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ON HB-RECURENT HYPERBOLIC KAEHLERIAN SPACES

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Abstract. We consider a hyperbolic Kaehlerian space with recurrent HB-tensor. We obtain some geometrical characterization of such a space. Particularly, we prove that HB-recurrent hyperbolic Kaehlerian space cannot be non-trivially Ricci-recurrent. Also, we give an analogue of Olszak's lemma.

0. Introduction. If an $n\,(=2m)$-dimensional pseudo-Riemannian space M_n with metric (g_{ij}) is provided by a nondegenerate structure tensor (F^i_j) satisfying the conditions

$$F_{j,k}=0,\quad F^i_jF^k_i=\delta^k_j,\quad F_{jk}=F^i_j\;g_{ik}=-F_{kj},$$

M_n is called a hyperbolic Kaehlerian space.

As it was proved in [**4**], a nondegenerate structure has n (the dimension of the space) linearly independent eigenvectors in the tangent space. In [**4**] was also proved

Proposition 1. (A) *Every vector in the tangent space of a hyperbolic Kaehlerian space is transformed by the structure into an orthogonal vector.* (B) *The scalar square of a vector-original is opposite to the scalar square of the vector-image.* □

In accordance with Proposition 1, eigenvectors of the structure are isotropic (null-vectors). As the structure has n linearly independent eigenvectors, there exists a basis of the tangent space of a hyperbolic Kaehlerian space where these isotropic vectors serve as basic vector fields. In such a basis, metric tensor is hybrid and the structure tensor is pure. Covariant structure tensor is also hybrid. Using this coordinate system, we can show that a hyperbolic Kaehlerian space admits isotropic vector fields which are not eigen for the structure. Moreover, such a coordinate system shows that a hyperbolic Kaehlerian space is naturally divided into two totally geodesic subspaces of equal dimension. Such a basis is called a

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separated basis. Also, according to Proposition 1(B) there exist vectors of positive scalar square (space-like vectors) and vectors of negative scalar square (time-like vectors). Space-like vectors may serve as a domain for the involution (F^i_j) and its codomain will be the set of time-like vectors. We may choose such a basis; then the metric tensor will be pure tensor of signature (n, n) and (F^i_j) will be a hybrid tensor. Such a basis is called an *adapted basis*.

For all the considerations in this paper we shall use an arbitrarily chosen basis – it will be neither separated nor adapted. However, all our results may be transferred into these special bases and some of them may look even simpler.

In [**4**] were investigated the properties of hyperbolic Kaehlerian space. Particularly, there was investigated a conformal connection (as there is no conformal transformation which can be introduced naturally) and a tensor which is an invariant for all conformal connections on a hyperbolic Kaehlerian space

$$\begin{aligned}\mathrm{HB}\,^i_{jkl} =& K^i_{jkl} - \frac{1}{n+4}\left[\delta^i_l K_{kj} - \delta^i_k K_{lj} + g_{kj}K^i_l - g_{lj}K^i_k\right.\\ &+ F^i_l S_{kj} - F^i_k S_{lj} + F_{kj}S^i_l - F_{lj}S^i_k + 2S^i_j F_{kl} + 2S_{kl}F^i_j\\ &\left.- \frac{K}{n+2}\left(\delta^i_l g_{kj} - \delta^i_k g_{lj} + F_{lj}F^i_k - F^i_l F_{kj} - 2F^i_j F_{kl}\right)\right].\end{aligned}$$

By K^i_{jkl} we denote the curvature tensor of the Levi-Civita connection for the metric (g_{ij}), by K_{ij} the corresponding Ricci tensor and by K the corresponding curvature scalar. We also have $S_{lj} = K_{la}F^a_j$. In [**4**] was proved that the tensor S_{lj} is skew-symmetric. The HB-tensor has algebraic properties similar to a curvature tensor

$$\begin{aligned}&\mathrm{HB}\,_{ijkl} = -\mathrm{HB}\,_{jikl},\ \ \mathrm{HB}\,_{ijkl} = -\mathrm{HB}\,_{ijlk},\ \ \mathrm{HB}\,_{ijkl} = \mathrm{HB}\,_{klij}\\ &\mathrm{HB}\,_{ijkl} + \mathrm{HB}\,_{iklj} + \mathrm{HB}\,_{iljk} = 0,\ \ \mathrm{HB}\,^t_{jkt} = 0,\ \ \mathrm{HB}\,^i_{tkl}F^t_j - \mathrm{HB}\,^t_{jkl}F^i_t = 0.\end{aligned} \tag{0.1}$$

Here we are going to investigate a hyperbolic Kaehlerian space satisfying

$$\mathrm{HB}\,_{ijkl,s} = \kappa_s \mathrm{HB}\,_{ijkl} \tag{0.2}$$

for some nonzero vector field κ_s.

