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SOME HERMITE METRICS IN COMPLEX FINSLER SPACES

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Abstract. In many papers and books (as [**1**], [**3**]–[**8**] and others) the complex and almost complex structures defined on real spaces are examined. In this paper they are defined on complex Finsler spaces. The complex Finsler space E' is formed in such a way, that its tangent space $T(E')$ is equal to $T(F_1) \oplus iT(F_2)$, where F_1 and F_2 are two $2n$-dimensional Finsler spaces. Using the nonlinear connections N and $\bar{N}$ of F_1 and F_2 respectively, the adapted basis B' of $T(E')$ is formed. In $T(E')$ different almost complex structures are given and the form of the corresponding Hermite metrics is determined.

1. Complex Finsler spaces. Let E be a $4n$-dimensional, real C^∞ differentiable manifold E. In a local chart, $u \in E$ has coordinates $(x^a, y^a, \dot{x}^a, \dot{y}^a)$, $a, b, c, d, e, f, g = 1, 2, \dots, n$.

The allowable coordinate transformations in E have the form

$$\begin{aligned} x^{a'} &= x^{a'}(x), & \dot{x}^{a'} &= A_a^{a'}(x)\dot{x}^a, & A_a^{a'} &= \partial x^{a'}/\partial x^a, \\ y^{a'} &= y^{a'}(y), & \dot{y}^{a'} &= B_a^{a'}(y)\dot{y}^a, & B_a^{a'} &= \partial y^{a'}/\partial y^a, \end{aligned} \tag{1.1}$$

where $\operatorname{rank}[A_a^{a'}] = n$, $\operatorname{rank}[B_a^{a'}] = n$.

We shall use the notations $A_{b\,c}^{a'} = \partial^2 x^{a'}/\partial x^b \partial x^c$, $B_{b\,c}^{a'} = \partial^2 y^{a'}/\partial y^b \partial y^c$.

Let us consider such a complex space E', which is in 1–1 correspondence with E. If the notation

$$\begin{aligned} z^a &= x^a + iy^a, & \dot{z}^a &= \dot{x}^a + i\dot{y}^a, \\ \bar{z}^a &= x^a - iy^a, & \dot{\bar{z}} &= \dot{x}^a - i\dot{y}^a \end{aligned} \tag{2.2}$$

is used, then to each $u = (x^a, y^a, \dot{x}^a, \dot{y}^a) \in E$ corresponds one and only one $u' = (z^a, \bar{z}^a, \dot{z}^a, \dot{\bar{z}}^a) \in E'$. The allowable coordinate transformation in E' are determined by (1.1) and (1.2). Such a complex space E' of real dimension $4n$ will be called complex Finsler space.

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The functions $z^{a'} = x^{a'} + iy^{a'} = z^{a'}(x,y)$, $\dot{z}^{a'} = \dot{x}^{a'} + i\dot{y}^{a'} = \dot{z}^{a'}(x,y,\dot{x},\dot{y})$ in the general case of (1.1) are not complex analytic, i.e. the equations

$$\frac{\partial z^{a'}}{\partial \bar{z}^k} = 0, \quad \frac{\partial \dot{z}^{a'}}{\partial \bar{z}^k} = 0, \quad \frac{\partial z^{a'}}{\partial \dot{\bar{z}}^k} = 0, \quad \frac{\partial \dot{z}^{a'}}{\partial \dot{\bar{z}}^k} = 0 \tag{1.3}$$

are not satisfied. They would have this property only when $A_a^{a'}(x) = B_a^{a'}(y) = A_a^{a'} =$ const., but this case of linear transformations for the Finsler geometry is not interesting.

$T(E)$ can be considered as a direct summ of $T(F_1)$ and $T(F_2)$, where $F_1 = F_1(x,\dot{x})$, $F_2 = F_2(y,\dot{y})$ are two Finsler spaces, with the allowable coordinate transformations (1.1).

