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# ON THE AC-CONTACT BOCHNER CURVATURE TENSOR FIELD ON ALMOST COSYMPLECTIC MANIFOLDS

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**Abstract.** On an almost cosymplectic manifold we define a new modified contact Bochner curvature tensor field which is invariant with respect to $D$-homothetic deformation. Then we generalize a theorem of Olszak [**5**] and describe some manifolds with vanishing its new modified contact Bochner curvature tensor field.

**1. Introduction.** Olszak [**5**, Theorem 6.2] got the necessary and sufficient condition for a conformally flat almost cosymplectic manifold to be cosymplectic. On the other hand, Matsumoto and Chuman [**4**] defined contact Bochner curvature tensor in Sasakian manifolds (see also Yano [**8**]). This tensor is invariant with respect to $D$-homothetic deformations (see Tanno [**7**] about $D$-homothetic deformations). In this paper we modify contact Bochner curvature tensor and define a new modified contact Bochner curvature tensor field which is invariant with respect to $D$-homothetic deformations of an almost cosymplectic manifold. We call it $AC$-contact Bochner curvature tensor. Then, by using $AC$-contact Bochner curvature, we get a generalization of an Olszak's theorem [**5**, Theorem 6.2]. Moreover, we consider an almost cosymplectic manifold with constant $\phi$-sectional curvature and another one with vanishing $AC$-contact Bochner curvature.

**2. Preliminaries**. Let $(M,\phi,\xi,\eta,g)$ be a $(2n+1)$-dimensional almost contact Riemannian manifold, that is, let $M$ be a differentiable manifold and $(\phi,\xi,\eta,g)$ an almost contact Riemannian structure on $M$, formed by tensor fields $\phi,\xi,\eta$, of type (1,1), (1,0) and (0,1), respectively, and a Riemannian metric $g$ such that

$$\begin{gathered}\phi^2=-I+\eta\otimes\xi,\quad \phi\xi=0,\quad \eta\cdot\phi=0,\quad \eta(\xi)=1,\\ \eta(X)=g(X,\xi),\quad g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y).\end{gathered}\tag{2.1}$$

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On such a manifold we may always define 2-form $\Phi$ by $\Phi(X,Y) = g(\phi X, Y)$. Then $(M, \phi, \xi, \eta, g)$ is said to be an almost cosymplectic manifold if the forms $\Phi$ and $\eta$ are closed, i.e., if $d\Phi = 0$, $d\eta = 0$, where $d$ is a exterior differentiation. On an almost cosymplectic manifold we define an operator $h$ by $h = -\frac{1}{2}\mathcal{L}_\xi \phi$, where $\mathcal{L}$ denotes the Lie differentiation. Then we see that $h$ is symmetric, $h$ anti-commutes with $\phi$ (i.e., $\phi h + h\phi = 0$), $h\xi = 0$, $\nabla_X \xi = \phi h X$ and $\mathrm{Tr}\, h = 0$, where $\nabla$ is the covariant differentiation with respect to $g$ and $\mathrm{Tr}\, h$ is the trace of $h$ (see [**2**]). From $\phi h \xi = 0$, we notice

$$(\nabla_Y(\phi h))\xi = -h^2 Y \tag{2.2}$$

$$(\nabla_\xi(\phi h))X = R(\xi, X)\xi - h^2 X, \tag{2.3}$$

where $R$ is the curvature tensor ($R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$). Furthermore, the following are satisfied [**2**]:

$$g(R(Y,\xi)\xi, Z) + g(R(\phi Y, \xi)\xi, \phi Z) + 2g(hY, hZ) = 0 \tag{2.4}$$

$$g(Q\xi, \xi) = -\mathrm{Tr}\, h^2, \tag{2.5}$$

where $Q$ is the Ricci operator. If an almost contact structure of an almost cosymplectic manifold is normal, then it is said to be a cosymplectic manifold. As it is known, an almost contact metric structure is cosymplectic if and only if both $\nabla\eta$ and $\nabla\phi$ vanish ([**3**]; see also [**2**] and [**5**]). However, if we have $\nabla\phi = 0$, then, we can easily get $\nabla\eta = 0$ by taking the covariant differentiation of $\phi\xi = 0$. In a cosymplectic manifold $M$ with structure tensor $(\phi, \xi, \eta, g)$, from $\nabla\xi = 0$ we have