In order to avoid very long and complicated calculations, we introduce the following abbreviations:

$$\begin{aligned}\Pi_{kj} &= \frac{1}{n+4}\left[K_{kj} - \frac{K}{2(n+2)}g_{kj}\right],\ \ \Pi_{kj} = \Pi_{jk}\\ T_{kj} &= \frac{1}{n+4}\left[S_{kj} + \frac{K}{2(n+2)}F_{kj}\right],\ \ T_{kj} = -T_{jk}\end{aligned}$$

One can easily see that these tensors are related to each other by $T_{kj} = \Pi_{ka}F^a_j$. We can write

$$\mathrm{HB}\,^i_{jkl} = K^i_{jkl} - D^i_{jkl}, \tag{0.3}$$

where

$$D^i_{jkl} = \delta^i_l \Pi_{kj} - \delta^i_k \Pi_{lj} + g_{kj}\Pi^i_l - g_{lj}\Pi^i_k + F^i_l T_{kj} - F^i_k T_{lj} + F_{kj}T^i_l - F_{lj}T^i_k$$
$$+ 2T^i_j F_{kl} + 2T_{kl}F^i_j.$$

All these abbreviations are from [**4**].

We shall use the results of Prasad [**3**] about a Kaehlerian space with recurrent Bochner tensor, the results of Adati and Miyazawa [**1**] about (pseudo) Riemannian spaces with recurrent Weyl conformal curvature tensors, commenting the Roter's construction of the metric tensor for the essential case [**6**] and the results of Pušić [**5**] about hyperbolic Kaehlerian spaces which are HB-symmetric. Also, we shall prove a lemma analogous to Olszak's lemma [**2**].

1. An Einstein HB-recurrent hyperbolic Kaehlerian space. Supposse that the hyperbolic Kaehlerian space is an Einstein space, i.e. that

$$K_{ij} = \frac{K}{n} g_{ij}, \tag{1.1}$$

wherefrom

$$S_{ij} = -\frac{K}{n} F_{ij} \tag{1.2}$$

On an Einstein space the curvature scalar is a global constant so these two tensors are parallel.

Supposse, now, that the space is HB-recurrent that is, that (0.7) holds. Using (1.1) and (1.2) we can express the covariant derivative of the curvature tensor:

$$K_{ijkl,s} = \kappa_s \left[K_{ijkl} - \frac{K}{n(n+2)} (g_{li}g_{kj} - g_{ki}g_{lj} - F_{li}F_{kj} + F_{ki}F_{lj} - 2F_{ji}F_{kl}) \right] \tag{1.3}$$

As the space is an Einstein space, the well-known formula $K^s_{jkl,s} = K_{jk,l} - K_{jl,k}$ gives $K^s_{jkl,s} = 0$. Applying effectively the second Bianchi identity to the curvature tensor

$$K_{ijkl,s} + K_{ijls,k} + K_{ijsk,l} = 0$$

and taking into account (1.3), we obtain, after transvection by κ^s

$$\kappa_s \kappa^s \left[K_{ijkl} - \frac{K}{n(n+2)} (g_{li}g_{kj} - g_{ki}g_{lj} - F_{li}F_{kj} + F_{ki}F_{lj} - 2F_{kl}F_{ji}) \right] = 0. \tag{1.4}$$

So, we proved

THEOREM 1. *If a* HB *-recurrent hyperbolic Kaehlerian space is an Einstein space, then the formula* (1.4) *is valid and one of the following cases occurs:*

(a) *the recurrence vector* κ_s *vanishes and the space is* HB *-parallel,*

(b) $\kappa_s \kappa^s = 0$ *the recurrence vector is isotropic,*

(c) $H_{ijkl} = K_{ijkl} - \frac{K}{n(n+2)} (g_{li}g_{kj} - g_{ki}g_{lj} - F_{li}F_{kj} + F_{ki}F_{lj} - 2F_{kl}F_{ji}) = 0,$

which means that the space is a space of almost constant curvature. □

For further investigation we shall use two Walker' s lemmas [**7**]:

Lemma 1. *The curvature tensor satisfies the identity:*

$$K_{ijkl,m,s} - K_{ijkl,s,m} + K_{klms,i,j} - K_{klms,j,i} + K_{msij,k,l} - K_{msij,l,k} = 0. \square \tag{1.5}$$

