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ON PARACOMPACTNESS AND ALMOST CLOSED MAPPINGS

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Abstract. Some properties of paracompactness in spaces which are not necessarily regular and some properties of almost closed mappings are studied. It is shown that some properties of paracompactness of regular spaces are hold even if the space is not necessarily regular. Beside that, additional conditions are given for almost closed mappings to have a closed graph.

1. Preliminaries. Throughout the paper, spaces will always mean topological spaces without separation axioms unless explicitly stated.

A space X is *paracompact (nearly paracompact)* iff every open (regular open) cover of X has an open locally finite refinement, [**1**].

Let X be a space and A a subset of X. The set A is *α-paracompact (α-nearly paracompact)* iff every open (regular open) cover of A has an open locally finite refinement which covers A. The subset A is *β-paracompact* iff it is paracompact as a subspace. A space X is *locally paracompact* iff each point has an open neighbourhood U such that $\mathrm{Cl}\,(U)$ is β-paracompact [**10**].

A subset A of a space X is *α-Hausdorff* iff for any $a \in A$ and $b \in X\backslash A$, there are disjoint open sets U and V containing a and b respectively. A subset A of a space X is *α-regular* iff for any point $a \in A$ and any open set U containing a there exists an open set V such that $a \in V \subset \mathrm{Cl}\,(V) \subset U$ [**4**].

An open cover $\mathcal{U}$ is *even* iff there exists a neighbourhood V of the diagonal in $X \times X$ such that for each $x \in X$ $V[x] \subset U \quad (V[x] = \{y : (x,y) \in V\})$ for some $U \in \mathcal{U}$ [**3**].

A space X is *almost paracompact* iff for every open covering $\mathcal{U}$ of X there is a locally finite family $\mathcal{V}$ of open subsets of X which refines $\mathcal{U}$ such that the family of closures of members of $\mathcal{V}$ forms a covering of X [**8**].

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A mapping $f : X \to Y$ is *almost closed* iff for every regular closed set F of X the set $f(F)$ is closed [**7**]. A mapping $f : X \to Y$ has a *closed graph* $G(f)$ iff $G(f) = \{(x, f(x)) : x \in X\}$ is closed in $X \times Y$ [**2**].

THEOREM 1.1. *If A is an α-Hausdorff α-paracompact subset of a space X and $x \in X \backslash A$, then there are disjoint regular open neighbourhoods of x and A. Consequently, each α-Hausdorff α-paracompact subset of X is closed.*

THEOREM 1.2. [**4**] *If A is an α-regular α-paracompact subset of a space X, then* $\mathrm{Cl}\,(A)$ *is α-paracompact.*

THEOREM 1.3. [**3**] *If an open covering $\mathcal{U}$ has a closed locally finite refinement, then $\mathcal{U}$ is even.*

THEOREM 1.4. [**3**] *Let X be a space such that every open covering is even. If U is a neighbourhood of the diagonal in $X \times X$, then there is a symmetric neighbourhood V of the diagonal such that $V \circ V \subset U$.*

THEOREM 1.5. [**3**] *Let X be a space such that each open cover of X is even and let A be a locally finite (or a discrete) family of subsets of X. Then, there is an open neighbourhood V of the diagonal in $X \times X$ such that the family of all sets $V[A]$ ($V[A] = \bigcup\{V[x] : x \in A\}$) for A in $\mathcal{A}$ is locally finite (respectively discrete).*

2. Some characterizations of paracompactness. It is well-known that in a regular space X the following four statements are equivalent:

(a) X is paracompact,

(b) every open covering of X has a locally finite refinement,

(c) every open covering of X has a closed locally finite refinement,

(d) every open covering of X is even

(For a $\Leftrightarrow$ b $\Leftrightarrow$ c see Theorem 1 in [**5**] and for a $\Leftrightarrow$ d see Theorem 5.28 in [**3**])

In this section we consider a weaker conditions than regularity and show for dense α-regular subsets that analogous characterizations of α-paracompactness hold. Also the above characterizations of paracompactness of a space X hold if X contains a dense α-regular α-paracompact subset, while X need not be regular. If there is a dense α-paracompact subset D in a space X, then X need not always be paracompact, as the following example shows.

