\leq 2d+1, 1\leq j\leq d+1\}$ be a family of convex sets in R^d so that $\{K_{i,j}\}_{j=1}^{d+1}$ has an empty intersection for all $j=1,\dots,2d+1$. Then there exist different numbers $i_1,i_2,\dots,i_{d+1}$ so that the family $\{K_{i_k,K}\}_{k+1}^{d+1}$ has an empty intersection.

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