$$R(X,Y)\xi = 0 \tag{2.6}$$

for any vector fields $X$ and $Y$ on $M$, wherefrom

$$Q\xi = 0. \tag{2.7}$$

Using $\nabla\phi = 0$ and $R(X,Y)\phi Z = \nabla_X \nabla_Y(\phi Z) - \nabla_Y \nabla_X(\phi Z) - \nabla_{[X,Y]}\phi Z$, we find

$$R(X,Y)\phi Z = \phi R(X,Y)Z. \tag{2.8}$$

Thus, using the property of the curvature tensor, we get

$$R(\phi X, \phi Y)Z = R(X,Y)Z. \tag{2.9}$$

From (2.9), we find $R(\phi X, Y)Z = -R(X, \phi Y)Z$. Moreover we have

$$\phi Q = Q\phi, \tag{2.10}$$

where we used

$$\begin{aligned} g(Q\phi Y, \phi Z) &= \sum_{i=1}^{2n+1} g(R(E_i, \phi Y)\phi Z, E_i) = -\sum_{i=1}^{2n+1} g(R(\phi E_i, Y)\phi Z, E_i) \\ &= -\sum_{i=1}^{2n+1} g(\phi R(\phi E_i, Y)Z, E_i) = \sum_{i=1}^{2n+1} g(R(\phi E_i, Y)Z, \phi E_i) = g(QY, Z), \end{aligned}$$

where $\{E_i,\ 1 \leq i \leq 2n+1\}$ is a $\phi$-basis ($E_{n+t} = \phi E_t,\ 1 \leq t \leq n;\ E_{2n+1} = \xi$).

By $D$ we denote the distribution of an almost contact metric manifold $M$ defined by $\eta = 0$. $M$ is said to be of pointwise constant $\phi$-sectional curvature if at any point $x \in M$, the sectional curvature $K(X, \phi X)$ is independent of the choice of non-zero $X \in D_x$, In this case, the $\phi$-sectional curvature $K$ is a function on $M$.

An almost contact metric manifold is said to be $\eta$-Einstein if $Q = aI + b\eta \otimes \xi$, where $a$ and $b$ are smooth functions on $M$. Especially if $b = 0$, then $M$ is said to be Einstein.

On a $(2n+1)$-dimensional almost cosymplectic manifold $M$ the Weyl conformal curvature tensor of $M$ is the tensor field $C$ of type (1,3) defined by

$$\begin{aligned} C(X,Y)Z &= R(X,Y)Z \\ &+ \frac{1}{2n-1}(g(QX,Z)Y - g(QY,Z)X + g(X,Z)QY - g(Y,Z)QX) \\ &- \frac{S}{2n(2n-1)}(g(X,Z)Y - g(Y,Z)X) \end{aligned} \tag{2.11}$$

for any vector fields $X$, $Y$ and $Z$ on $M$ (where $S$ is the scalar curvature). Moreover we put

$$c(X,Y) = (\nabla_X Q)Y - (\nabla_Y Q)X - \frac{1}{2(2n-1)}((\nabla_X S)Y - (\nabla_Y S)X). \tag{2.12}$$

Then it is well-known that a necessary and sufficent condition for $M$ to be conformally flat is that $C = 0$ for $n > 3$ and $c = 0$ for $n = 3$ ($C$ vanishes identically for $n = 3$).