Lemma 2. *If $a_{\alpha\beta}$ and b_γ are two number sets with following properties:*

$$a_{\alpha\beta} = a_{\beta\alpha}, \ \ a_{\alpha\beta}b_\gamma + a_{\beta\gamma}b_\alpha + a_{\gamma\alpha}b_\beta = 0 \ \text{ for } \ \alpha, \beta, \gamma = 1, \dots , N \tag{1.6}$$

then either all $a_{\alpha\beta}$ vanish or all b_γ vanish. $\square$

According to (1.3), the following is valid:

$$K_{ijkl,s,m} = (\kappa_{s,m}+\kappa_s\kappa_m)H_{ijkl} \ \text{ or } \ K_{ijkl,s,m}-K_{ijkl,m,s} = (\kappa_{s,m}-\kappa_{m,s})H_{ijkl}. \tag{1.7}$$

One can easily see that the tensor H_{ijkl} has the property $H_{ijkl} = H_{klij}$. If we apply the formula (1.5), then we obtain

$$(\kappa_{m,s} - \kappa_{s,m})H_{ijkl} + (\kappa_{i,j} - \kappa_{j,i})H_{klms} + (\kappa_{k,l} - \kappa_{l,k})H_{msij} = 0.$$

If we arrange pairs of indices in this way: $(ms) \to \alpha$, $(ij) \to \beta$, $(kl) \to \gamma$; then the formula above gets the form (1.6) and, by Lemma 2 we obtain $\kappa_{m,s} = \kappa_{s,m}$ or $H_{ijkl} = 0$.

We have proved

Theorem 2. *If a* HB *-recurrent hyperbolic Kaehlerian space is an Einstein space, then either the recurrence vector is a gradient or the space is a space of almost constant curvature.* $\square$

2. HB-recurrent Ricci recurrent hyperbolic Kaehlerian space. Suppose now, that the hyperbolic Kaehlerian space, besides (0.7), satisfies the condition $K_{ij,k} = \kappa^*_k K_{ij}$. First, we can prove the following

Theorem 3. *At a* HB *-recurrent Ricci-recurrent hyperbolic Kaehlerian space the recurrence vector of the Ricci tensor is a gradient if and only if the recurrence vector of* HB *-tensor is a gradient.* $\square$

Proof. Using (0.3) we get $\mathrm{HB}_{ijkl} = K_{ijkl} - D_{ijkl}$ or

$$K_{ijkl,m,s} = (\kappa_{m,s} + \kappa_m\kappa_s)\mathrm{HB}_{ijkl} + (\kappa^*_{m,s} + \kappa^*_m\kappa^*_s)D_{ijkl}.$$

By Lemma 1, we obtain

$$\begin{aligned}(\kappa_{m,s} - \kappa_{m,s})\mathrm{HB}_{ijkl} + (\kappa_{i,j} - \kappa_{j,i}\mathrm{HB}_{klms} + (\kappa_{k,l} - \kappa_{l,k})\mathrm{HB}_{msij}\\ + (\kappa^*_{m,s} - \kappa^*_{s,m})D_{ijkl} + (\kappa^*_{i,j} - \kappa^*_{j,i})D_{klms} + (\kappa^*_{k,l} - \kappa^*_{l,k})D_{msij} = 0.\end{aligned}$$

If one of these two vectors, for example, κ_m, is a gradient, then the whole first row in the formula above vanishes. We can apply Lemma 2 to the second row. The tensor D_{ijkl} is also invariant with recpect to interchanging places of the first and the second pair of indices. Then either $D_{ijkl}=0$ and the space is recurrent (trivial case) or the vector κ^*_m is a gradient. ■

Further we shall prove

THEREM 4. *If a* HB*-recurrent hyperbolic Kaehlerian space, which is not flat is Ricci-flat, then the space is recurrent and the recurrence vector is zero or null-vector.* □

Proof. If the space is Ricci-flat, then the curvature scalar vanishes, HB-tensor is equal to curvature tensor and $K^i_{jkl,s}=\kappa_s K^i_{ijk}$. As the space is Ricci-flat, we have $K^a_{jkl,a}=K_{jk,l}-K_{jl,k}=0$. Further, as the curvature tensor is recurrent, the Bianchi identity will look this way

$$\kappa_s K^i_{jkl}+\kappa_k K^i_{jls}+\kappa_l K^i_{jsk}=0.$$

After transvection by κ^s, we obtain $\kappa^s\kappa_s K^i_{jkl}=0$ and the recurrence vector is either a zero vector or it is isotropic. ■