The basis vectors of $T(E)$ and $T(E')$ are connected by the relations

$$\begin{aligned}
\frac{\partial}{\partial z^a} &= \frac{1}{2}\left(\frac{\partial}{\partial x^a} - i\frac{\partial}{\partial y^a}\right), & \frac{\partial}{\partial x^a} &= \frac{\partial}{\partial z^a} + \frac{\partial}{\partial \bar{z}^a},\\
\frac{\partial}{\partial \bar{z}^a} &= \frac{1}{2}\left(\frac{\partial}{\partial x^a} + i\frac{\partial}{\partial y^a}\right), & \frac{\partial}{\partial y^a} &= \frac{1}{i}\left(\frac{\partial}{\partial \bar{z}^a} - \frac{\partial}{\partial z^a}\right).
\end{aligned}$$

The above relations are valid, when $(z,\bar{z},x,y)$ are substituted by $(\dot{z},\dot{\bar{z}},\dot{x},\dot{y})$.

The 1-forms from $T^*(E)$ and $T^*(E')$ are connected by the formulae:

$$\begin{aligned}
dz^a &= dx^a + i dy^a, & dx^a &= (dz^a + d\bar{z}^a)/2,\\
d\bar{z}^a &= dx^a - i dy^a, & dy^a &= (dz^a - d\bar{z}^a)/2i.
\end{aligned} \tag{1.4}$$

The above relations are valid, when $(z,\bar{z},x,y)$ are substituted by $(\dot{z},\dot{\bar{z}},\dot{x},\dot{y})$.

To obtain the adapted basis in $T(E)$ we introduce two kinds of nonlinear connections: $N_b^a(x,\dot{x})$ and $\bar{N}_b^a(y,\dot{y})$, as arbitrary functions, which, with respect to (1.1), satisfy the relations

$$\begin{aligned}
N_b^a(x,\dot{x}) &= N_{b'}^{a'}(x',\dot{x}')A_b^{b'}A_{a'}^a - A_{b'}{}^a{}_{c'}\dot{x}^{c'}A_b^{b'},\\
\bar{N}_b^a(y,\dot{y}) &= \bar{N}_{b'}^{a'}(y',\dot{y}')B_b^{b'}B_{a'}^a - B_{b'}{}^a{}_{c'}\dot{y}^{c'}B_b^{b'}.
\end{aligned} \tag{1.5}$$

Using (1.5), the adapted basis $B = \left\{\frac{\delta}{\delta x^a}, \frac{\delta}{\delta y^a}, \frac{\partial}{\partial \dot{x}^a}, \frac{\partial}{\partial \dot{y}^a}\right\}$ of $T(E)$ is formed, where

$$\frac{\delta}{\delta x^a} = \frac{\partial}{\partial x^a} - N_a^b(x,\dot{x})\frac{\partial}{\partial \dot{x}^b}, \quad \frac{\delta}{\delta y^a} = \frac{\partial}{\partial y^a} - \bar{N}_a^b(y,\dot{y})\frac{\partial}{\partial \dot{y}^b}.$$

The adapted basis of $T^*(E)$ is $B^* = \{dx^a, dy^a, \delta\dot{x}^a, \delta\dot{y}^a\}$, where dx^a, dy^a, $d\dot{x}^a$, $d\dot{y}^a$ are determined by (1.4) and

$$\delta\dot{x}^a = d\dot{x}^a + N_b^a(x,\dot{x})dx^b, \quad \delta\dot{y}^a = d\dot{y}^a + \bar{N}_b^a(y,\dot{y})dy^b.$$

The basis B of $T(E)$ induces in the similar way the adapted basis B' of $T(E')$, where

$$B' = \left\{ \frac{\delta}{\delta z^a}, \frac{\delta}{\delta \bar{z}^a}, \frac{\delta}{\delta \dot{z}^a}, \frac{\delta}{\delta \dot{\bar{z}}^a} \right\},$$

$$\frac{\delta}{\delta z^a} = \left(\frac{\partial}{\partial \bar{z}^a} + \frac{\partial}{\partial z^a} \right) - N_a^b(x, \dot{x}) \left(\frac{\partial}{\partial \dot{\bar{z}}^b} + \frac{\partial}{\partial \dot{z}^b} \right), \quad \frac{\delta}{\delta \dot{z}^a} = \frac{\partial}{\partial \dot{\bar{z}}^a} + \frac{\partial}{\partial \dot{z}^a},$$

$$\frac{\delta}{\delta \bar{z}^a} = \left(\frac{\partial}{\partial \bar{z}^a} - \frac{\partial}{\partial z^a} \right) - \bar{N}_a^b(y, \dot{y}) \left(\frac{\partial}{\partial \dot{\bar{z}}^b} - \frac{\partial}{\partial \dot{z}^b} \right), \quad \frac{\delta}{\delta \dot{\bar{z}}^a} = \frac{\partial}{\partial \dot{\bar{z}}^a} - \frac{\partial}{\partial \dot{z}^a}.$$