**3. $D$-homothetic deformations**. Let $M$ be an $(m+1)$-dimensional ($m = 2n$) almost cosymplectic manifold. Now we define a tensor field $B^{ac}$ on $M$ by

$$\begin{aligned} B^{ac}(X,Y) &= R(X,Y) + \phi hX \wedge \phi hY \\ &+ \frac{1}{2(m+4)}(QY \wedge X - (\phi Q\phi Y) \wedge X + \frac{1}{2}(\eta(Y)Q\xi \wedge X + \eta(QY)\xi \wedge X) \\ &- QX \wedge Y + (\phi Q\phi X) \wedge Y - \frac{1}{2}(\eta(X)Q\xi \wedge Y + \eta(QX)\xi \wedge Y) + (Q\phi Y) \wedge \phi X \\ &+ (\phi QY) \wedge \phi X - (Q\phi X) \wedge \phi Y - (\phi QX) \wedge \phi Y + 2g(Q\phi X, Y)\phi \\ &+ 2g(\phi QX, Y)\phi + 2g(\phi X, Y)\phi Q + 2g(\phi X, Y)Q\phi - \eta(X)QY \wedge \xi \\ &+ \eta(X)(\phi Q\phi Y) \wedge \xi + \eta(Y)QX \wedge \xi - \eta(Y)(\phi Q\phi X) \wedge \xi), \end{aligned} \tag{3.1}$$

where $(X \wedge Y)Z = g(Y,Z)X - g(X,Z)Y$ (c.f. [**4**]). Using (3.1), $B^{ac}$ satisfies the following identities in an almost cosymplectic manifold $M$.

$$\begin{gathered} B^{ac}(X,Y)Z = -B^{ac}(Y,X)Z, \\ B^{ac}(X,Y)Z + B^{ac}(Y,Z)X + B^{ac}(Z,X)Y = 0 \\ g(B^{ac}(X,Y)Z, W) = -g(Z, B^{ac}(X,Y)W), \\ g(B^{ac}(X,Y)Z, W) = g(B^{ac}(Z,W)X, Y). \end{gathered} \tag{3.2}$$

If $M$ is a cosymplectic manifold, then $B^{ac}$ turns into the following $B^c$ because of (2.7), (2.10) and (3.1).

$$\begin{aligned} B^c(X,Y) = R(X,Y) + \frac{1}{m+4}(&QY \wedge X - QX \wedge Y + Q\phi Y \wedge \phi X - Q\phi X \wedge \phi Y \\ &+ 2g(Q\phi X, Y)\phi + 2g(\phi X, Y)Q\phi + \eta(Y)QX \wedge \xi + \eta(X)\xi \wedge QY). \end{aligned}$$

$B^c$ is the main part of the contact Bochner curvature tensor (Matsumoto and Chuman [**4**]). Moreover, the following are satisfied in a cosymplectic manifold $M$.

$$\begin{aligned} B^{ac}(\xi,Y)Z = B^c(\xi,Y)Z = B^c(X,Y)\xi = B^{ac}(X,Y)\xi = 0 \\ B^{ac}(\phi X,\phi Y)Z = B^c(\phi X,\phi Y)Z = B^c(X,Y)Z = B^{ac}(X,Y)Z, \end{aligned} \tag{3.3}$$

where we used (2.4), (2.6), (2.8) and (2.9).

We consider a $D$-homothetic deformation $g^* = \alpha g + \alpha(\alpha-1)\eta\otimes\eta$, $\phi^* = \phi$, $\xi^* = \alpha^{-1}\xi$, $\eta^* = \alpha\eta$ on an almost cosymplectic manifold $M$, where $\alpha$ is a positive constant. For a $D$-homothetic deformation we say that $M(\phi,\xi,\eta,g)$ is $D$-homothetic to $M(\phi^*,\xi^*,\eta^*,g^*)$. It is easy to see that if an almost cosymplectic manifold $M(\phi,\xi,\eta,g)$ is $D$-homothetic to $M(\phi^*,\xi^*,\eta^*,g^*)$, then $M(\phi^*,\xi^*,\eta^*,g^*)$ is an almost cosymplectic manifold. Moreover if $M(\phi,\xi,\eta,g)$ is a cosymplectic manifold, then $M(\phi^*,\xi^*,\eta^*,g^*)$ is also a cosymplectic manifold. Denoting by $W^i_{jk}$ the difference $\Gamma^{*i}_{jk} - \Gamma^i_{jk}$ of Christoffel symbols, by $\nabla_X\xi = \phi hX$ (hence $(\nabla_X\eta)Y = (\nabla_Y\eta)X$) we have in an almost cosymplectic manifold $M$ [**6**]