Now we shall prove that both κ_m and κ^*_m are gradient vector fields. We need

LEMMA 3. (Olszak [**2**]) *Let $\sigma_1,\dots,\sigma_N$ and $\omega_1,\dots,\omega_N$ are two sets of numbers which are linearly independent as elements of $\mathbf{R}^N$. Let T_{AB} and S_{AB} are numbers such that $T_{AB}=T_{BA}$ and $S_{AB}=S_{BA}$ and let the condition*

$$T_{AB}\sigma_C+T_{BC}\sigma_A+T_{CA}\sigma_B+S_{AB}\omega_C S_{BC}\omega_A+S_{CA}\omega_B=0$$

be satisfied. Then there exists a set of numbers $\vartheta_1,\dots,\vartheta_N$ such that

$$T_{AB}=\omega_A\vartheta_B-\omega_B\vartheta_A;\ \ S_{AB}=\sigma_A\vartheta_B+\sigma_B\vartheta_A.\ \square$$

Lemma 3 is a generalization of Lemma 2. Now we can prove the following theorem:

THEOREM 5. *Let HK_n be a hyperbolic Kaehlerian space of dimension greater or equal to* 4 *and let U_1* (*respectively U_2*) *be a subset of HK_n consisting of the points where Ricci tensor* (*respectively* HB*-tensor*) *does not vanish. Suppose that*

$$K_{ij,k,l}-K_{ij,l,k}=a_{kl}K_{ij}\ \ \text{on } U_1$$

for a skew-symmetric tensor a, and

$$\mathrm{HB}_{ijkl,m,s}-\mathrm{HB}_{ijkl,s,m}=\mathrm{HB}_{ijkl}b_{ms}\ \ \text{on } U_2$$

for another skew-symmetric tensor b. Then $a_{ij}=0$ everywhere on U_1 and $b_{ij}=0$ everywhere on U_2. □

Proof. We shall first consider the set $U_1\backslash U_2$. On this set, $\mathrm{HB}_{ijkl}=0$ and $\mathrm{HB}_{ijkl,m,s}-\mathrm{HB}_{ijkl,s,m}=0$. Then $K_{ijkl,m,s}-K_{ijkl,s,m}=K_{ijkl}a_{ms}$. Usig Lemma 1 we obtain

$$K_{ijkl}a_{ms}+K_{klms}a_{ij}+K_{msij}a_{kl}=0. \tag{2.1}$$

If we arrange pairs of indices in this way: $(ij) \to \alpha$, $(kl) \to \beta$, $(ms) \to \gamma$ and use Lemma 2, then from (2.1) we obtain that either the space is flat (what is impossible as the point belongs to U_1) or $a_{ms} = 0$.

In the next step, we consider the set $U_2 \backslash U_1$. On this set $K_{ij} = 0$ and, consequently, $K_{ij,k,l} - K_{ij,l,k} = 0$. Then $K_{ijkl} = \mathrm{HB}_{ijkl}$ and $K_{ijkl,m,s} - K_{ijkl,s,m} = \mathrm{HB}_{ijkl} b_{ms}$. Using Lemma 1 and Lemma 2 again, we obtain that either $\mathrm{HB}_{ijkl} = 0$ (what is impossible as the point belongs to U_2), or $b_{ms} = 0$. Now we shall consider the set $U_1 \cap U_2$, if it is not empty. We have

$$K_{ijkl,m,s} - K_{ijkl,s,m} = (K_{ijkl} - \mathrm{HB}_{ijkl}) a_{ms} + \mathrm{HB}_{ijkl} b_{ms}$$

and, according to Lemma 1

$$(2.2)\qquad \begin{aligned}(K_{ijkl} - \mathrm{HB}_{ijkl}) a_{ms} + (K_{klms} - \mathrm{HB}_{klms}) a_{ij} + (K_{msij} - \mathrm{HB}_{msij}) a_{kl} \\ + \mathrm{HB}_{ijkl} b_{ms} + \mathrm{HB}_{klms} b_{ij} + \mathrm{HB}_{msij} b_{kl} = 0.\end{aligned}$$