$T(E')$ is a vector space of real dimension $4n$ over the field of real numbers.

The adapted bases B of $T(E)$ and B' of $T(E')$ are connected by the relations:

$$\frac{\delta}{\delta z^a} = \frac{\delta}{\delta x^a}, \qquad \frac{\delta}{\delta \bar{z}^a} = i \frac{\delta}{\delta y^a},$$

$$\frac{\delta}{\delta \dot{z}^a} = \frac{\partial}{\partial \dot{x}^a}, \qquad \frac{\delta}{\delta \dot{\bar{z}}^a} = i \frac{\partial}{\partial \dot{y}^a}.$$

The basis B^* of $T^*(E)$ induces the adapted basis

$$B'^* = \{\delta z^a, \delta \bar{z}^a, \delta \dot{z}^a, \delta \dot{\bar{z}}^a\} = \{dx^a, -idy^a, \delta \dot{x}^a, -i\delta \dot{y}^a\}$$

of $T^*(E')$. In what follows we shall use five kinds of indices:

$$a, b, c, d, e, f, g = 1, 2, \dots, n, \quad i, j, h, k, l, m, p, q = n+1, \dots, 2n,$$

$$A, B, C, D, E, F, G = 2n+1, \dots, 3n, \quad I, J, H, K, L, M, P, Q = 3n+1, \dots, 4n,$$

$$\alpha, \beta, \gamma, \delta, \chi, \upsilon, \mu = 1, 2, \dots, 4n.$$

Using these indices the adapted basis B' of $T(E')$ gets the form

$$B' = \left\{ \frac{\delta}{\delta z^a}, \frac{\delta}{\delta \bar{z}^i}, \frac{\delta}{\delta \dot{z}^A}, \frac{\delta}{\delta \dot{\bar{z}}^I} \right\} = \left\{ \frac{\delta}{\delta x^a}, i\frac{\delta}{\delta y^i}, \frac{\partial}{\partial \dot{x}^A}, i\frac{\partial}{\partial \dot{y}^I} \right\} = \{\partial_\alpha\}.$$

To every field T on E, there corresponds, by definition one and only one tensor field on E', denoted also by T, such that the product of coefficients of T on E' and the corresponding tensor product of basic vectors on E' are equal to the corresponding product on E. The coefficients of T on E' are the same as the corresponding coefficients on E, or differ from them by the factor -1, i or $-i$. Each product of the coefficient of T and the corresponding basic vector T on E' should be real. That means, that the coefficients of any tensor field T on E' are real or imaginary functions. The metric tensor g on E' is defined in such a way by (1.12).

The complex Finsler space in which the coefficients of tensors have both real and imaginary parts has the real dimension $8n$. In such a space the Cauchy–Riemann conditions (1.3) should be satisfied but, as it was noted earlier, it follows from these equations that only linear transformations of x, y, $\dot{x}$, $\dot{y}$ are allowed, which are not interesting for Finsler spaces. That is the reason, why we restrict our attention to such complex Finsler spaces in which the tensor fields have real or imaginary coefficients.