$$W(X,Y) = \frac{\alpha-1}{\alpha}(\nabla_X\eta)(Y)\xi = \frac{\alpha-1}{\alpha}g(\phi hX,Y)\xi.$$

Putting this into

$$\begin{aligned} R^*(X,Y)Z = R(X,Y)Z &+ (\nabla_X W)(Z,Y) - (\nabla_Y W)(Z,X) \\ &+ W(W(Z,Y),X) - W(W(Z,X),Y) \end{aligned}$$

and using $\nabla_X\xi = \phi hX$, we find

$$\begin{aligned} R^*(X,Y)Z = R(X,Y)Z &+ \frac{\alpha-1}{\alpha}(g(Y,(\nabla_X(\phi h))Z)\xi \\ &- g(X,(\nabla_Y(\phi h))Z)\xi + g(Y,\phi hZ)\phi hX - g(X,\phi hZ)\phi hY). \end{aligned} \tag{3.4}$$

Here, choosing $\phi^*$-basis with respect to $g^*$ and using (2.2) and (2.3), we get

$$\mathrm{Ric}^*(Y,Z) = \mathrm{Ric}(Y,Z) - \frac{\alpha-1}{\alpha}(g(R(Z,\xi)\xi,Y) + g(h^2Y,Z)), \tag{3.5}$$

where Ric is the Ricci curvature of $M$. From (3.5) we have

$$Q^*Y = \frac{1}{\alpha}QY - \frac{\alpha-1}{\alpha^2}\eta(QY)\xi - \frac{\alpha-1}{\alpha^2}(h^2Y - R(\xi,Y)\xi), \tag{3.6}$$

where we used $\eta(Q^*Y) = \alpha^{-2}\eta(QY)$. By virtue of (3.5) we find

$$S^* = \frac{1}{\alpha}S - 2\frac{\alpha-1}{\alpha^2}\mathrm{Ric}(\xi,\xi). \tag{3.7}$$

Moreover if we consider the $D$-homothetic deformation of $\mathcal{L}_\xi\phi$, we find

$$h^* = \frac{1}{\alpha}h, \tag{3.8}$$

wherefrom we get

$$\operatorname{Tr} h^{*^2} = \frac{1}{\alpha^2}\operatorname{Tr} h^2. \tag{3.9}$$

After a clumsy computations we obtain, by means of (2.4), (3.1), (3.4), (3.5), (3.6) and (3.8), the following

$$\begin{aligned}\overset{*}{B}{}^{ac}(X,Y)Z = {} & B^{ac}(X,Y)Z + \frac{\alpha-1}{\alpha}(g(Y,(\nabla_X(\phi h))Z)\xi - g(X,(\nabla_Y(\phi h))Z)\xi) \\ & + \frac{1}{2}\frac{\alpha-1}{\alpha}(\eta(X)g(Q\xi,\xi)g(Y,Z)\xi - \eta(Y)g(Q\xi,\xi)g(X,Z)\xi) \\ & + \frac{3}{2}\frac{\alpha-1}{\alpha}(g(Y,Z)\eta(QX)\xi - g(X,Z)\eta(QY)\xi \\ & \qquad + \eta(X)\eta(Z)\eta(QY)\xi - \eta(Y)\eta(Z)\eta(QX)\xi) \\ & + \frac{\alpha-1}{\alpha}(\eta(Q\phi X)g(\phi Y,Z)\xi - \eta(Q\phi Y)g(\phi X,Z)\xi - 2\eta(Q\phi Z)g(\phi X,Y)\xi).\end{aligned} \tag{3.10}$$

Now we shall introduce the AC-contact Bochner curvature tensor in $M$ by

$$AC(X,Y)Z = B^{ac}(X,Y)Z - \eta(B^{ac}(X,Y)Z)\xi. \tag{3.11}$$

In particular, if $M$ is a cosymplectic manifold, by the definition of $B^c$, (3.2) and (3.3), we have $AC = B^c$.