Example 2.1. Let $X = \{a_i : i = 1, 2, 3, \dots\}$, $A = \{a_1, a_2, a_3\}$ and let each point of A be isolated. If the fundamental system of neighbourhoods of a_i, $i = 4, 5, \dots$, is the set $\{a_i\} \cup A$, then the set A is α-paracompact (in fact compact) such that $\mathrm{Cl}\,(A) = X$. The space X is not paracompact, since the family $\mathcal{U} = \{\{a_i\} \cup A : i = 4, 5, \dots\}$ is an open covering of X which admits no locally finite open refinement.

Let A be an α-paracompact α-Hausdorff subset of a space X such that $\mathrm{Cl}\,(A) = X$. Then $A = X$ (i.e. in the space X, there is no proper α-Hausdorff α-paracompact dense subset, since such a subset is closed). Hence, if there is a dense

α-paracompact subset A in a Hausdorff space X, then it is paracompact. If there is a dense α-regular α-paracompact subset of a space X, then X is paracompact (the closure of an α-regular α-paracompact subset is α-paracompact). Hence, if there is a dense α-paracompact subset D in a regular space X, then it is paracompact.

The following example shows that a space containing a dense α-paracompact subset need not be regular.

Example 2.2. Let $X = \{a, b, a_i : i = 1, 2, \dots\}$. Let each point a_i be isolated. If $\{V^n(a) : n = 1, 2, \dots\}$ is a fundamental system of neighbourhoods of a, where $V^n(a) = \{a, a_i : i \geq n\}$, and $\{U^n(b) : n = 1, 2, \dots\}$ is a fundamental system of neighbourhoods of b, where $U^n(b) = \{b, a, a_i : i \geq n\}$, then the set $D = \{b, a_i : i = 1, 2, \dots\}$ is a dense α-regular α-paracompact subset of X. The set X is not regular at a, hence it is not regular.

THEOREM 2.1. *If in a space X there is a dense α-paracompact subset A, then X is almost paracompact.*

Proof. Let $\mathcal{U}$ be any open covering of X. Since A is α-paracompact there exists an open locally finite family $\mathcal{V}$ which refines $\mathcal{U}$ such that $A \subset \bigcup\{V : V \in \mathcal{V}\}$. Then, $X = \mathrm{Cl}\,(A) \subset \mathrm{Cl}\,(\bigcup\{V : V \in \mathcal{V}\}) = \bigcup\{\mathrm{Cl}\,(V) : V \in \mathcal{V}\}$. We can show, however, that the converse of Theorem 2.1 is not necessarily true. The following example will serve the purpose.

Example 2.3. [**7**] Let $X = \{a_{ij}, b_{ij}, c_i, a : i, j = 1, 2, \dots\}$. Let each point a_{ij} and b_{ij} be isolated. If the fundamental system of neighbourhoods of c_i is $\{U^n(c_i) : n = 1, 2, \dots\}$, where $U^n(c_i) = \{c_i, a_{ij}, b_{ij} : j \geq n\}$ and that of a is $\{U^n(a) : n = 1, 2, \dots\}$, where $U^n(a) = \{a, a_{ij} : i, j \geq n\}$, then X is a Hausdorff space which is almost paracompact but is not paracompact, hence there is no dense α-paracompact subset A in X.

THEOREM 2.2. *Let A be a dense α-regular α-paracompact subset of a space X. Then, every open covering of A has a closed locally finite refinement.*

Proof. Let $\mathcal{U} = \{U_i : i \in I\}$ be any open covering of the set A. For each $x \in A$, there exists an open set V_x such that $x \in V_x \subset \mathrm{Cl}\,(V_x) \subset U_{i(x)}$ for some $i(x) \in I$. Now, $\mathcal{V} = \{V_x : x \in A\}$ is an open covering of A, hence there exists an open locally finite family $\mathcal{H} = \{H_j : j \in J\}$ which refines $\mathcal{V}$ and covers A. Since, $H_j \subset V_{x(j)}$ for some $x(j) \in A$, then $\mathrm{Cl}\,(H_j) \subset \mathrm{Cl}\,(V_{x(j)}) \subset U_{i(x(j))}$. It follows that $\{\mathrm{Cl}\,(H_j) : j \in J\}$ is a closed locally finite covering of A which refines $\mathcal{U}$. (Since $X = \mathrm{Cl}\,(A) = \bigcup\{\mathrm{Cl}\,(H_j) : j \in J\} \subset \bigcup\{U_i : i \in I\}$, it follows that every open covering of A is an open covering of X).