The generalized linear connection

$$\nabla : T(E') \otimes T(E') \to T(E')((X,Y) \to \nabla_X Y, X, Y, \nabla_X Y \in T(E'))$$

is defined in the following way

$$\nabla_{\partial_\alpha} \partial_\beta = \Gamma_{\beta\ \alpha}^{\ \gamma} \partial_\gamma. \tag{1.6}$$

We shall use the notation

$$\Gamma_{\beta\ \alpha}^{\ \gamma} = \begin{cases} F_{\beta\ \alpha}^{\ \gamma}, & \text{for } \alpha = 1, \dots, n; \\ F_{\beta\ i}^{\ \gamma}, & \text{for } \alpha = n+1, \dots, 2n; \\ C_{\beta\ A}^{\ \gamma}, & \text{for } \alpha = 2n+1, \dots, 3n; \\ C_{\beta\ I}^{\ \gamma}, & \text{for } \alpha = 3n+1, \dots, 4n. \end{cases} \tag{1.7}$$

The torsion tensor $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$ for the generalized connection ∇ has the form $T(X,Y) = T_{\alpha\ \beta}^{\ \gamma} Y^\alpha X^\beta \partial_\gamma$, where

$$T_{\alpha\ \beta}^{\ \gamma} = \Gamma_{\alpha\ \beta}^{\ \gamma} - \Gamma_{\beta\ \alpha}^{\ \gamma} \tag{1.8}$$

except the following components:

$$\begin{aligned} T_{b\ a}^{\ C} &= F_{b\ a}^{\ C} - F_{a\ b}^{\ C} + \delta_b N_a^C - \delta_a N_b^C, \quad T_{b\ C}^{\ D} = C_{b\ C}^{\ D} - F_{C\ b}^{\ D} + \delta_C N_b^D, \\ T_{j\ i}^{\ H} &= F_{j\ i}^{\ H} - F_{i\ j}^{\ H} + \delta_j \bar{N}_i^H - \delta_i \bar{N}_j^H, \quad T_{i\ K}^{\ H} = C_{i\ K}^{\ H} - F_{K\ i}^{\ H} + \delta_K \bar{N}_i^H. \end{aligned} \tag{1.9}$$

THEOREM 1.1 *The distribution $T(E')$ is involutive, i.e. $[\partial_\alpha, \partial_\beta] = 0$ for all $\alpha, \beta = 1, \dots, 4n$ iff*

$$\begin{aligned} K_{b\ a}^{\ c}(x,\dot{x}) &= \frac{\partial N_a^c(x,\dot{x})}{\partial \dot{x}^b} = 0, \quad R_{b\ a}^{\ c}(x,\dot{x}) = \frac{\delta N_a^c(x,\dot{x})}{\delta x^b} - \frac{\delta N_b^c(x,\dot{x})}{\delta x^a} = 0, \\ \bar{K}_{b\ a}^{\ c}(y,\dot{y}) &= \frac{\partial \bar{N}_a^c(y,\dot{y})}{\partial \dot{y}^b} = 0, \quad \bar{R}_{b\ a}^{\ c}(y,\dot{y}) = \frac{\delta \bar{N}_a^c(y,\dot{y})}{\delta y^b} - \frac{\delta \bar{N}_b^c(y,\dot{y})}{\delta y^b} = 0. \end{aligned} \tag{1.10}$$

Proof. By direct calculation we obtain

$$\begin{gathered} \left[\frac{\delta}{\delta x^a}, \frac{\delta}{\delta x^b}\right] = R_{b\ a}^{\ C}(x,\dot{x}) \frac{\partial}{\partial \dot{x}^C}, \quad \left[i\frac{\delta}{\delta y^i}, i\frac{\delta}{\delta y^j}\right] = i\bar{R}_{j\ i}^{\ H}(y,\dot{y}) i\frac{\partial}{\partial \dot{y}^H}, \\ \left[\frac{\delta}{\delta x^a}, \frac{\partial}{\partial \dot{x}^B}\right] = K_{B\ a}^{\ C}(x,\dot{x}) \frac{\partial}{\partial \dot{x}^C}, \quad \left[i\frac{\delta}{\delta y^i}, i\frac{\partial}{\partial \dot{y}^J}\right] = i\bar{K}_{J\ i}^{\ H}(y,\dot{y}) i\frac{\partial}{\partial \dot{y}^H}, \\ \left[\frac{\delta}{\delta x^a}, i\frac{\delta}{\delta y^j}\right] = 0, \quad \left[\frac{\delta}{\delta x^a}, i\frac{\partial}{\partial \dot{y}^J}\right] = 0, \quad \left[\frac{\partial}{\partial \dot{x}^A}, \frac{\partial}{\partial \dot{x}^B}\right] = 0, \\ \left[\frac{\partial}{\partial \dot{x}^A}, i\frac{\partial}{\partial \dot{y}^J}\right] = 0, \quad \left[i\frac{\partial}{\partial \dot{y}^I}, i\frac{\partial}{\partial \dot{y}^J}\right] = 0. \end{gathered} \tag{1.11}$$

