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ON IMBEDDINGS OF WEIGHTED SOBOLEV SPACES ON AN UNBOUNDED DOMAIN

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Abstract. We obtained (necessary and sufficient) conditions on the weight functions v_0, v_1 and w for the imbedding $W^{1,p}(\Omega; v_0, v_1) \hookrightarrow W^{1,p}(\Omega; w)$ where Ω is an unbounded domain with nonempty boundary. It is shown that in the case when $v_0 = v_1$ the imbedding holds under weaker conditions.

1. Introduction. In the present paper, we are concerned with certain conditions on the weight functions v_0, v_1 and w for the continuous (and compact) imbedding

$$W^{1,p}(\Omega; v_0, v_1) \hookrightarrow W^{1,p}(\Omega; w), \quad \Omega \subset \mathbf{R}^N \text{ unbounded domain.} \tag{1.1}$$

It has been observed that the sufficient conditions for (1.1) are not necessary. Moreover, in the particular case when $v_0 = v_1 = v$, the sufficient as well as the necessary conditions for the imbedding

$$W^{1,p}(\Omega; v, v) \hookrightarrow W^{1,p}(\Omega; w) \tag{1.2}$$

are estabilished under weaker conditions than those for (1.1).

Further, Gurka and Opic [**4**] obtained conditions for the imbedding

$$W^{1,p}(\Omega; v_0, v_1) \hookrightarrow L^q(\Omega; w), \quad \Omega \subset \mathbf{R}^N \text{ unbounded domain.} \tag{1.3}$$

We also show that if $v_0 = v_1 = v$, then the imbedding

$$W^{1,p}(\Omega; v, v) \hookrightarrow L^q(\Omega; w) \tag{1.4}$$

holds, again under weaker conditions than those for (1.3).

We give notations and terminology in Section 2, the lemmas which are required in the proofs of our main results are given in Section 3. In Section 4, we

AMS Subject Classification (1991): Primary 46 E 35 discuss the continuous imbeddings while in Section 5 the compact imbeddings are considered.

2. Notations and terminology. Let Ω be a domain in $\mathbf{R}^N$. We denote by S, the set of weight functions on Ω, where a weight function is a function measurable and positive almost everywhere (a.e.) in Ω.

For $w \in S$, let us denote by $L^p(\Omega; w)$, $1 \le p < \infty$, the set of all functions $u = u(x)$ on Ω such that

$$\|u\|_{p;w} = \left(\int_\Omega |u(x)|^p w(x) dx \right)^{1/p} < \infty. \tag{2.1}$$

Also, for v_0, $v_1 \in S$, let us write

$$W^{1,p}(\Omega; v_0, v_1) = \{u \in L^p(\Omega; v_0)\colon \partial u/\partial x_i \in L^p(\Omega; v_1),\ \ i = 1, 2, \dots, N\}.$$

The spaces $L^p(\Omega; v)$ and $W^{1,p}(\Omega; v_0, v_1)$ are, respectively, known as weighted Lebesgue space and weighted Sobolev space. The two spaces are Banach spaces. The former one with the norm (2.1) while the later one with the norm

$$\|u\|_{1,p,v_0,v_1} = \left(\|u\|^p_{p,v_0} + \sum \left\| \frac{\partial u}{\partial x_i} \right\|^p_{p,v_1} \right)^{1/p}.$$

For various properties and applications of such spaces, one may refer to [**1**,**2**,**6**,**7**,**8**].

Given $x \in \mathbf{R}^N$, $R > 0$ and $h > 0$, write $B(x, R) = \{y \in \mathbf{R}^N;\ |x - y| < R\}$ and $hB(x, R) = B(x, hR)$.

Throughout this paper, it will be assumed that v_0, $v_1 \in S \bigcap L^1_{\text{loc}}(\Omega)$, $v_0^{-1/p}$, $v_1^{-1/p} \in L^{p'}_{\text{loc}}(\Omega)$, $[p' = p/(p-1)]$. Also, we shall be taking the domain Ω such that

$$\Omega = \bigcup_{n=1}^{\infty} \Omega_n, \tag{2.2}$$

where Ω_n are domains in $\mathbf{R}^N$ satisfying

$$\Omega_n \subset \Omega_{n+1} \subset \Omega, \qquad \Omega_{n+1} \neq \Omega \tag{2.3}$$

and we write $\Omega^n = \Omega \backslash \Omega_n$, $n \in \mathbf{N}$.

