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FUNCTIONAL EQUATIONS AND TEMPERED ULTRADISTRIBUTIONS

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Abstract. We introduce a method of solving functional equations based on the theory of ultradistributions.

1. Introduction. In [**1**] Baker studied complex valued functions and distributions f, of several real variables, satisfying functional equations of the form

$$\sum_{\alpha=0}^{m} a_\alpha f(x+k_\alpha) = Q(x), \qquad x \in \mathbf{R}^d,$$

where $a_0, a_1, \dots a_m \in \mathbf{C}, k_0, k_1, \dots k_m \in \mathbf{R}^d$, $m \in \mathbf{N}$, and Q is a polynomial. He introduced a method for soloving such an equation based on the theory of distributions. In this paper we investigate a wider class of functional equations and complex valued functions and ultradistributions which are solutions of a functional equation of the form

$$\sum_{\alpha=0}^{\infty} a_\alpha f(x+k_\alpha) = Q(x), \qquad x \in \mathbf{R}^d, \tag{1.1}$$

where $\{a_\alpha\}_\alpha$ and $\{k_\alpha\}_\alpha$ are sequences in $\mathbf{C}$ and $\mathbf{R}^d$, respectively, and Q is an entire function with the appropriate growth rate.

We investigate the equation (1.1) in the frame of the theory of tempered ultradistributions. Our method is based on Komatsu's theory of Beurling (resp. Roumieu) type ultradistributions and the properties of the Fourier transform on the spaces of tempered ultradistributions, which were obtained in [**5**] and [**4**]. Following an idea analogous to Baker's one for distributions [**1**], we will show that under certain assumptions on sequences $\{a_\alpha\}_\alpha$ and $\{k_\alpha\}_\alpha$, if a solution f of (1.1)

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is a function which satisfies some mild regularity conditions, then f is almost everywhere equal to an entire function with appropriate growth rate. The method is illustrated by two examples.

2. Notation and background. The symbols $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{R}$ and $\mathbf{C}$ denote the sets of natural numbers, integers, real numbers and complex numbers, respectively, and $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$. The letter d denotes a fixed element of $\mathbf{N}$. In the paper we suppose that:

1. $\{M_\alpha\}_\alpha$ is the sequence of positive numbers which satisfy the following conditions (see [**2**]):

(M.1) $M_\alpha^2 \le M_{\alpha-1} M_{\alpha+1}$, $\alpha \in \mathbf{N}$;

(M.2) $M_\alpha \le AH^\alpha \min_{0\le\beta\le\alpha} M_{\alpha-\beta} M_\beta, \quad \alpha,\beta \in \mathbf{N}_0$, for some $A, H \ge 0$;

(M.3) $\displaystyle\sum_{\alpha=\beta+1}^{\infty} \frac{M_{\alpha-1}}{M_\alpha} \le A\beta \frac{M_\beta}{M_{\beta+1}}$, $\beta \in \mathbf{N}$.

2. So-called associated function M for the sequence $\{M_\alpha\}_\alpha$, is given by

$$M(\rho) = \sup_{\alpha\in\mathbf{N}_0} \log(\rho^\alpha / M_\alpha), \qquad \rho > 0.$$

An example of a sequence $\{M_\alpha\}_\alpha$, which satisfies the conditions (M.1), (M.2) and (M.3), is the Gevrey sequence $\{\alpha^{s\alpha}\}_\alpha$, $s > 1$. In this special case, $M(\rho)$ behaves as $\rho^{1/s}$, when ρ tends to infinity.

We denote by $\mathcal{D}^{(M_p)}$ (resp. $\mathcal{D}^{\{M_p\}}$) the space of ultradifferentiable functions, on $\mathbf{R}^d$, of Beurling (resp. Roumieu) type and by $\mathcal{D}'^{(M_p)}$ (resp. $D'^{\{M_p\}}$) spaces introduced in Komatsu's theory of Beurling (resp. Roumieu) type ultradistributions on $\mathbf{R}^d$. For the definitions and properties of the spaces we refer to [**2**]. The common notation for the symbols (M_p) and $\{M_p\}$ will be $*$.