THEOREM 3.1. *The AC-contact Bochner curvature tensor is invariant with respect to the $D$-homothetic deformation $M(\phi,\xi,\eta,g) \to M(\phi^*,\xi^*,\eta^*,g^*)$ on an almost cosymplectic manifold $M$.*

*Proof.* Using (3.10), we find

$$\begin{aligned}\overset{*}{B}{}^{ac}(X,Y)Z - \eta^*(B^{ac}(X,Y)Z)\xi^* &= \overset{*}{B}{}^{ac}(X,Y)Z - \eta(\overset{*}{B}{}^{ac}(X,Y)Z)\xi \\ &= B^{ac}(X,Y)Z - \eta(B^{ac}(X,Y)Z)\xi.\end{aligned}$$

Thus we get $AC^*(X,Y)Z = AC(X,Y)Z$.

**4. Some results.** We define $s^\sharp = \sum_{i,j=1}^{2n+1} g(R(E_i,E_j)\phi E_j, \phi E_i)$, where $\{E_i\}$ is an orthonormal frame.

LEMMA 4.1. [**5**]. *For each almost cosymplectic manifold $M$ we have*

$$S - s^\sharp - g(Q\xi,\xi) + \frac{1}{2}\|\nabla\phi\|^2 = 0.$$

Using this lemma, we prove the following.

THEOREM 4.1. *Let $M$ be an almost cosymplectic manifold with vanishing $AC$-contact Bochner curvature tensor. Then $M$ is a cosymplectic manifold and the scalar curvature of $M$ vanishes.*

*Proof.* Since the AC-contact Bochner curvature tensor of $M$ vanishes, we have

$$g(B^{ac}(X,Y)Z,W) = \eta(B^{ac}(X,Y)Z)\eta(W). \tag{4.1}$$

Taking $X = E_i$, $Y = E_j$, $Z = \phi E_j$, $W = \phi E_i$ ($\{E_i\}$ is a $\phi$-basis) into the each member of (4.1), using (3.1) and summing over $i$ and $j$, we have

$$\sum_{i,j=1}^{2n+1} g(B^{ac}(E_i,E_j)\phi E_j, \phi E_i) = s^\sharp - \operatorname{Tr} h^2 - \frac{2(n+1)}{n+2}(S - g(Q\xi,\xi)) = 0. \tag{4.2}$$

On the other hand, using (3.1), (4.1) and (3.2), we find

$$\sum_{i=1}^{2n+1} g(B^{ac}(E_i,\xi)\xi, E_i) = \frac{2}{(n+2)}g(Q\xi,\xi) = \sum_{i=1}^{2n+1} \eta(B^{ac}(E_i,\xi)\xi)\eta(E_i) = 0. \tag{4.3}$$

Moreover, calculating $\sum_{i,j=1}^{2n+1} g(B^{ac}(E_i,E_j)E_j,E_i)$ by means of (3.1), and using (4.1), (3.2) and (4.3), we get

$$\begin{aligned}\sum_{i,j=1}^{2n+1} g(B^{ac}(E_i,E_j)E_j,E_i) &= -\frac{n}{(n+2)}S - \operatorname{Tr} h^2 + \frac{2}{(n+2)}g(Q\xi,\xi)\\ =\sum_{i,j=1}^{2n+1} \eta(B^{ac}(E_i,E_j)E_j)\eta(E_i) &= \frac{2}{(n+2)}g(Q\xi,\xi) = 0.\end{aligned} \tag{4.4}$$

By Lemma 4.1, (2.5), (4.2), (4.3) and (4.4) we obtain our result.