Suppose $n \in \mathbf{N}$, $I = (n, \infty)$ and $r\colon I \to (0, \infty)$. For $x \in \mathbf{R}$, write $I(x) = [x - r(x), x + r(x)]$.

We say that the function r, given above has the property $V(n)$ and write $r \in V(n)$ if

(i) r is continuous and nondecreasing on I

(ii) $x - r(x)$ is nondecreasing on I

(iii) $\lim_{x\to\infty}[x - r(x)] = \infty$, $\lim_{x\to n^-} r(x) > 0$

(iv) $r(x) \leq x/2$ for $x \in I$

(v) there is a constant $c_r \geq 1$ such that $c_r^{-1} \leq r(y)/r(x) \leq c_r$, for all $x \in I$ and all $y \in I(x) \bigcap I$.

Finally, we shall use the symbols $\hookrightarrow$ and $\hookrightarrow\hookrightarrow$, respectively, for continuous and compact imbeddings.

3. Lemmas. In this section, we collect certain results in the form of lemmas on which we rely heavily for the proofs of our main results.

LEMMA 1. [**5**] *Let $X(\Omega)$ and $Y(\Omega)$ be two Banach spaces of functions defined on Ω, where Ω is a domain satisfying* (2.2) *and* (2.3). *If*

$$X(\Omega_n) \hookrightarrow Y(\Omega_n), \qquad n \in \mathbf{N}$$

then a necessary and sufficient condition for the imbedding $x(\Omega) \hookrightarrow Y(\Omega)$ to hold is that

$$\lim_{n\to\infty} \left\{ \sup_{\|f\|_{x,\Omega}\leq 1} \|u\|_{Y,\Omega^n} \right\} < \infty. \tag{3.1}$$

LEMMA 2. [**5**] *Let $X(\Omega)$ and $Y(\Omega)$ be as in Lemma* 1. *If*

$$X(\Omega_n) \hookrightarrow\hookrightarrow Y(\Omega_n), \qquad n \in \mathbf{N}$$

then a necessary and sufficient condition for the imbedding $X(\Omega) \hookrightarrow\hookrightarrow Y(\Omega)$ to hold is that

$$\lim_{n\to\infty} \left\{ \sup_{\|f\|_{X,\Omega}\leq 1} \|u\|_{Y,\Omega^n} \right\} = 0.$$

LEMMA 3. [**4**] *Let $n \in \mathbf{N}$, $I = (n, \infty)$ and*

$$M = \left\{ x \in \mathbf{R}^N;\ |x| > n \right\}.$$

If $r\colon I \to (0,\infty)$ is a function which satisfies (i), (ii) and (iii) of the definition of $V(n)$, then there is a sequence $\{x_{ki}\} \subset M$ such that the following properties are satisfied:

(a) *$M \subset \bigcup_{k,i=1}^{\infty} B_{k,i}$, where $B_{k,i} = B(x_{ki}, r(|x_{ki}|))$*

(b) *there exists a number τ depending only on the dimension N such that*

$$\sum_{k,i=1}^{\infty} \chi_{B_{k,i}}(z) \leq \tau, \qquad \forall z \in \mathbf{R}^N.$$

LEMMA 4. [3] *Let* $1 \le p,\ q < \infty$, $1/N \ge 1/p - 1/q$, $R > 0$ *and* $x \in \mathbf{R}^N$. *Then for* $u \in W^{1,p}(B(x,R))$,

$$\left(\int\limits_{B(x,R)} |u(y)|^q dy\right)^{1/q}$$
$$\le K R^{N/q-n/p+1} \left(R^{-p} \int\limits_{B(x,R)} |u(y)|^p dy + \int\limits_{B(x,R)} |\nabla u(y)|^p dy\right)^{1/p},$$

where $|\nabla u(y)|^p = \sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}(y)\right|^p$ *and* $K > 0$ *(independent of* x, R *and* u*).*

From now on Ω will be an unbounded domain and $\Omega_n = \{z \in \Omega; |z| < n\}$, $n \in \mathbf{N}$.