If

$$\begin{aligned} &i = a+n, \quad A = a+2n, \quad I = a+3n, \\ &j = b+n, \quad B = b+2n, \quad J = b+3n, \\ &h = c+n, \quad C = C+2n, \quad H = c+3n, \end{aligned}$$

then $R_{b\ a}^{C} = R_{b\ a}^{c}$, $K_{B\ a}^{C} = K_{b\ a}^{c}$, $\bar{R}_{j\ i}^{H} = \bar{R}_{b\ a}^{c}$, $\bar{K}_{j\ i}^{H} = \bar{K}_{b\ a}^{c}$. From these relations and (1.11) follows (1.10).

More about the generalized connection and torsion tensor in complex Finsler spaces can be found in [**2**].

The metric tensor on $T^*(E) \otimes T^*(E)$ expressed in the basis B is given by

$$g = [dx^a, dy^i, \delta\dot{x}^A, \delta\dot{y}^I] \begin{bmatrix} \bar{g}_{ab} & \bar{g}_{aj} & \bar{g}_{aB} & \bar{g}_{aJ} \\ \bar{g}_{ib} & \bar{g}_{ij} & \bar{g}_{iB} & \bar{g}_{iJ} \\ \bar{g}_{Ab} & \bar{g}_{Aj} & \bar{g}_{AB} & \bar{g}_{AJ} \\ \bar{g}_{Ib} & \bar{g}_{Ij} & \bar{g}_{IB} & \bar{g}_{IJ} \end{bmatrix} \otimes \begin{bmatrix} dx^b \\ dy^j \\ \delta\dot{x}^B \\ \delta\dot{y}^J \end{bmatrix},$$

$$\begin{aligned} &i = a+n, \quad A = a+2n, \quad I = a+3n, \quad a = 1, \dots, n, \\ &j = b+n, \quad B = b+2n, \quad J = b+3n, \quad b = 1, \dots, n. \end{aligned}$$

We shall suppose that the above matrix has rank $4n$, is symmetric and positively definite.

The metric g on $T^*(E') \otimes T^*(E')$ in the basis B'^* has the form

$$g = [dx^a, -idy^i, \delta\dot{x}^A, -i\delta\dot{y}^I] \begin{bmatrix} g_{ab} & g_{aj} & g_{aB} & g_{aJ} \\ g_{ib} & g_{ij} & g_{iB} & g_{iJ} \\ g_{Ab} & g_{Aj} & g_{AB} & g_{AJ} \\ g_{Ib} & g_{Ij} & g_{IB} & g_{IJ} \end{bmatrix} \otimes \begin{bmatrix} dx^b \\ -idy^j \\ \delta\dot{x}^B \\ -i\delta\dot{y}^J \end{bmatrix},$$

where

$$\begin{aligned} g_{ab} = \bar{g}_{ab}, \quad g_{aj} = i\bar{g}_{aj}, \quad g_{aB} &= \bar{g}_{aB}, \quad g_{aJ} = i\bar{g}_{aJ}, \\ g_{ij} = -\bar{g}_{ij}, \quad g_{iB} &= i\bar{g}_{iB}, \quad g_{iJ} = -\bar{g}_{iJ}, \\ g_{AB} &= \bar{g}_{AB}, \quad g_{AJ} = i\bar{g}_{AJ}, \\ & \qquad\qquad g_{IJ} = -\bar{g}_{IJ}. \end{aligned}$$

Definition. The $4n$-dimensional differentiable manifold E', in which the allowable coordinate transformations are induced by (1.1), the adapted basis B' of $T(E')$ is formed by $N(x,\dot{x})$, $\bar{N}(y,\dot{y})$ ((1.5)) the generalized connection ∇ is defined by (1.6), (1.7), the torsion tensor T by (1.8), (1.9), the metric tensor g by (1.12), will be called complex Finsler space $E'(z, \bar{z}, \dot{z}, \dot{\bar{z}}, N, \bar{N}, \nabla, T, g)$.