Now, let $M$ be a conformally flat almost cosymplectic manifold of dimension $(2n+1) \geq 5$. Then the following identities are known (see (3.1) in [**2**] and (6.4) in [**5**]), that is,

$$\begin{aligned}&(2n-3)(\operatorname{Ric}(X,X) + \operatorname{Ric}(\phi X,\phi X))\\ &\qquad = -\frac{S}{n} - (2n-1)(\|(\nabla\phi)(X)\|^2 + \|hX\|^2)\end{aligned} \tag{4.5}$$

$$\frac{2n-2}{2n-1}S + \frac{2n-3}{2n-1}\|\nabla\xi\|^2 + \frac{1}{2}\|\nabla\phi\|^2 = 0, \tag{4.6}$$

where $X$ is a vector such that $X \in D_x$, $\|X\| = 1$. Here, we get the following theorem.

THEOREM 4.2. *Let $M^{2n+1}$ be a conformally flat almost cosymplectic manifold of dimension $(2n+1) \geq 5$. Then the following conditions are equivalent:* (1) *$M$ is locally flat* (2) *The $AC$-contact Bochner curvature of $M$ vanishes* (3) *$M$ is cosymplectic* (4) *The Ricci curvature of $M$ is flat* (5) *The scalar curvature of $M$ vanishes*

*Proof.* (1) $\Rightarrow$ (2): From (4.5) we have $S = \mathrm{Ric}\,(X,X) = \mathrm{Ric}\,(\phi X, \phi X) = 0$. Thus $M$ is cosymplectic, wherefrom $h = 0$, so that $B^{ac} = 0$. Therefore $AC = B^c = 0$.

(2) $\Rightarrow$ (3): This follows from Theorem 4.1.

(3) $\Rightarrow$ (4): By (4.6) we find $S = 0$, so that, from (4.5) we get $g(QX,X) = g(\phi Q\phi X, X)$ for $X \in D_x$. However, from (2.10) and (2.7) we obtain $g(QX,X) = 0$ for $X \in D$. Using the polarization identity, we have $g(QX,Y) = 0$ for any $X, Y \in D$. Moreover, by (2.7) we obtain $g(QX,Y) = 0$ for any vector field $X$ and $Y$, that is, the Ricci curvature of $M$ is flat.

(4) $\Rightarrow$ (5): Trivial.

(5) $\Rightarrow$ (1): Using (4.6), we see that $M$ is cosymplectic. Therefore, from (4.5) we get $g(QX,X) = (\phi Q\phi X, X)$ for $X \in D_x$, so that the Ricci curvature of $M$ is flat. By (2.11) we get our result.

*Remark* 4.1. Theorem 4.2 is a generalization of Theorem 6 in [**5**].

Next, we consider an almost cosymplectic manifold with constant $\phi$-sectional curvature $K$. Suppose that $X$ is a vector such that $X \in D_x$, $\|X\| = 1$. Then we have the following (see (2.3) and Remark 2.1 in [**2**]).

$$\mathrm{Ric}\,(X,X) + \mathrm{Ric}\,(\phi X, \phi X) = (n+1)K - \frac{3}{4}\|(\nabla\phi)(X)\|^2 - \frac{5}{4}\|hX\|^2 \tag{4.7}$$

$$\begin{aligned} S &= \mathrm{Ric}\,(\xi,\xi) + \sum_{\alpha} \mathrm{Ric}\,(e_\alpha, e_\alpha) + \sum_{\alpha} \mathrm{Ric}\,(\phi e_\alpha, \phi e_\alpha) \\ &= -\|h\|^2 + (n+1)nK - \frac{3}{4}\sum_{\alpha=1}^{n} \|(\nabla\phi)(e_\alpha)\|^2 - \frac{5}{4}\sum_{\alpha=1}^{n} \|he_\alpha\|^2. \end{aligned} \tag{4.8}$$

From (4.7) and (4.8) we have the following theorem.