We shall prove that tensors a_{ij} and b_{ij} are linearly dependent (proportional). Supposse that they are linearly independent. Then, using Lemma 3, we obtain

$$(2.3)\qquad \mathrm{HB}_{ijkl} - K_{ijkl} = b_{ij} c_{kl} + b_{kl} c_{ij}$$

where c_{ij} is another skew-symmetric tensor field. Then we get

$$(2.4)\qquad \begin{aligned}&g_{li} K_{kj} - g_{ki} K_{lj} + g_{kj} K_{li} - g_{lj} K_{ki} + F_{li} S_{kj} - F_{ki} S_{lj} + F_{kj} S_{li} - F_{lj} S_{ki} \\ &+ 2S_{ji} F_{kl} + 2S_{kl} F_{ji} - \frac{K}{n+2}(g_{li} g_{kj} - g_{ki} g_{lj} + F_{lj} F_{ki} - F_{li} F_{kj} - 2F_{ji} F_{kl}) \\ &= (n+4)(b_{ij} c_{kl} + b_{kl} c_{ij}).\end{aligned}$$

Supposse, now, that $K_{sr} X^s X^r = 0$ for any isotropic vector field X. As there exists a basis consisting of isotropis vector fields only, then $K_{ij} = K/n\, g_{ij}$ and (2.4) gives

$$\frac{K}{n(n+2)}(g_{li} g_{kj} - g_{ki} g_{lj} + F_{lj} F_{ki} - F_{li} F_{kj} - 2F_{ji} F_{kl}) = b_{ij} c_{kl} + b_{kl} c_{ij}.$$

Transvecting this by $g^{li} g^{kj}$, we obtain $K = 2b_{ij} c^{ij}$. On the other hand, transvecting (2.3) by $g^{li} g^{kj}$, we obtain $K = -2b_{ij} c^{ij}$ and this means that $K = 0$. As the space is an Einstein space, then it is Ricci-flat, what is impossible. Then we know that there exists at least one isotropic vector field X such that $K_{sr} X^s X^r \neq 0$. Now we transvect (2.4) by $X^j X^k$ and obtain

$$(2.5)\qquad \begin{aligned}g_{li} K_{kj} X^k X^j =& X_i \left(K_{kl} X^k - \frac{K}{2(n+2)} X_l\right) + X_l \left(K_{ki} X^k - \frac{K}{2(n+2)} X_i\right) \\ &+3\tilde{X}_i \left(K_{la} \tilde{X}^a - \frac{K}{2(n+2)} \tilde{X}_l\right) + 3\tilde{X}_l \left(K_{ia} \tilde{X}^a - \frac{K}{2(n+2)} \tilde{X}_i\right) \\ &+2(n+4) b_{lj} c_{ki} X^j X^k\end{aligned}$$

One can easily prove that $b_{ij}c_{kl}X^jX^k + b_{kl}c_{ij}X^jX^k = 2b_{lj}c_{ki}X^jX^k$. Components of the vector $\tilde{X}$, which is the image of the vector X by the structure, are denoted by $\tilde{X}^i$. The vector $\tilde{X}$ is also isotropic. Now we transvect (2.5) by g^{li} and obtain

$$K_{kj}X^kX^j = 2b_{lj}c^l_kX^jX^k.$$

On the other hand, if we transvect (2.3) by $g^{li}X^kX^j$, then we get

$$-K_{jk}X^jX^k = 2b_{lj}c^l_kX^jX^k$$

and then $K_{jk}X^jX^k = 0$, a contradiction. So, tensors a and b are mutually proportional.

As $a_{ij} = \lambda b_{ij}$, (2.2) gets the form

$$\begin{aligned}[\lambda K_{ijkl} + (1-\lambda)\mathrm{HB}_{\,ijkl}]b_{ms} + [\lambda K_{klms} + (1-\lambda)\mathrm{HB}_{\,klms}]b_{ij}& \\ + [\lambda K_{msij} + (1-\lambda)\mathrm{HB}_{\,msij}] = 0.& \end{aligned} \tag{2.6}$$

We can see that (2.6) satisfies the conditions of Lemma 2. If $\lambda K_{ijkl} + (1-\lambda)$ $\mathrm{HB}_{\,ijkl} = 0$, then $K_{ijkl} = \sigma\mathrm{HB}_{\,ijkl}$ and the space is Ricci-flat. But the considered point belonges to U_2. Then $b_{ms} = 0$ and $a_{ms} = 0$ as they are proportional. ■

The following theorem is a corollary of Theorem 5:

Theorem 6. *If a hyperbolic Kaehlerian space of dimension greater then* 4 *is nontrivially* HB*-recurrent and Ricci-recurrent, then both recurrence vectors are gradients.* □

Proof. As the space is HB-recurrent and Ricci-recurrent, then

$$K_{ij,k,l} - K_{ij,l,k} = (\kappa^*_{k,l} - \kappa^*_{l,k})K_{ij}, \mathrm{HB}_{\,ijkl,m,s} - \mathrm{HB}_{\,ijkl,s,m} = (\kappa_{m,s} - \kappa_{s,m})\mathrm{HB}_{\,ijkl}.$$

