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MULTIMENSIONAL HERMITE POLYNOMIALS OF COMPLEX GAUSSIAN VARIABLES

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Abstract. Multidimensional Hermite polynomials of complex Gaussian variables are defined and some of their basic properties are investigated.

We assume all random variables are centered at the expectations. Let $X_1, X_2, \dots, X_p$ be a sequence of real Gaussian variables, not necessarily different. Multidimensional Hermite polynomial $H_p(X_1, X_2, \dots, X_p)$ of degree $p \geq 1$ is defined as the polynomial orthogonal to any polynomial $P_q(Y_1, Y_2, \dots, Y_q)$, $q < p$, of any Gaussian variables $Y_1, Y_2, \dots, Y_q$, [**3**]. The explicit expression follows from the definition ([**3**], [**1**]):

$$\begin{aligned} H_p(X_1, X_2, \dots, X_p) = X_1 X_2 \dots X_p &- \sum_{k \neq m,n} b_{mn} X_{k(1)} X_{k(2)} \dots X_{k(p-2)} \\ &+ \sum_{k \neq m,n} b_{m(1)n(1)} b_{m(2)n(2)} X_{k(1)} X_{k(2)} \dots X_{k(p-4)} - \dots, \\ b_{mn} &= EX_m X_n, \end{aligned} \tag{1}$$

where the first sum is over all combinations (m,n) of the elements of the set $\{1, 2, \dots, p\}$, the second sum is over all pairs (m_1, n_1), (m_2, n_2) and so on. For example, $H_1(X) = X$, $H_2(X_1, X_2) = X_1 X_2 - b_{12}$, $H_3(X_1, X_2, X_3) = X_1 X_2 X_3 - b_{12} X_3 - b_{13} X_2 - b_{23} X_1$. We denote $H_p(X) = H_p \underbrace{(X, X, \dots, X)}_{p}$ and call it one dimensional Hermite polynomial. It follows from (1) that the Hermite polymial is a multilinear function, i.e.

$$H_p(aX_1, X_2, \dots, X_p) = aH_p(X_1, X_2, \dots, X_p), a - \text{real number}, \tag{2}$$

$$H_p(Y_1 + Y_2, X_2, \dots, X_p) = H_p(Y_1, X_2, \dots, X_p) + H_p(Y_2, X_2, \dots, X_p). \tag{3}$$

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$H_p(X_1, X_2, \dots X_p)$ being a symmetric function, it follows from (3) that

$$H_p\left(\sum_1^{n(1)} X_{k(1)}, \sum_1^{n(2)}, X_{k(2)}, \dots, \sum_1^{n(p)} X_{k(p)}\right) = \sum H_p\left(X_{k(1)}, X_{k(2)}, \dots, X_{k(p)}\right) \quad (4)$$

where the summation on the right hand side of (4) is over all arrangements ($n(1)n(2)\dots n(p)$ in the numbers) of the indices $k(1), k(2), \dots, k(p)$.

One important property is the commutability of the Hermite polynomial and the conditional expectation. Let $\{X_t, t \in T\}$ be a set of real Gaussian variables. Denote by XS the σ-algebra generated by $\{X_t, t \in S\}$, $S \subset T$. We have [**1**]:

$$\begin{aligned} E(H_p(X_{t(1)},&X_{t(2)}, \dots, X_{t(p)}) \mid XS) \\ &= H_p(E(X_{t(1)}|XS), E(X_{t(2)}|XS), \dots, E(x_{t(p)}|XS)). \end{aligned} \quad (5)$$

Let $\{X_t, t \geq 0\}$, $x_0 = 0$, be real mean square continuous Gaussian process and let $\mathcal{H}_t$ be the mean square linear closure of all random variables with finite variances, measurable with respect to $\{X_s, s \leq t\}$. Then the family $\{E_t, t \geq 0\}$ of projectors E_t onto $\mathcal{H}_t$ defines the resolution of the identity in separable Hilbert space $\mathcal{H} = \overline{\bigvee_t \mathcal{H}_t}$. The nonlinear time domain analysis of $\{X_t\}$ is the determination of the spectral type of $\{E_t\}$, [**2**]. The main tool in this analysis is property (5). It is more convenient to consider the Hilbert space $\mathcal{H}$ over the complex numbers in the nonlinear time domain analysis. For instance, the proper values and vectors correspond to the discontinuity points of the spectral type. This is the reason for introducing Hermite polynomials of complex Gaussian variables.