The space $\mathcal{S}^{(M_p)}$ (resp. $\mathcal{S}^{\{M_p\}}$) is projective (resp. inductive) limit of $\mathcal{S}_2^{M_p,m}$, $m > 0$, where $\mathcal{S}^{M_p,m}$ is the space of all smooth functions φ on $\mathbf{R}^d$ such that

$$\sigma_{m,2}(\varphi) = \left(\sum_{\alpha,\beta\in\mathbf{N}_0^d} \int_{\mathbf{R}^d} \left| \frac{m^{|\alpha|+|\beta|}}{M_{|\alpha|} M_{|\beta|}} x^\beta \varphi^{(\alpha)}(x) \right|^2 dx \right)^{1/2} < \infty, \tag{2.1}$$

where if $\alpha = (\alpha_1, \ldots \alpha_d) \in \mathbf{N}_0^d$, then $|\alpha|$ denotes $\alpha_1 + \cdots + \alpha_d$. Its dual space, the space of tempered Beurling (resp. Roumieu) type ultradistributions, is denoted by $\mathcal{S}'^{(M_p)}$ (resp. $\mathcal{S}'^{\{M_p\}}$. These spaces were investigated in [**4**] and [**6**], and in the case $M_\alpha = \alpha^{s\alpha}$, $s > 1$, in [**5**].

A function Q is an ultrapolynomial of class (M_p) (resp. $\{M_p\}$), if it is entire and satifies the following condition: there exist $l > 0$ and $\mathcal{C} > 0$ (resp. for each $l > 0$ there exists $\mathcal{C} > 0$), such that

$$|Q(x)| \le \mathcal{C} \exp(M(l|x|)), \qquad x \in \mathbf{R}^d, \tag{2.2}$$

or equivalently [**2**]:

$$Q(x) = \sum_{\beta \in \mathbf{N}_0^d} b_\beta x^\beta, \qquad x \in \mathbf{R}^d,$$

where $b_\beta \in \mathbf{C}$, for each $\beta \in \mathbf{N}^d$, and for some $L > 0$ and $\mathcal{C} > 0$ (resp. for each $L > 0$ there exists $\mathcal{C} > 0$)

$$|b_\beta| \leq \mathcal{C} L^{|\beta|} M_{|\beta|}, \beta \in \mathbf{N}_0^d. \tag{2.3}$$

In the special case when $M_\alpha = \alpha^{s\alpha}$, $\alpha \in \mathbf{N}$, $s > 1$, the condition (2.2) is equivalent to

$$|Q(x)| \leq \mathcal{C} \exp((lx)^{1/2}), \qquad x \in \mathbf{R}.$$

We say that a function $f\colon \mathbf{R}^d \to \mathbf{C}$ is a regular tempered ultradistribution of class $*$, if it is locally integrable and the regular ultradistribution determined by f, via the mapping

$$\varphi \to \int_{\mathbf{R}^d} f\varphi, \qquad \varphi \in \mathcal{S}^*,$$

is a tempered ultradistribution of class $*$. For this it is sufficient that f is measurable and of ultrapolynomial growth of class (M_p) (resp. $\{M_p\}$), i.e. that it satisfies the estimation of the form (2.2).

A regular ultradistribution determined by a function f will also be denoted by f.

If f is a function $\mathbf{R}^d \to \mathbf{C}$ and $k \in \mathbf{R}^d$, then $\mathcal{T}_k f$ denotes the function defined by

$$(\mathcal{T}_k f)(x) = f(x + k), \qquad x \in \mathbf{R}^d.$$

If $\phi \in \mathcal{S}^*$, then the Fourier transform of ϕ is defined by

$$\mathcal{F}\phi(x) = \int_{\mathbf{R}^d} \phi(t) \exp(-ixt) dt, \qquad x \in \mathbf{R}^d.$$

The Fourier transform of a tempered ultradistribution f is defined by

$$\langle \mathcal{F} f, \phi \rangle = \langle f, \mathcal{F}\phi(x) \rangle, \qquad \phi \in \mathcal{S}^*.$$

It is a topological automorphism of space $\mathcal{S}'^*$ ([**4**], [**5**]).