As a Corollary to the above Theorem we have the following statement:

COROLLARY 2.1. *If a space X contains a dense α-regular α-paracompact subset, then every open covering of X has a closed locally finite refinement.*

Similarly to the proof of Theorem 2.2, the following result can be established.

Theorem 2.3. *Let D be a dense α-regular subset of a space X. If every open covering of D has a locally finite refinement, then every open covering of D has a closed locally finite refinement.*

As a Corollary to the above Theorem we have the following statement.

Corollary 2.2. *If every open cover of a dense α-regular subset D of a space X has a locally finite refinement which covers it, then every open cover of D is an open covering of X, and every open covering of the space X has a closed locally finite refinement.*

In general every open covering of a dense α-regular subset D of a space X is not always an open covering of X. The following example will serve the purpose.

Example 2.4. Let $X = \{a_i, a : i = 1, 2, \dots\}$. Let each point a_i be isolated. If the fundamental system of neighbourhoods of a is the set $\{V^n(a) : n = 1, 2, \dots\}$, where $V^n(a) = \{a, a_i : i \geq n\}$, then the set $D = \{a_i : i = 1, 2, \dots\}$ is α-regular. The family consisting of all $\{a_i\}$ is an open cover of D, but is not an open covering of X. Note that D is not α-paracompact.

Theorem 2.4. *Let X be a space and D be its dense α-regular subset. If every open cover of D has a locally finite refinement which covers D, then D is α-paracompact, so X is paracompact.*

Proof. Let $\mathcal{U} = \{U_i : i \in I\}$ be any open cover of D. There exists a locally finite family $\mathcal{A}$ which refines $\mathcal{U}$ and covers D (i.e. covers X). By Theorem 1.5 and Theorem 1.3, there exists an open neighbourhood V of the diagonal in $X \times X$ such that $\{V[A], A \in \mathcal{A}\}$ is an open locally finite covering of X. For each $A \in \mathcal{A}$ pick the subset U_A of $\mathcal{U}$ which contains A. Let $W_A = U_A \cap V[A]$ and $\mathcal{W} = \{W_A : A \in \mathcal{A}\}$. $\mathcal{W}$ is an open locally finite family which refines $\mathcal{U}$ and covers D, hence D is α-paracompact, so X is paracompact.

Example 2.2 shows that a space which satisfies the conditions of Theorem 2.3 need not be regular.

In Theorem 2.3 the condition “every open covering of D has a locally finite refinement” cannot be replaced by “every open covering of X has a locally finite refinement”.

The following example serves the purpose.

Example 2.5. Let $X = \{a, b, a_i, b_i : i = 1, 2, \dots\}$. Let each point a_i be isolated. If the fundamental system of neighbourhoods of a is the set $\{V^n(a) : n = 1, 2, \dots\}$, where $V^n(a) = \{a, a_i : i \geq n\}$, the fundamental system of neighbourhoods of b is the set $\{\{b\} \cup V^n(a) : n = 1, 2, \dots\}$, and the fundamental system of neighbourhoods of b_i is the set $\{U^n(b_i) : n = 1, 2, \dots\}$, where $U^n(b_i) = \{b_i, a_j : j \geq n\}$, then the set $A = \{a_i : i = 1, 2, \dots\}$ is α-regular. X is not regular at a, hence X is not regular. The subset A is not α-paracompact. X is not paracompact, since the family consisting of the sets $V^n(a)$, $\{b\} \cup V^n(a)$, $U^i(b_i)$ for all i and all $\{a_i\}$ is an open covering of X which admits no locally finite open refinement. But every open covering of X has a locally finite refinement.

Note that for a dense α-regular subset A, there is an open covering which admits no X-locally finite refinement.

COROLLARY 2.3. [**5**] *A regular space X is paracompact iff every open covering of X has a locally finite refinement.*

3. Some properties of almost closed mappings. Hamlett and Herrington [**2**] proved the following result:

THEOREM A. *Let $f : X \to Y$ be a closed function, where X is a regular topological space. If $f^{-1}(y)$ is closed for every $y \in Y$, then f has a closed graph.*

It would be interesting to see whether similar statements hold for almost closed mappings. Some results will be given in that direction.