Let $Z_k = X_k + iY_k$, $k = \overline{1,p}$, $i^2 = -1$, be a sequence of complex Gaussian variables.

Proposition 1. *The symmetric polynomial $H_p(Z_1, \dots, Z_p)$ defined by*

$$H_p(X_1 + iY_2, X_2, \dots, X_p) = H_p(X_1, X_2, \dots, X_p) + iH_p(Y_1, X_2, \dots, X_p) \quad (6)$$

is the Hermite polynomial.

For example,

$$H_p(X + iY) = \sum_{k=0}^{p} i^k \binom{p}{k} H_p\left(\underbrace{X, \dots, X}_{p-k}, \underbrace{Y, \dots, Y}_{k}\right).$$

Proof. The real and imaginary parts of the polynomial $H_p(Z, \dots Z_p)$ are the linear combinations of the Hermite polynomials $\mathcal{H}'_p$ of degree p of the real variables $X_1, \dots, X_p, Y_1, \dots, Y_p$. Also, the real and imaginary parts of any polynomial P_q, $q < p$, of complex variables, are the polynomials P'_q of real variables. So, the inner product $\langle H_p, P_q \rangle = EH_p\bar{P}_q$ is the complex number whose real and imaginary parts are linear combinations of $\langle H'_p, P'_q \rangle = 0$ i.e. $\langle H_p, P_q \rangle = 0$. $\square$

Similarly it is easy to verify that properties (2) - (5) are also satisfied for Hermite polynomials of complex Gaussian variables. We add one more for the conjugates:

$$\bar{H}_p\left(Z_1, Z_2, \ldots, Z_p\right) = H_p\left(\bar{Z}_1, \bar{Z}_2, \ldots, \bar{Z}_p\right). \tag{7}$$

Consider a complex-valued second order random field $\{Z(\mathbf{t}), \mathbf{t} \in R^d\}$ as one application of the previous notations. It is well known that the orthogonality of $\{Z(\mathbf{t})\}$ is characterized by

$$\langle Z(\mathbf{s}), Z(\mathbf{t})\rangle = \|Z(\mathbf{s} \wedge \mathbf{t})\|^2. \tag{8}$$

Suppose that $\{Z(\mathbf{t})\}$ is Gaussian field. Then (8) means that $E(Z(\mathbf{t})|Z(\mathbf{s})) = Z(\mathbf{s} \wedge \mathbf{t})$. Consider the field $\{H_p(Z(\mathbf{t})), \mathbf{t} \in R^d\}$, where $H_p(\cdot)$ is one-dimensional Hermite polynomial.

PROPOSITION 2. *The field $H_p(Z(\mathbf{t}))$ is the orthogonal field.*

Proof. We have

$$\begin{aligned}
&\langle H_p(Z(\mathbf{s})), H_p(Z(\mathbf{t}))\rangle = EH_p(Z(\mathbf{s}))\bar{H}_p(Z(\mathbf{t})) \\
&\stackrel{1}{=} EH_p(Z(\mathbf{s}))E(H_p(Z(\mathbf{s}))\overline{E(H_p(Z(\mathbf{t}))|Z(\mathbf{s}))}) \\
&\stackrel{2}{=} EH_p(Z(\mathbf{s}))\bar{H}_p(Z(\mathbf{s} \wedge \mathbf{t})) \stackrel{3}{=} E\bar{H}_p(Z(\mathbf{s} \wedge \mathbf{t})E(H_p(Z(\mathbf{s})|Z(\mathbf{s} \wedge \mathbf{t}))) \\
&\stackrel{4}{=} E\bar{H}_p(Z(\mathbf{s} \wedge \mathbf{t}))H_p(Z(\mathbf{s} \wedge \mathbf{t})) = \|H_p(Z(\mathbf{s} \wedge \mathbf{t}))\|^2.
\end{aligned}$$

The equalities 1 and 3 follow from the properties of conditional expectation and the equalities 2 and 4 follow from the commutability property (5).

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