3. Main theorem. In this section we will give precise formulation of the assertion announced in the Introduction.

THEOREM 1. *Let T be a nonempty subset of $\mathbf{R}^d$. Assume that for each $t_0 \in T$ there exist a sequence $\{k_{t_0,\alpha}\}_\alpha$ in $\mathbf{R}^d$, and a sequence of complex numbers $\{a_{t_0,\alpha}\}_\alpha$, such that there exist $l > 0$ and $\mathcal{C} > 0$ in case (M_p) (resp. for each $l > 0$ there exists $\mathcal{C} > 0$ in case $\{M_p\}$), such that $|a_{t_0,\alpha}| \leq \mathcal{C} l^\alpha / M_\alpha$, $\quad \alpha \in \mathbf{N}$. Put $Z_{t_0} = \{x \in \mathbf{R}^d, \sum_{\alpha=0}^\infty a_{t_0,\alpha} \exp(ik_{t_0,\alpha} x) = 0\}$, and suppose that $\bigcap_{t_0 \in T} Z_{t_0}$ is a subset of $\{0\} \subset \mathbf{R}^d$.*

(a) *Suppose that* $f \in \mathcal{S}'^*$ *and that for each* $t_0 \in T$ *there exists an ultrapolynomial* Q_{t_0} *of class* $*$*, such that the following equation holds in* $\mathcal{S}'^*$*:*

$$\sum_{\alpha=0}^{\infty} a_{t_0,\alpha} \mathcal{T}_{k_{t_0,\alpha}} f = Q_{t_0}. \tag{3.1}$$

Then, f *is an ultrapolynomial of class* $*$.

(b) *Let* $f: \mathbf{R}^d \to \mathbf{C}$ *be a regular rempered ultradistribution of class* $*$ *and let for each* $t_0 \in T$ *there exist an ultrapolynomial* Q_{t_0}*, such that*

$$\sum_{\alpha=0}^{\infty} a_{t_0,\alpha} f(x + k_{t_0,\alpha}) = Q_{t_0}(x), \qquad \textit{for almost all } x \in \mathbf{R}^d. \tag{3.2}$$

Then, there is an ultrapolynomial P *of class* $*$*, such that* $f(x) = P(x)$*, for almost all* $x \in \mathbf{R}^d$*. If moreover,* f *is continuous, then* f *is an ultrapolynomial of class* $*$.

Proof. (a) Suppose that $f \in \mathcal{S}'^*$, and that (3.1) holds. From the assumption of the theorem it follows that the function Q_{t_0} is an element of the space $\mathcal{S}'^*$, and that

$$\mathcal{F}\left(\sum_{\alpha=0}^{\infty} a_{t_0,\alpha} \mathcal{T}_{k_{t_0},\alpha} f\right) = \mathcal{F} Q_{t_0}.$$

Since, the Fourier transform is continuous and automorphism of the space $\mathcal{S}'^*$, we have

$$\mathcal{F}\left(\sum_{\alpha=0}^{\infty} a_{t_0,\alpha} \mathcal{T}_{k_{t_0},\alpha} f\right) = \sum_{\alpha=0}^{\infty} a_{t_0,\alpha} \mathcal{F}(\mathcal{T}_{k_{t_0},\alpha} f).$$

Using the following equality

$$\mathcal{F}(\mathcal{T}_{k_{t_0},\alpha} f)(\xi) = \exp(ik_{t_0,\alpha}\xi) \mathcal{F} f(\xi), \qquad \xi \in \mathbf{R}^d,$$

we obtain

$$\left(\sum_{\alpha=0}^{\infty} a_\alpha \exp(ik_{t_0,\alpha} x)\right) \mathcal{F} f = \mathcal{F}(Q_{t_0}). \tag{3.3}$$