THEOREM 3.1. *Let $f : X \to Y$ be an almost closed mapping of a space X onto a space Y such that $f^{-1}(y)$ is an α-Hausdorff α-nearly paracompact subset of X for each $y \in Y$. Then f has a closed graph.*

Proof. Let $(x, y) \notin G(f)$ be any point. Since $y \neq f(x)$, then $x \notin f^{-1}(y)$. Since $f^{-1}(y)$ is α-Hausdorff α-nearly paracompact, by Theorem 1.1 there are regular open disjoint sets U and V of X such that $x \in U$, $f^{-1}(y) \subset V$. Since f is almost closed, by Lemma 3 of [**6**], there is an open neighbourhood W of y such that $f^{-1}(y) \subset f^{-1}(W) \subset V$. Thus $(x, y) \in U \times W$, $U \times W \cap G(f) = \emptyset$. Hence, $G(f)$ is closed.

COROLLARY 3.1. [**2**] *Let $f : X \to Y$ be an almost closed injection from a Hausdorff space X onto a space Y. Then f has a closed graph.*

COROLLARY 3.2. *If $f : X \to Y$ is an almost closed mapping of a space X onto a compact space Y such that $f^{-1}(y)$ is α-Hausdorff α-nearly paracompact for each $y \in Y$; then f is continuous.*

Proof. Since $G(f)$ is closed and Y is compact, then f is continuous.

The following two examples will show that the assumptions "$f^{-1}(y)$ is α-paracompact" and "$f^{-1}(y)$ is closed" are independent even if the mapping f is almost closed and X is regular.

Example 3.1. Let $X = \{a_1, a_2, a_3, a_4\}$ and $\tau = \{\emptyset, \{a_1, a_2\}, \{a_3, a_4\}, X\}$. The space (X, τ) is regular. The identity map $i : X \to X$ is almost closed and $i^{-1}(x) = x$ is α-paracompact (in fact compact) for each $x \in X$, but $i^{-1}(a_1) = a_1$ is not closed. Hence, i has not a closed graph.

Example 3.2. Let X_1 be a regular but not paracompact space and $X_2 = \{a_1, a_2, a_3, a_4\}$, where $a_i \notin X_1$ for each $i \in \{1, 2, 3, 4\}$. Let $\tau_{X_2} = \{\emptyset, \{a_1, a_2\}, \{a_3, a_4\}, X_2\}$ and $X = X_1 + X_2$ be the topological sum of the spaces X_1 and X_2. Let $Y = \{m, n\}$ be the discrete space. Define a mapping $f : X \to Y$ by

$$f(x) = \begin{cases} m, & x \in X_1 \\ n, & x \in X_2 \end{cases}$$

f is an almost closed mapping from the regular space X onto the space Y such that $f^{-1}(y)$ is closed for each $y \in Y$, but $f^{-1}(m) = X_1$ is not α-paracompact ($G(f)$ is closed).

THEOREM 3.2. *Let $f : X \to Y$ be an almost closed mapping of a space X onto a space Y such that $f^{-1}(y)$ is closed, for each $y \in Y$. Then, for each α-regular α-paracompact subset A of X, $f(A)$ is closed.*

Proof. Let $y \notin f(A)$ be any point. Since $f^{-1}(y)$ is closed and A is α-regular α-paracompact, by Theorem 2.5 of [**4**], there is a regular open set V such that $A \subset V \subset \mathrm{Cl}\,(V) \subset X \backslash f^{-1}(y)$. Thus $f^{-1}(y) \subset X \backslash \mathrm{Cl}\,(V)$. Since $X \backslash \mathrm{Cl}\,(V)$ is regular open and f is almost closed, there is an open neighbourhood W of y such that $f^{-1}(y) \subset f^{-1}(W) \subset X \backslash \mathrm{Cl}\,(V)$. Thus, W is an open neighbourhood of y such that $W \cap f(A) = \emptyset$. Hence, $f(A)$ is closed.

Definition 3.1. A subset A of a space X is *α-normal* iff for every open neighbourhood U of A, there is an open neighbourhood V of A such that

$$A \subset V \subset \mathrm{Cl}\,(V) \subset U.$$

THEOREM 3.3. *Let $f : X \to Y$ be an almost closed mapping of a space X onto a space Y such that $f^{-1}(y)$ is an α-normal subset for each $y \in Y$. Then:*
(a) *f is closed;* (b) *if X is T_1, then f has a closed graph.*