According to Theorem 5, we have $\kappa^*_{k,l} = \kappa^*_{l,k}$ and $\kappa_{m,s} = \kappa_{s,m}$ what proves the statement. ■

Now we are able to give a classification of HB-recurrent Ricci-recurrent hyperbolic Kaehlerian spaces. Using (0.3) and the Ricci identity, we obtain

$$\begin{aligned}&\mathrm{HB}^{\,h}_{\,tlm}\mathrm{HB}^{\,t}_{\,ijk} - \mathrm{HB}^{\,t}_{\,ilm}\mathrm{HB}^{\,h}_{\,tjk} - \mathrm{HB}^{\,t}_{\,jlm}\mathrm{HB}^{\,h}_{\,itk} - \mathrm{HB}^{\,t}_{\,klm}\mathrm{HB}^{\,h}_{\,itj}\\ &\qquad + D^h_{tlm}\mathrm{HB}^{\,t}_{\,ijk} - D^t_{ilm}\mathrm{HB}^{\,h}_{\,tjk} - D^t_{jlm}\mathrm{HB}^{\,h}_{\,itk} - D^t_{klm}\mathrm{HB}^{\,h}_{\,itj} = 0.\end{aligned}$$

Differentiating this covariantly, we obtain

$$\begin{aligned}&2\kappa_s(\mathrm{HB}_{\,tlm}\mathrm{HB}_{\,ijk} - \mathrm{HB}_{\,ilm}\mathrm{HB}_{\,tjk} - \mathrm{HB}_{\,jlm}\mathrm{HB}_{\,itk} - \mathrm{HB}_{\,klm}\mathrm{HB}_{\,itj})\\ &\quad + (\kappa_s + \kappa^*_s)(D^h_{tlm}\mathrm{HB}^{\,t}_{\,ijk} - D^t_{ilm}\mathrm{HB}^{\,h}_{\,tjk} - D^t_{jlm}\mathrm{HB}^{\,h}_{\,itk} - D^t_{klm}\mathrm{HB}^{\,h}_{\,ijt}) = 0,\end{aligned}$$

wherefrom

$$(\kappa^*_s - \kappa_s)(D^h_{tlm}\mathrm{HB}^{\,t}_{\,ijk} - D^t_{ilm}\mathrm{HB}^{\,h}_{\,tjk} - D^t_{jlm}\mathrm{HB}^{\,h}_{\,itk} - D^t_{klm}\mathrm{HB}^{\,h}_{\,ijt}) = 0. \tag{2.7}$$

The last relation gives $\kappa^*_s = \kappa_s$ and then the space is recurrent, or

$$D^h_{tlm}\mathrm{HB}^{\,t}_{\,ijk} - D^t_{ilm}\mathrm{HB}^{\,h}_{\,tjk} - D^t_{jlm}\mathrm{HB}^{\,h}_{\,itk} - D^t_{klm}\mathrm{HB}^{\,h}_{\,ijt} = 0.$$

Tensors Π_{ij}, T_{ij} and F_{ij} are related to HB-tensor by relations proved in [**4**]. From above we obtain

$$(n+4)\Pi_{lt}\mathrm{HB}\,{}^{t}_{ijk} + \Pi\mathrm{HB}\,{}_{lijk} = 0. \tag{2.8}$$

As the tensor Π_{ij} is also recurrent, we have

$$\kappa^{*a}\Pi_{la} = \frac{n+1}{n+4}\kappa^{*}_{l}\Pi \qquad \left(\Pi = \Pi_{ij}g^{ij} = \frac{K}{2(n+2)}\right).$$

Transvecting (2.8) by κ^{*l}, we obtain either

$$(a) \quad \Pi = 0 (\Leftrightarrow K = 0), \qquad \text{or} \qquad (b) \quad \kappa^{*}_{t}\mathrm{HB}\,{}^{t}_{ijk} = 0. \tag{2.9}$$

Supposse that (2.9) (b) holds. Then we have $D^{h}_{tlm}\kappa^{*}_{h}\kappa^{*m} = 0$. After transvection by $\mathrm{HB}\,{}^{t}_{ijk}$ and using formula (2.8), we get

$$(n+2)\kappa^{*}_{a}\kappa^{*a}\Pi\mathrm{HB}\,{}_{lijk} = 0.$$

If (2.9) (a) is satisfied, then (2.8) becomes $\Pi_{lt}\mathrm{HB}\,{}^{t}_{ijk} = 0$. Transvecting (2.7) by Π_{hs}, we obtain $D^{h}_{tlm}\mathrm{HB}\,{}^{t}_{ijk}\Pi_{hs} = 0$. This gives

$$\mathrm{HB}\,{}_{lijk}(\Pi_{bs}\Pi^{b}_{m}) - \mathrm{HB}\,{}_{mijk}(\Pi_{bs}\Pi^{b}_{l}) = 0.$$

Then, transvecting by g^{sm} we obtain that either the space is HB-flat or $\Pi_{ba}\Pi^{ba} = 0$.