Clearly, the sequence $\{\Omega_n\}$ satisfies (2.2) and (2.3) and consequently Lemmas 1 and 2 hold for this Ω and $\{\Omega_n\}$ also.

LEMMA 5. [4] *Let* $n_0 \in \mathbf{N}$, $I = (n_0, \infty)$ *and* $r: I \to (0,\infty)$ *be a function such that* $r(y) \le y/2$, $y \in I$. *If* $n \ge n_0$, $B(x, r(|x|)) \bigcap \Omega^{3n} \ne 0$, *then* $|z| > n$, *for every* $z \in B(x, r(|x|))$.

4. Continuous imbeddings. In this section, we discuss the conditions under which the continuous imbedding (1.2) holds. Also, particular cases of the imbeddings (1.2) and (1.3) are discussed. We first prove

THEOREM 1. *Let* Ω *be a domain in* $\mathbf{R}^N$, $1 \le p \le \infty$ *and let the following conditions be satisfied:*

S1 *there exists* $n_0 \in \mathbf{N}$ *such that* $\Omega^{n_0} = \{x \in \mathbf{R}^N;\ |x| > n_0\}$;

S2 $W^{1,p}(\Omega_n; v_0, v_1) \hookrightarrow W^{1,p}(\Omega_n; w)$, $n \ge n_0$;

S3 *there exists positive measurable functions* a_0, a_1 *defined on* Ω^{n_0} *and a function* $r \in V(n_0)$ *such that*

$$w(y) \le a_0(x), \tag{4.1}$$
$$\left(1 + r^{-p}(|y|)\right) a_1(x) \le v_1(y); \tag{4.2}$$

for all $x \in \Omega^{n_0}$ *and for a.e.* $y \in B(x, r(|x|))$.

S4 *there exists a constant* $K_0 > 0$ *such that*

$$v_1(x) r^{-p}(|x|) \le K_0 v_0(x), \qquad \text{for a.e. } x \in \Omega^{n_0}.$$

S5 $\lim_{n\to\infty} A_n = A < \infty$, *where*

$$A_n = \sup_{x \in \Omega^n} \frac{a_0(x)}{a_1(x)} r^p(|x|). \tag{4.3}$$

Then

$$W^{1,p}(\Omega; v_0, v_1) \hookrightarrow W^{1,p}(\Omega; w). \tag{4.4}$$

Proof. Let $X(\Omega) = W^{1,p}(\Omega; v_0, v_1)$ and $Y(\Omega) = W^{1,p}(\Omega; w)$. Then, by Lemma 1, it is sufficient to verify (3.1) with Ω^{3n} instead of Ω^n. Taking $M = \Omega^{n_0}$ in Lemma 3, there exists a sequence $\{x_{ki}\} \subset M$ such that

$$\Omega^{n_0} \subset \bigcup_{k,i=1}^{\infty} B_{ki}, \tag{4.5}$$

and there exists a number τ depending only on the dimension N with

$$\sum_{k,i=1}^{\infty} \chi_{B_{k,i}}(z) \leq \tau, \qquad z \in \mathbf{R}^N. \tag{4.6}$$

Let $n \geq n_0$ be fixed. Write

$$\mathbf{K}_n = \left\{(k,i) \in \mathbf{N} \times \mathbf{N};\ B_{ki} \bigcap \Omega^{3n} \neq 0\right\}.$$

Then, by Lemma 5, we have

$$\bigcup_{(k,i)\in\mathbf{K}_n} B_{ki} \subset \Omega^n \subseteq \Omega^{n_0}. \tag{4.7}$$

In view of (4.5), we get

$$\|u\|^p_{1,p,\Omega^{3n},w} \leq \sum_{(k,i)\in\mathbf{K}_n} \left(\int\limits_{B_{ki}} |u(y)|^p w(y) dy + \int\limits_{B_{ki}} |\nabla u(y)|^p w(y) dy \right). \tag{4.8}$$