Proof. (a) Let A be a closed subset of a space X and $y \notin f(A)$ be any point of Y. Then, $f^{-1}(y) \subset X \backslash A$. Since $f^{-1}(y)$ is α-normal, there is an open subset V of X such that $f^{-1}(y) \subset V \subset \mathrm{Cl}\,(V) \subset X \backslash A$. Then $U = \alpha(V) = \mathrm{Int}\,(\mathrm{Cl}\,(V))$ is a regular open subset of X such that $f^{-1}(y) \subset U \subset \mathrm{Cl}\,(U) \subset X \backslash A$. The rest of the proof is similar to the one given in Theorem 3.2. (b) Let $(x, y) \notin G(f)$ be any point. Since $y \neq f(x)$, it follows that $x \notin f^{-1}(y)$. Since X is T_1 and $f^{-1}(y)$ is α-normal, there is a regular open subset U of X such that $f^{-1}(y) \subset U \subset \mathrm{Cl}\,(U) \subset X \backslash \{x\}$. Since f is almost closed, there is an open neighbourhood W of y such that $f^{-1}(y) \subset f^{-1}(W) \subset U$. Let $V = X \backslash \mathrm{Cl}\,(U)$. Then V is an open neighbourhood of x such that $f(V) \cap W = \emptyset$. Thus, $(x, y) \in V \times W$, $V \times W \cap G(f) = \emptyset$. Hence, $G(f)$ is closed.

The following example shows that in Theorem 3.3 the space X need not be either normal or regular.

Example 3.3. Let $X = \{a, b, a_i : i = 1, 2, \dots\}$. Let each point a_i be isolated. If $\{U^n(a) : n \in N\}$ is a fundamental system of neighbourhoods of a, where $U^n(a) = \{a, a_i : i \geq n\}$ and $\{V^n(b) : n \in N\}$ is a fundamental system of neighbourhoods of b, where $V^n(b) = \{b, a_i : i \geq n\}$, then the space X is T_1 but X is not a regular space since $\mathrm{Cl}\,(U^n(a)) = U^n(a) \cup \{b\}$, hence X is not normal.

If $Y = \{m, b_i : i = 1, 2, \dots\}$ is equiped with the discrete topology, then the mapping $f : X \to Y$ of the space X onto the space Y defined by $f(a_i) = b_i$, $i = 1, 2, \dots$; $f(a) = f(b) = m$, satisfies all the conditions of Theorem 3.2. Hence f has a closed graph.

By combining Theorems 3.1 and 3.3 (b) we obtain the following result.

THEOREM 3.4. *Let X_1 be a space and X_2 be a regular space. Let $X = X_1 + X_2$ be a topological sum of the spaces X_1 and X_2. Let $f : X \to Y$ be an almost closed mapping of the space X onto a space Y such that for each $y \in Y$,*

(a) *if $f^{-1}(y) \cap X_1 \neq \emptyset$, then $f^{-1}(y) \cap X_1$ is an α-Hausdorff α-nearly paracompact subset of X*

(b) *if $f^{-1}(y) \cap X_2 \neq \emptyset$, then $f^{-1}(y) \cap X_2$ is closed.*

Then f has a closed graph.

REFERENCES

[1] J. Dieudonné, *Une généralisation des espaces compacts*, J. Math. Pures. Appl. **23** (1944), 65–76.

[2] T.R. Hamlett and L.L. Herrington, *The closed graph and P-closed graph properties in general topology*, Amer. Math. Soc., Providence, R.I. 1981, 1–68.

[3] J.L. Kelley, *General Topology*, Van Nostrand, Princeton, N.J., 1957.

[4] I. Kovačević, *Subsets and paracompactness*, Univ. u N. Sadu Zb. Rad. Prir. Mat. Fak. Ser. Mat. **14–2** (1984), 79–87.

[5] E. Michael, *A note on paracompact spaces*, Proc. Amer. Mat. Soc. **4** (1953), 831–838.

[6] G.T. Noiri, *Almost continuous and some separation axioms*, Glasnik Mat. **9(29)** (1974), 131–135.

[7] M.K. Singal and A.R. Singal, *Almost continuous mappings*, Yokohama Math. J. **16** (1968), 63–73.

[8] M.K. Singal and S.P. Arya, *On m-paracompact spaces*, Math. Ann. **181** (1969), 119–133.

[9] M.K. Singal and S.P. Arya, *On nearly paracompact spaces*, Mat. Vesnik **6(21)** (1969), 3–16.

[10] J.D. Wine, *Locally paracompact spaces*, Glasnik Mat. **10(30)** (1975), 351–357.

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