We have proved

THEOREM 7. *If a hyperbolic Kaehlerian space is* HB*-recurrent and Ricci-recurrent, then one of the following cases occurs:*

(1) $\Pi = 0 \Longleftrightarrow K = 0$ and $K_{ab}K^{ab} = 0$;

(2) $\kappa_a = 0$ (Ricci-parallel);

(3) HB $-$ flat;

(4) $\kappa^{*}_{a}\kappa^{*a} = 0$. □

3. Effective expression for covariant derivative of the Ricci tensor. Using relations which connect the curvature tensor of the hyperbolic Kaehlerian space with the structure, we obtain: $S_{ij} = K_{bija}F^{ab}$ and, using this,

$$S_{ij} = -1/2F^{sm}K_{msij}. \tag{3.2}$$

Using the Ricci identity and the relation (0.1), we obtain

$$K_{kb,a} = F^{l}_{b}F^{j}_{a}(K_{kj,l} - K_{lj,k}). \tag{3.1}$$

Differentiating covariantly $\mathrm{HB}\,{}^{i}_{jkl}$ by κ^{m} and contracting indices i and m, we obtain

$$\begin{aligned} K_{jk,l} - K_{jl,k} = {} & \frac{1}{n+4}\Big[4(K_{kj,l} - K_{lj,k}) \\ & + \frac{n}{2(n+2)}(K_{,l}g_{kj} - K_{,k}g_{lj} + K_{,a}F^{a}_{k}F_{lj} - K_{,a}F^{a}_{l}F_{kj} - 2K_{,a}F^{a}_{j}F_{kl})\Big] + \kappa^{i}\mathrm{HB}\,{}^{i}_{jkl} \end{aligned}$$

and finally

$$K_{kj,l} - K_{lj,k} = \frac{1}{2(n+2)}(K_{,l}g_{kj} - K_{,l}g_{lj} + K_{,a}F_k^aF_{lj} - K_{,a}F_l^aF_{kj} - 2K_{,a}F_j^aF_{kl}) + \frac{n+4}{n}\kappa_i \mathrm{HB}\,^i_{jkl}$$

Then, by (3.1)

$$\tag{3.2} \begin{aligned} K_{kj,l} = &\frac{1}{2(n+2)}(K_{,k}g_{jl} + K_{,j}g_{kl} + K_{,a}F_k^aF_{lj} + K_{,a}F_j^aF_{lk} + 2K_{,l}g_{kj}) \\ &+ \frac{n+4}{n}\kappa^i \mathrm{HB}\,_{iakb}F_j^aF_l^b. \end{aligned}$$

4. Existence of essentially HB-recurrent Ricci-recurrent hyperbolic Kaehlerian space. We want to find out does there exist a nontrivially HB-recurrent Ricci-recurrent hyperbolic Kaehlerian space (essential case). For this purpose, suppose $\Pi \neq 0$ and according to 2, we have (2.9) (b). Using this, we obtain:

$$\tag{4.1} \begin{aligned} \kappa^{*i}K_{ijkl} = &\frac{1}{n+4}[\kappa_l^*K_{kj} - \kappa_k^*K_{lj} + g_{kj}\kappa^{*i}K_{li} - g_{lj}\kappa^{*i}K_{ki} + \hat{\kappa}_l^*S_{kj} - \hat{\kappa}_k^*S_{lj} \\ &+ F_{kj}S_{li}\kappa^{*i} - F_{lj}S_{ki}\kappa^{*i} + 2S_{ji}\kappa^{*i}F_{kl} + 2\hat{\kappa}_j^*S_{kl} \\ &- \frac{K}{n+2}(\kappa_l^*g_{kj} - \kappa_k^*g_{lj} + F_{lj}\hat{\kappa}_k^* - F_{kj}\hat{\kappa}_l^* - 2\hat{\kappa}_j^*F_{kl})] \end{aligned}$$