As usual the connection ∇ in the complex Finsler space is called metric connection iff $\nabla_X g = 0$ for all $X \in T(E')$. Under the conditions (1.10) and $\Gamma_{\beta\ \alpha}^{\ \gamma} = \Gamma_{\alpha\ \beta}^{\ \gamma}$ $\forall \alpha, \beta, \gamma = 1, 2, \dots, 4n$ the coefficients of the metric connections are given by $2\Gamma_{\alpha\ \beta}^{\ \gamma} = g^{\gamma\delta}(\partial_\alpha g_{\delta\beta} + \partial_\beta g_{\alpha\delta} - \partial_\delta g_{\beta\alpha})$.

It looks very similar to the Levi–Civita connection, but the summation goes over all $\delta = 1, \dots, 4n$.

2. Different almost complex structures and the corresponding Hermite metrics. It is known, that the almost complex structure F on E' is a tensor field of type (1,1), such that at every point $u' \in E'$, $F^2 = -I$, where I denotes the identity transformation of $T_{u'}(E')$. The metric tensor g defined on E' is called Hermitian iff $g(FX, FY) = g(X,Y)$. The complex Finsler space $E'(z, \bar{z}, \dot{z}, \dot{\bar{z}}, N, \bar{N}, \nabla, T, g)$ endowed with the almost complex structure F and the Hermite metric g, will be denoted by $E'(z, \bar{z}, \dot{z}, \dot{\bar{z}}, N, \bar{N}, \nabla, T, Hg, F)$ or shorter $E'(Hg, F)$.

In the complex Finsler space $E'(z, \bar{z}, \dot{z}, \dot{\bar{z}}, N, \bar{N}, \nabla, T, g)$, some Hermite metric for the almost complex structure F, can be obtained in the following way [**1**]:

$$Hg(X,Y) = g(X,Y) + g(FX, FY),$$

where g is determined by (1.12).

The torsion tensor of the almost complex structure F, or the Nijenhuis tensor of F is defined by

$$[F,F](X,Y) = N(X,Y) = [FX,FY] - [X,Y] - F[FX,Y] - F[X,FY]$$

for any $X, Y \in T(E')$. If the Nijenhuis tensor of F vanishes identically on E', we say that F is a complex structure on E'.

If the almost complex structure F is parallel with respect to the generalized metric connection ∇, i.e. $\nabla_X g = 0$, $\nabla_X F = 0$ for all $X \in T(E')$, then $E'(Hg, F)$ is called Kähler complex Finsler space endowed with Hermite metric.

Remark 1. For every almost complex structure F, it is obvious, that $-F$ is also almost complex structure, i.e. $(-F)^2 = -I$.

Remark 2. The Hermite metric for the almost complex structure $-F$ concides with the Hermite metric for F, i.e. from $g(FX, FY) = g(X,Y)$ it follows $g(-FX, -FY) = g(X,Y)$.

Remark 3. For the Hermite metric g and almost complex structure F the relation $g(X, FX) = 0$ is valid for all $X \in T(E')$.

Proposition 2.1. *The structure J defined on $T(E')$ by:*

$$\begin{aligned} J\left(\frac{\delta}{\delta x^a}\right) &= i\frac{\partial}{\partial \dot{y}^I}, \qquad & J\left(i\frac{\delta}{\delta y^i}\right) &= -\frac{\partial}{\partial \dot{x}^A}, \\ J\left(\frac{\partial}{\partial \dot{x}^A}\right) &= i\frac{\delta}{\delta y^i}, \qquad & J\left(i\frac{\partial}{\partial \dot{y}^I}\right) &= -\frac{\delta}{\delta x^a}, \end{aligned} \tag{2.1}$$

satisfies the relation $J^2 = -I$. To the almost complex structure J in the basis $B' = \{\delta/\delta x^a, i\delta/\delta y^i, \partial/\partial \dot{x}^A, i\partial/\partial \dot{y}^I\}$ corresponds the matrix

$$J \Rightarrow \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

(1 stands instead of $n \times n$ matrix I).