Using (4.1) and Lemma 4 with $p = q$, we obtain

$$\begin{aligned} &\int\limits_{B_{ki}} |u(y)|^p w(y) dy + \int\limits_{B_{ki}} |\nabla u(y)|^p w(y) dy \\ &\leq a_0(x_{ki}) \left(\int\limits_{B_{ki}} |u(y)|^p dy + \int\limits_{B_{ki}} |\nabla u(y)|^p dy \right) \\ &\leq a_0(x_{ki}) \left((Kr(|x_{ki}|))^p \left\{ r^{-p}(|x_{ki}|) \int\limits_{B_{ki}} |u(y)|^p dy \right.\right. \\ &\qquad \left.\left. + \int\limits_{B_{ki}} |\nabla u(y)|^p dy \right\} + \int\limits_{B_{ki}} |\nabla u(y)|^p dy \right) \\ &\leq a_0(x_{ki}) K_1 r^p(|x_{ki}|) \left(r^{-p}(|x_{ki}|) \int\limits_{B_{ki}} |u(y)|^p dy \right. \\ &\qquad \left. + (1 + r^{-p}(|x_{ki}|)) \int\limits_{B_{ki}} |\nabla u(y)|^p dy \right) \end{aligned} \tag{4.9}$$

where $K_1 = \max(K^p, 1)$.

Since $r \in V(n_0)$, as a consequence of the definition of $V(n)$, for $x \in \mathbf{R}^N$, $|x| > n_0$ and for $y \in B(x, r(|x|))$, $|y| > n_0$ we have

$$c_r^{-1} \leq \frac{r(|y|)}{r(|x|)} \leq c_r. \tag{4.10}$$

But, in view of (4.7), (4.10) also holds for $y \in B_{ki}$. Using this fact along with S4 and (4.2), (4.9) takes the form

$$\begin{aligned}
&\int\limits_{B_{ki}} |u(y)|^p w(y) dy + \int\limits_{B_{ki}} |\nabla u(y)|^p w(y) dy \\
&\leq a_0(x_{ki}) K_1 r^p(|x_{ki}|) \left(c_r^p \int\limits_{B_{ki}} |u(y)|^p \frac{dy}{r^p(|y|)} + c_r^p \int\limits_{B_{ki}} |\nabla u(y)|^p \left(1 + \frac{1}{r^p(|y|)}\right) dy \right) \\
&\leq \frac{a_0(x_{ki})}{a_1(x_{ki})} K_1 c_r^p r^p(|x_{ki}|) \left(\int\limits_{B_{ki}} |u(y)|^p v_0(y) dy + \int\limits_{B_{ki}} |\nabla u(y)|^p v_1(y) dy \right) \\
&\leq K_2 A_n \left(\int\limits_{B_{ki}} |u(y)|^p v_0(y) dy + \int\limits_{B_{ki}} |\nabla u(y)|^p v_1(y) dy \right),
\end{aligned}$$

using (4.3), where $K_2 = K_1 c_r^p$. Thus, (4.8) reduces to

$$\|u\|^p_{1,p,\Omega^{3n},w} \leq \tau K_2 A_n \|u\|^p_{1,p,\Omega,v_0,v_1}, \tag{4.11}$$

in view of (4.6), which on using S5 proves the assertion. □

Towards the converse of Theorem 1, we prove the following

THEOREM 2. *Let Ω be a domain in $\mathbf{R}^N$. Let* S1 *and the following conditions be satisfied:*

N3 *there exist positive measurable functions $\hat{a}_0$, $\hat{a}_1$ defined on Ω^{n_0} and a function $r \in V(n_0)$ such that $w(y) \geq \hat{a}_0(x)$, $\hat{a}_1(x) \geq v_1(y)$, for all $x \in \Omega^{n_0}$ and for a.e. $y \in B(x, r(|x|))$.*

N4 *there exists a constant $K_0 > 0$ such that*

$$K_0 v_0(x) \leq v_1(x) r^{-p}(|x|), \qquad \textit{for a.e. } x \in \Omega^{n_0}.$$

N5 *$\lim_{n\to\infty} \hat{A}_n = \infty$, where*

$$\hat{A}_n = \sup_{x \in \Omega^n} \frac{\hat{a}_0(x)}{\hat{a}_1(x)} r^p(|x|).$$

Then $W^{1,p}(\Omega, v_0, v_1)$ is not imbedded in $W^{1,p}(\Omega; w)$.