where $\hat{\kappa}_k^* = \kappa_a F_k^a$. Using the formula

$$\tag{4.2} \kappa^{*i}K_{il} = K\kappa_l^*/2$$

we obtain $S_{ji}\kappa^{*i} = -\frac{1}{2}K\kappa_j^*$. We can express (4.1) in the following form:

$$\tag{4.3} \kappa^{*i}K_{ijkl} = \frac{1}{n+4}(\kappa_l^*M_{kj} - \kappa_k^*M_{lj} + \hat{\kappa}_l^*N_{kj} - \hat{\kappa}_k^*N_{lj} + 2\hat{\kappa}_j^*N_{kl})$$

where $M_{kj} = K_{kj} + \frac{n}{2(n+2)}Kg_{kj}$ and $N_{kj} = S_{kj} - \frac{n}{2(n+2)}KF_{kj}$. M_{kj} and N_{kj} are related by $N_{kj} = M_{ka}F_j^a$. The tensor M_{kj} is symmetric and the tensor N_{kj} is skew-symmetric. Let us transvect (4.3) by g^{jk}. Then

$$\kappa^{*i}K_{il} = \frac{1}{n+4}[\kappa_l^*M - \kappa_k^*M_l^k]$$

where

$$M = M_{jk}g^{jk} = K\frac{n^2+2n+4}{2(n+2)} \quad \text{and} \quad \kappa_k^*M_l^k = \frac{n+1}{n+2}K\kappa_l^*.$$

In such a way, we obtain

$$\tag{4.4} \kappa^{*i}K_{il} = \frac{n^2+2}{2(n+2)(n+4)}K\kappa_l^*.$$

In comparison of (4.4) and (4.2) there yields either $K = 0$ or $\kappa^*_j = 0$ The last relation, in fact, means that the space is not Ricci-recurrent, but Ricci-parallel. This is a simpler case.

By this investigation we in fact loose the case (4) from Theorem 7 as one of possible cases which even may be an essential case. There remains only case (1) as a possible origin of essential cases. Such a situation was geometrically expected. Roter [**6**] constructed an essentially conformally recurrent Ricci-recurrent (pseudo) Riemannian space (i.e. he constructed its metric tensor) whose curvature scalar vanished and the recurrence vector was not isotropic.

Now we are going to look for nontrivial (essential) HB-recurrent Ricci-recurrent hyperbolic Kaehlerian space with vanishing curvature scalar. First, from (3.2) we have

$$K_{kj,l} = \frac{n+4}{n}\kappa^i \mathrm{HB}_{iakb} F^a_j F^b_l. \tag{4.5}$$

Further, if $K = 0$, then $K_{ab}K^{ab} = 0$ (case (1)) and $\kappa^i \mathrm{HB}_{iakb} F^a_j F^b_l = 0$. As the Ricci tensor is symmetric, then

$$\kappa^i \mathrm{HB}_{iakb} F^a_j F^b_l = \kappa^i \mathrm{HB}_{iajb} F^a_k F^b_l$$

After transvection by $F^l_m F^k_t$ we obtain

$$\kappa^i \mathrm{HB}_{iakm} F^a_j F^k_t = \kappa^i \mathrm{HB}_{itjm} \quad \text{and} \quad \kappa^i \mathrm{HB}_{iakm} F^a_j F^t_k = -\kappa^i \mathrm{HB}_{itjm}.$$

By (4.5), the last relation means that $K_{mj,t} = -\kappa^i \mathrm{HB}_{itjm}$. As the Ricci tensor is symmetric and HB-tensor is skew-symmetric with respect to indices j, m we get either $\kappa^*_t = 0$ (the space is Ricci-parallel and recurrent) or $K_{mj} = 0$ (the space is Ricci-flat and recurrent). Now we have

Corollary 1. *There does not exist nontrivially* HB-*recurrent hyperbolic Kaehlerian space which is nontrivially Ricci-recurrent.* □

Instead of Theorem 7, we can state

Theorem 8. *If a hyperbolic Kaehlerian space is* HB-*recurrent and Ricci-recurrent, then one of the following cases occurs:*

(1) *the recurrence vectors of* HB-*tensor and Ricci tensor are equal and the space is recurrent*
(2) *the space is* HB-*symmetric and Ricci-recurrent*
(3) *the space is* HB-*flat and Ricci-recurrent*
(4) *the space is Ricci-flat*
(5) *the curvature scalar of the space vanishes and there one of the following cases occurs:*
   (a) *the space is Ricci-parallel*
   (b) *the space is Ricci-flat*
   (c) *the space is recurrent and it is not a flat extension of HK_2.* □

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