PROPOSITION 2.2 *The metric g determined by* (1.12) *is a Hermite metric with respect to J iff*

$$\begin{aligned} &g_{ab} = g_{IJ}, \quad g_{ij} = g_{AB}, \quad g_{aJ} = g_{Ib} = 0, \quad g_{Aj} = g_{iB} = 0, \\ &g_{aj} = g_{ib} = -g_{AJ} = -g_{IB}, \quad g_{aB} = g_{Ab} = g_{iJ} = g_{Ij}. \end{aligned} \tag{2.3}$$

Proof. The first relation in (2.3) follows from

$$g\left(\frac{\delta}{\delta x^a}, \frac{\delta}{\delta x^b}\right) = g_{ab}, \quad g\left(J\frac{\delta}{\delta x^a}, J\frac{\delta}{\delta x^b}\right) = g\left(i\frac{\partial}{\partial \dot{y}^I}, i\frac{\partial}{\partial \dot{y}^J}\right) = -\bar{g}_{IJ} = g_{IJ}.$$

The others can be obtained in a similar way.

From (2.3), it follows that to the Hermite metric g, with respect to the almost complex structure J given by (2.1), in the basis B'^* corresponds the matrix

$$Hg(J) \Rightarrow \begin{bmatrix} A & B & C & 0 \\ B & E & 0 & C \\ C & 0 & E & -B \\ 0 & C & -B & A \end{bmatrix}, \tag{2.4}$$

where $A = [g_{ab}]$, $B = [g_{aj}]$, $C = [g_{aB}]$, $E = [g_{ij}]$. Remarks 1, 2 and 3 are valid for the almost complex structure J and for the Hermite metric $Hg(J)$.

PROPOSITION 2.3 *The structures L_i given bellow, are almost complex structures and the matrices $Hg(L_i)$ are the corresponding Hermite metrics for L_i, $i = \overline{1,8}$ (both expressed in the basis B' and its dual B'^*).*

$$\begin{array}{ccccc}
i: & 1 & 2 & 3 & 4 \\
L_i: & \begin{bmatrix} i&0&0&0\\0&i&0&0\\0&0&i&0\\0&0&0&i \end{bmatrix} & \begin{bmatrix} -i&0&0&0\\0&i&0&0\\0&0&i&0\\0&0&0&i \end{bmatrix} & \begin{bmatrix} i&0&0&0\\0&-i&0&0\\0&0&i&0\\0&0&0&i \end{bmatrix} & \begin{bmatrix} i&0&0&0\\0&i&0&0\\0&0&-i&0\\0&0&0&i \end{bmatrix} \\
Hg(L_i): & \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix} & \begin{bmatrix} 0&B&C&D\\B&0&0&0\\C&0&0&0\\D&0&0&0 \end{bmatrix} & \begin{bmatrix} 0&B&0&0\\B&0&F&G\\0&F&0&0\\0&G&0&0 \end{bmatrix} & \begin{bmatrix} 0&0&C&0\\0&0&F&0\\C&F&0&I\\0&0&I&0 \end{bmatrix}
\end{array}$$

$$\begin{array}{ccccc}
i: & 5 & 6 & 7 & 8 \\
L_i: & \begin{bmatrix} i&0&0&0\\0&i&0&0\\0&0&i&0\\0&0&0&-i \end{bmatrix} & \begin{bmatrix} -i&0&0&0\\0&-i&0&0\\0&0&i&0\\0&0&0&i \end{bmatrix} & \begin{bmatrix} -i&0&0&0\\0&i&0&0\\0&0&-i&0\\0&0&0&i \end{bmatrix} & \begin{bmatrix} -i&0&0&0\\0&i&0&0\\0&0&i&0\\0&0&0&-i \end{bmatrix} \\
Hg(L_i): & \begin{bmatrix} 0&0&0&D\\0&0&0&G\\0&0&0&I\\D&G&I&0 \end{bmatrix} & \begin{bmatrix} 0&0&C&D\\0&0&F&G\\C&F&0&0\\D&G&0&0 \end{bmatrix} & \begin{bmatrix} 0&B&0&D\\B&0&F&0\\0&F&0&I\\D&0&I&0 \end{bmatrix} & \begin{bmatrix} 0&B&C&D\\B&0&0&G\\C&0&0&I\\D&G&I&0 \end{bmatrix}
\end{array}$$