Proof. In view of N5, there exist an increasing sequence of natural numbers $\{n_k\}$ and a sequence $\{x_k\}$ with $x_k \in \Omega^{n_k}$ such that

$$\frac{\hat{a}_0(x_k)}{\hat{a}_1(x_k)} r^p(|x_k|) > \kappa, \qquad k \in \mathbf{N}, \tag{4.12}$$

If we set

$$u_\kappa = R_{r(x_k)/8} \chi_{3/4B_k}, \qquad k = 1, 2, \ldots$$

where R_ε is a mollifier with radius ε defined in sence of Gurka and Opic [**3**, Theorem 2.4], then the functions u_k have the following properties:

(i) $u_k \in c_0^\infty(B_k)$, $0 \leq u_k \leq 1$

(ii) $u_k \equiv 1$ on $\frac{1}{2}B_k$

(iii) there exists $c > 0$ such that

$$\left|\frac{\partial u_k}{\partial x_i}(x)\right| \leq \frac{c}{r(|x_k|)}, \qquad x \in \Omega, \quad i = 1, 2, \ldots, N$$

(iv) $u_k \in W^{1,p}(\Omega; v_0, v_1)$.

The property (ii) above gives that $\frac{\partial u_k}{\partial x_i} = 0$, on $\frac{1}{2}B_k$ and as a consequence we have

$$|\nabla u_k(y)| = 0 \qquad \text{on} \qquad \frac{1}{2}B_k. \tag{4.13}$$

Now, by N3, (4.13) and the property (ii) of the function u_k, we get

$$\begin{aligned} \int_\Omega |u_k(y)|^p w(y)dy + \int_\Omega |\nabla u_k(y)|^p w(y)dy &\geq \int_{(1/2)B_k} w(y)dy \\ &\geq 2^{-N}|B(0,1)|\hat{a}_0(x_k) r^N(|x_k|). \end{aligned} \tag{4.14}$$

Also, it can be shown that

$$\int_\Omega |u_k(y)|^p v_0(y)dy + \int_\Omega |\nabla u_k(y)|^p v_1(y)dy \leq L r^{N-p}(|x_k|)\hat{a}_1(x_k), \tag{4.15}$$

where $L = (K_0^{-1} c_r^p + N c^p)|B(0,1)$ with c_r given by (4.10).

Now, suppose that the imbedding

$$W^{1,p}(\Omega; v_0, v_1) \hookrightarrow W^{1,p}(\Omega; w), \tag{4.16}$$

holds; then (4.14) and (4.15) give

$$2^{-N}|B(0,1)|\hat{a}_0(x_k) r^N(|x_k|) \leq \tilde{K} L r^{N-p}(|x_k|)\hat{a}_1(x_k),$$

for $k \in \mathbf{N}$, where $\tilde{K}$ is the norm of the imbedding operator from (4.16). This is a contradiction to (4.12) and hence the theorem follows. $\square$

Remark 1. It can be observed from the proof of Theorem 2, that not only the space $W^{1,p}(\Omega; v_0, v_1)$, but also the space $W_0^{1,p}(\Omega; v_0, v_1)$ is not continuously imbedded in $W^{1,p}(\Omega; w)$.

In the case $v_0 = v_1$, the imbedding (4.4) holds under weaker conditions in the sence that we do not require (4.10) for the imbedding, and as a consequence, the definition of $V(n)$ can be weakened. Moreover, in this case, the inequality (4.2) can also be replaced by a weaker one.

We say that the function r has the property $\nabla(n)$ (written $r \in \nabla(n)$) if r satisfies the conditions (i) – (iv) in the difinition of $V(n)$.

In the light of the above discussion, we have the following result.