The function Q_{t_0} is ultrapolynomial, therefore

$$\mathcal{F} Q_{t_0} = \sum_{\beta \in \mathbf{N}^d} b_\beta \delta^{(\beta)},$$

where for some $L > 0$ and $\mathcal{C} > 0$ (resp. for each $L > 0$ there exists $\mathcal{C} > 0$) the constants $b_\beta, \beta \in \mathbf{N}^d$, satisfy the estimation (2.3). From the above equality and [**3**, Theorem 3.1] it follows that $\operatorname{supp} \mathcal{F}(Q_{t_0}) \subseteq \{0\}$. Elements of the sum $\sum_{\alpha=0}^{\infty} a_\alpha \exp(ik_{t_0,\alpha} x)$ are smooth functions since the sum converges absolutely, therefore it is a smooth function. Since $\mathcal{F} f$ and $\mathcal{F} Q_{t_0}$ are elements of $\mathcal{S}'^*$, from above and (3.3) it follows that:

$$\operatorname{supp} \mathcal{F} f \subseteq \left\{x \in \mathbf{R}^d, \sum_{\alpha=0}^{\infty} a_{t_0,\alpha} \exp(ik_{t_0,\alpha} x) = 0\right\} \cup \{0\}.$$

In other words we proved that

$$\mathrm{supp}\mathcal{F}f \subseteq \bigcap_{t_0 \in T} (Z_{t_0} \cup \{0\}) = \{0\}.$$

By [**3**, Theorem 3.1], it follows,

$$\mathcal{F}f = \sum_{\alpha \in \mathbf{N}^d} c_\alpha \delta^{(\alpha)},$$

where for some $L > 0$ and $C > 0$ (resp. for each $L > 0$ there exists $C > 0$) the constants C_α, $\alpha \in \mathbf{N}^d$, satisfy the estimation of the form (2.3). Applying the inverse Fourier transform on both sides of the above equation, we obtain

$$f = \frac{1}{2\pi} \sum_{\alpha \in \mathbf{N}^d} c_\alpha x^{(\alpha)}.$$

Thus, f is an ultrapolynomial of class $*$.

(b) Suppose that (3.2) holds and that the function $f\colon \mathbf{R}^d \to \mathbf{C}$ is a regular tempered ultradistribution of class $*$. From the first part of this theorem it follows that there exists an ultrapolynomial P of class $*$, such that

$$\int_{\mathbf{R}^d} P\varphi dx = \int_{\mathbf{R}^d} f\varphi dx, \qquad \varphi \in \mathcal{S}^*.$$

Therefore $f(x) = P(x)$ for almost all $x \in \mathbf{R}^d$. □

4. Applications to functinal equations. As an illustration of our method, we prove the following two theorems, in which we consider a wider class of functional equations and their solutions than those considered in [**1**, Theorem 1 and Theorem 6].

THEOREM 2. *Let $\{b_\alpha\}_\alpha$ be a sequence of positive real numbers such that*

$$|b_\alpha| \leq \mathcal{C} l^\alpha / M_\alpha, \qquad \alpha \in \mathbf{N}, \tag{4.1}$$

for some $l > 0$ and $C > 0$ (resp. for each $l > 0$ there exists $C > 0$). Let $\sum_{\alpha=1}^\infty b_\alpha = 1$, and $\{h_\alpha\}_\alpha$ be a sequence in $\mathbf{R}^d$, such that the following implication holds:

$$\text{If } x \in \mathbf{R}^d \text{ and } h_\alpha x \in 2\pi\mathbf{Z} \text{ for all } \alpha \in \mathbf{N}, \text{ then } x = 0.$$

(In case $d = 1$ it is sufficient that h_α and h_β are rationally independent for some α and β such that $1 \leq \alpha \leq \beta$).

(a) *If $f \in \mathcal{S}'^*, Q$ is an ultrapolynomial of class $*$, and*

$$f = \sum_{\alpha=1}^\infty b_\alpha \mathcal{T}_{h_\alpha} f + Q,$$

then there exists an ultrapolynomial P such that $f = P$ in $\mathcal{S}'^*$.