(i stands instead of $n \times n$ matrix iI, $B = [g_{aj}]$, $C = [g_{aB}]$, $D = [g_{aJ}]$, $F = [g_{iB}]$, $G = [g_{iJ}]$, $I = [g_{AJ}]$. Remarks 1–3 are valid for L_i, $i = \overline{1,8}$.

Proposition 2.4 *The structure F defined on $T(E')$ by*

$$\begin{aligned} F\left(\frac{\delta}{\delta x^a}\right) &= a\frac{\delta}{\delta x^a} - b^{-1}(1+a^2)\frac{\partial}{\partial \dot{x}^A}, \\ F\left(i\frac{\delta}{\delta y^i}\right) &= c\left(i\frac{\delta}{\delta y^i}\right) - d^{-1}(1+c^2)\left(i\frac{\partial}{\partial \dot{y}^I}\right), \\ F\left(\frac{\partial}{\partial \dot{x}^A}\right) &= b\frac{\delta}{\delta x^a} - a\frac{\partial}{\partial \dot{x}^A}, \\ F\left(i\frac{\partial}{\partial \dot{y}^I}\right) &= d\left(i\frac{\delta}{\delta y^i}\right) - c\left(i\frac{\partial}{\partial \dot{y}^I}\right) \end{aligned} \tag{2.6}$$

satisfies the relation $F^2 = -I$ and in the basis B' is determined by the matrix

$$F \Rightarrow \begin{bmatrix} a & 0 & b & 0 \\ 0 & c & 0 & d \\ -b^{-1}(1+a^2) & 0 & -a & 0 \\ 0 & -d^{-1}(1+c^2) & 0 & -c \end{bmatrix}.$$

In the above matrix every of the real scalar fields a b, c, d, $b \neq 0$, $d \neq 0$ denotes the corresponding scalar matrix of type $n \times n$.

The almost complex structure F defined by (2.7) is the generalization of the almost complex structure J defined by Ichiyó in [**3**], (2.1).

Proposition 2.5 *The metric g determined by* (1.12) *is a Hermite metric with respect to F iff its matrix in the basis B'^* has the form:*

$$\begin{bmatrix} A & B & C & D \\ B & E & F & G \\ C & F & H & I \\ D & G & I & J \end{bmatrix},$$

where

$$\begin{gathered} A = [g_{ab}], \quad B = (a+c)^{-1}[b^{-1}(a^2+1)F + d^{-1}(c^2+1)D] \\ C = (1+a^2)^{-1}abA, \quad D = [g_{aJ}] \quad E = [g_{ij}], \quad F = [g_{Aj}] \\ G = (1+c^2)^{-1}cdE, \quad H = (1+a^2)^{-1}b^2A \\ I = (a+c)^{-1}(dF + bD), \quad J = (1+c^2)^{-1}d^2E. \end{gathered}$$

Theorem 2.1. *The almost complex structures J ((2.1), (2.2)), L_i $i = \overline{1,8}$* (2.6) *and F ((2.6), (2.7)) are complex structures on E' iff the relations* (1.10) *are valid.*

Proof. By calculation of the Nijenhuis tensors $N\left(\frac{\delta}{\delta x^a}, \frac{\delta}{\delta x^b}\right), \dots, N\left(\frac{i\partial}{\partial \dot{y}^I}, \frac{i\partial}{\partial \dot{y}^J}\right)$ for the almost complex structures J, L_i $(i = \overline{1,8})$ and F we obtain some linear combinations of the Lie brackets, which appear in (1.11). From (1.11) and Theorem 1.1 it follows that the Nijenhuis tensor for the almost complex structures $J, L_i \quad (i = \overline{1,8})$ and F is equal to zero iff (1.10) is satisfied. $\square$

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