THEOREM 3. *Let Ω be a domain in $\mathbf{R}^N$, $1 \le p < \infty$. Let* S2 *and* S4 *with $v_0 = v_1 = v$ along with* S1, S5 *and the following condition be satisfied*

$\overline{\text{S3}}$ *there exist positive measurable functions a_0, a_1 defined on Ω^{n_0} and a function $r \in \overline{\nabla}(n_0)$ such that* (4.1) *and*

$$a_1(x) \le v(y) \tag{$\overline{4.2}$}$$

hold for all $x \in \Omega^{n_0}$ and for a.e. $y \in B(x, r(|x|))$. Then the following imbeding holds

$$W^{1,p}(\Omega; v, v) \hookrightarrow W^{1,p}(\Omega; w). \tag{4.17}$$

Proof. Using ($\overline{4.2}$) in (4.9), we have

$$\begin{aligned}
&\int\limits_{B_{ki}} |u(y)|^p w(y)dy + \int\limits_{B_{ki}} |\nabla u(y)|^p w(y)dy \\
&\qquad \le a_0(x_{ki}) K_3 r^p(|x_{ki}|) \left(\int\limits_{B_{ki}} |u(y)|^p dy + \int\limits_{B_{ki}} |\nabla u(y)|^p dy \right) \\
&\qquad \le \frac{a_0(x_{ki})}{a_1(x_{ki})} K_3 r^p(|x_{ki}|) \left(\int\limits_{B_{ki}} |u(y)|^p v(y)dy + \int\limits_{B_{ki}} |\nabla u(y)|^p v(y)dy \right) \\
&\qquad \le K_3 A_n \left(\int\limits_{B_{ki}} |u(y)|^p v(y)dy + \int\limits_{B_{ki}} |\nabla u(y)|^p v(y)dy \right),
\end{aligned}$$

where $K_3 = K_1 \max(K_0, 1 + K_0)$. The proof now follows along the same lines as that of Theorem 1.

Towards the converse of Theorem 3, we prove the following result.

THEOREM 4. *Let* N3 *with $v_0 = v_1 = v$ along with* S1 *and* S5 *be satisfied. Then the space $W^{1,p}(\Omega, v, v)$ is not continuously imbedded in $W^{1,p}(\Omega, w)$.*

Proof. It follows on the lines of the proof of Theorem 2. □

In view of Theorems 1 and 2 (and similarly Theorems 3 and 4), it may be pointed out that necessary and sufficient conditions for the imbedding (4.4) [resp. (4.17)] are different. It remains open to achieve suitable sets of conditions which are both necessary and sufficient.

Gurka and Opic proved the following [**4**, Theorem 12.1].

THEOREM A. *Let* Ω *be a domain in* $\mathbf{R}^N$, $1 \le p \le q < \infty$, $1/N \ge 1/p - 1/q$. *Let* S1, S4 *and the following conditions be satisfied:*

S2′ $\quad W^{1,p}(\Omega_n; v_0, v_1) \hookrightarrow L^q(\Omega_n; w)$, $n \ge n_0$;

S3′ *there exist positive measurable functions* a_0, a_1 *defined on* Ω^{n_0} *and a function* $r \in V(n_0)$ *such that* $w(y) \le a_0(x)$, $a_1(x) \le v_1(y)$ *for all* $x \in \Omega^{n_0}$ *and for a.e.* $y \in B(x, r(|x|))$.

S5′ $\quad \lim_{n\to\infty} A'_n < \infty$, *where*

$$A'_n = \sup_{x\in\Omega^n} \frac{a_0^{1/q}(x)}{a_1^{1/p}(x)} r^{\frac{N}{q}-\frac{N}{p}+1}(|x|).$$

Then the imbedding $W^{1,p}(\Omega; v_0, v_1) \hookrightarrow L^q(\Omega; w)$ *holds.*

THEOREM B. *Let* Ω *be a domain in* $\mathbf{R}^N$, $1 \le p$, $q < \infty$. *Let* S1, N3, N4 *and the condition*

N5′ $\quad \lim_{n\to\infty} \hat{A}'_n < \infty$, *where*

$$\hat{A}'_n = \sup_{x\in\Omega^n} \frac{\hat{a}_0^{1/q}(x)}{\hat{a}_1^{1/p}(x)} r^{\frac{N}{q}-\frac{N}{p}+1}(|x|)$$

be satisfied. Then the space $W^{1,p}(\Omega; v_0, v_1)$ *is not continuously imbedded in the space* $L^q(\Omega; w)$.