(b) *If a function* $f\colon \mathbf{R}^d \to \mathbf{C}$ *is a regular tempered ultradistribution of class* $*$, *which satisfies*

$$f(x) = \sum_{\alpha=1}^{\infty} b_\alpha f(x+h_\alpha) + Q(x), \ \textit{ for almost all } x \in \mathbf{R}^d, \qquad (4.2)$$

where Q *is an ultrapolynomial of class* $*$, *then there exists an ultrapolynomial* P *of class* $*$ *such that* $f(x) = P(x)$, *for almost all* $x \in \mathbf{R}^d$.

Proof. (a) The equation (4.2) can be written equivalently as

$$\sum_{\alpha=0}^{\infty} a_\alpha \mathcal{T}_{k_\alpha} f = Q,$$

where $a_0 = 1$, $k_0 = 0$, $a_\alpha = -b_\alpha, k_\alpha = h_\alpha$, for $\alpha \in \mathbf{N}$. We will prove that the assumptions of Theorem 1 (a) are fulfiled, which implies that f is an ultrapolynomial of class $*$.

Put

$$Z = \left\{ x \in \mathbf{R}^d, \sum_{\alpha=0}^{\infty} a_\alpha \exp(ik_\alpha x) = 0 \right\},$$

and

$$g(x) = 1 - \sum_{\alpha=1}^{\infty} b_\alpha \exp(ih_\alpha x), \qquad x \in \mathbf{R}^d.$$

Suppose $x \in Z$, i.e. $x \in \mathbf{R}^d$ and $g(x) = 0$. Then

$$\sum_{\alpha=1}^{\infty} b_\alpha \exp(ih_\alpha x) = 1 = \sum_{\alpha=1}^{\infty} b_\alpha = \sum_{\alpha=1}^{\infty} b_\alpha |\exp(ih_\alpha x)|,$$

which implies that

$$\lim_{n\to\infty} \sum_{\alpha=1}^{n} b_\alpha (\exp(ih_\alpha x) - 1) = 0.$$

Since $b_\alpha > 0$, $\alpha \in \mathbf{N}$, we must have $\exp(ih_\alpha x) = 1$. Thus $h_\alpha x \in 2\pi\mathbf{Z}$, $\alpha \geq 1$. It follows $x = 0$.

We have proved that $Z \subset \{0\}$. This completes the proof of the assertion (a).

(b) As in (a) one can prove that the assumptions for the assertion (b) imply the assumptions of the assertion (b) in Theorem 2. Thus (b) follows from Theorem 2 (b). $\square$

As an illustration of the above assertion we consider the equation

$$\sum_{\alpha=1}^{d} (f(x+ke_\alpha) + f(x-ke_\alpha)) = 2df(x), \qquad x \in \mathbf{R}^d, \qquad (4.3)$$

where $k \in \mathbf{R}$, and $\{e_1, \dots e_d\}$ is the usual basis for $\mathbf{R}^d$. We will show that its classical solutions which are of ultrapolynomial growth are ultrapolynomials. Notice that the difference equation (4.3) is an analog of the Laplace equation in the d-dimensional case.

THEOREM 3. *Suppose that* $0 < a < b$ *and that* a/b *is irrational. Moreover suppose that a function* $f\colon \mathbf{R}^d \to \mathbf{C}$ *is a regular tempered ultradistribution of class* $*$ *and that* (4.3) *holds for almost all* $x \in \mathbf{R}^d$. *If* $k = a$ *or* $k = b$, *then there exists an ultrapolynomial* P *of class* $*$, *such that* $f(x) = P(x)$ *for almost all* $x \in \mathbf{R}^d$.

Proof. Assume $k = a$. The equation (4.3) can be written equivalently in the form

$$f(x) = \sum_{\alpha=1}^{2d} b_\alpha f(x + h_\alpha), \qquad x \in \mathbf{R}^d,$$

where

$$b_\alpha = \frac{1}{2d}, \ \alpha \in \{1, 2, \dots, 2d\}, \qquad h_\alpha = \begin{cases} ke_\alpha, & \alpha \in \{1, 2, \dots, d\}, \\ -ke_\alpha, & \alpha \in \{d+1, d+2, \dots, 2d\} \end{cases}$$

Now we can apply the assertion (b) in the previous theorem. □

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