Here, the particular case when $v_0 = v_1$ also needs attention i.e. we can weaken the conditions of Theorems A and B as in Theorem 3 and 4, respectively, although in our case the necessary and sufficient conditions are not the same. More precisely, we have the following

THEOREM 5. *Let* Ω *be a domain in* $\mathbf{R}^N$, $1 \le p \le q < \infty$ *and* $1/N \ge 1/p - 1/q$. *Let* S2′ *and* S4 *with* $v_0 = v_1 = v$ *along with* S1, $\overline{\text{S3}}$ *and* S5′ *be satisfied. Then the imbedding* $W^{1,p}(\Omega; v, v) \hookrightarrow L^q(\Omega; w)$ *holds.*

Proof. If we replace $\|u\|^p_{1,p,\Omega^{3n},w}$ by $\|u\|^q_{q,\Omega^{3n},w}$ in (4.8) and use Lemma 4, then the proof goes along the same lines as that of Theorem 3. □

THEOREM 6. *Let* Ω, p, q *be as in Theorem B. Let* N3 *and with* $v_0 = v_1 = v$ *along with* S1 *and* N5′ *be satisfied. Then* $W^{1,p}(\Omega; v, v)$ [*and also* $W_0^{1,p}(\Omega; v, v)$] *is not continuously imbedded in* $W^q(\Omega; w)$.

5. Compact imbeddings. The discussion made in Section 4 about the continuous imbeddings is carried over in this section for compact imbeddings. We omit the details for conciseness.

THEOREM 7. *Let* S1, S3, S4 *and the following conditions be satisfied:*

S2∗ $W^{1,p}(\Omega_n; v_0, v_1) \hookrightarrow\hookrightarrow W^{1,p}(\Omega_n; w)$, $n \geq n_0$.

S5∗ $\lim_{n\to\infty} A_n = 0$, *where* A_n *is given by* (4.3).

Then $W^{1,p}(\Omega; v_0, v_1) \hookrightarrow\hookrightarrow W^{1,p}(\Omega; w)$.

Proof. If we use S5∗ in (4.11), then the assertion follows immediately in view of Lemma 2.□

THEOREM 8. *Let* S1, N3, N4 *and*

N5∗ $\lim_{n\to\infty} \hat{A}_n < \infty$, *where*

$$\hat{A}_n = \sup_{x\in\Omega^n} \frac{\hat{a}_0(x)}{\hat{a}_1(x)} r^p(|x|)$$

be satisfied. Then the space $W^{1,p}(\Omega; v_0, v_1)$ [*and also* $W_0^{1,p}(\Omega; v_0, v_1)$] *is not compactly imbedded in* $W^{1,p}(\Omega; w)$.

THEOREM 9. *Let the condition* S2 *and* S4 *with* $v_0 = v_1 = v$ *along with* S1, S$\overline{3}$ *and* S5∗ *be satisfied. Then the imbedding* $W^{1,p}(\Omega; v, v) \hookrightarrow\hookrightarrow W^{1,p}(\Omega; w)$ *holds.*

THEOREM 10. *Let* N3 *with* $v_0 = v_0 = v$ *along with* S1 *and* N5∗ *be satisfied. Then* $W^{1,p}(\Omega; v, v)$ [*and also* $W_0^{1,p}(\Omega; v, v)$] *is not compactly imbedded in* $W^{1,p}(\Omega; w)$.

Remark 3. It is again open to seal the gap between the sets of necessary conditions obrained in Theorem 7 and sufficient conditions obtained in Theorem 8 (and similarly in Theorems 9 and 10).

Further, subsequent results analogous to Theorems 5 and 6 can also be obtained in respect of the compact imbedding.

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