THEOREM 4.3. *Let $M$ be an almost cosymplectic manifold with constant $\phi$-sectional curvature. Then a necessary and sufficent condition for $M$ to be locally flat is that the $AC$-contact Bochner curvature of $M$ vanishes.*

*Proof.* Suppose that the AC-contact Bochner curvature of $M$ vanishes. Then, from Theorem 4.1 $M$ is cosymplectic and $S = 0$. This result and (4.8) lead to $K = 0$. Therefore, by (4.7) it follows that $\mathrm{Ric}\,(X,Y) = 0$ for any vector fields $X$ and $Y$. Considering $B^{ac} = B^c = 0$, we can see that $M$ is locally flat.

Conversely, suppose that $M$ is locally flat. Then $K = \mathrm{Ric}\,(X,Y) = 0$, so that, by (4.7) we see that $M$ is cosymplectic, wherefrom $B^{ac} = B^c = 0$. Therefore $AC = 0$.

Last we consider an almost cosymplectic manifold with vanishing AC-contact Bochner curvature tensor. Then we obtain the following theorem.

THEOREM 4.4. *Let $M^{2n+1} (n \neq 1)$ be an almost cosymplectic manifold with vanishing $AC$-contact Bochner curvature tensor. Then the following conditions are* *equivalent:* (1) *$M$ has a constant $\phi$-sectional curvature* 0, (2) *$M$ has a constant $\phi$-sectional curvature,* (3) *$M$ is Ricci flat,* (4) *$M$ is $\eta$-Einstein,* (5) *$M$ is locally flat* (6) *$M$ is conformally flat.*

*Proof.* First of all, since the AC-contact Bochner curvature tensor of $M$ vanishes, $M$ is cosymplectic and $S = 0$. Then:

(1) $\Rightarrow$ (2) trivial;

(2) $\Rightarrow$ (3) from (4.8) we have $K = 0$, so that, by (4.7) we get the result;

(3) $\Rightarrow$ (4) trivial;

(4) $\Rightarrow$ (1) since $M$ is $\eta$-Einstein, by two definitions of $\eta$-Einstein manifold and $B^c$, (2.7) and (2.10), we get

$$\begin{aligned} g(R(X,Y)Z,W) = &-\frac{a}{2n+4}(2g(X,Z)g(Y,W) - 2g(X,W)g(Y,Z) \\ &+ 2g(\phi X,Z)g(\phi Y,W) - 2g(\phi X,W)g(\phi Y,Z) + 4g(\phi Z,W)g(\phi X,Y) \\ &+ \eta(Y)\eta(Z)g(X,W) - \eta(Y)\eta(W)g(X,Z) + \eta(X)\eta(W)g(Y,Z) \\ &- \eta(X)\eta(Z)g(Y,W)) - \frac{b}{2n+4}(g(X,Z)\eta(Y)\eta(W) - g(X,W)\eta(Y)\eta(Z) \\ &- g(Y,Z)\eta(X)\eta(W) + g(Y,W)\eta(X)\eta(Z)). \end{aligned} \tag{4.9}$$

Taking $X \in T_x(M)$ such that $\|X\| = 1$, $X \perp \xi$, and calculating $g(R(X,\phi X)\phi X,X)$ by using (4.9), we get $g(R(X,\phi X)\phi X,X) = \frac{4}{n+2}a$. On the other hand, from $\mathrm{Ric}\,(X,Y) = ag(X,Y) + b\eta(X)\eta(Y)$ we find $S = (2n+1)a + b$. We also have $\mathrm{Ric}\,(\xi,\xi) = a + b$. However, by (2.7) we get $b = -a$. Thus $S = 2na$, wherefrom we find $g(R(X,\phi X)\phi X,X) = \frac{2S}{n(n+2)}$, which completes this proof;

(5) $\Rightarrow$ (1): trivial;

(1) $\Rightarrow$ (5): from (4.7) we have $\mathrm{Ric}\,(X,Y) = 0$ for any vector fields $X$ and $Y$. Considering $B^c = 0$, we get the result;

(6) $\Rightarrow$ (5): by (4.5) we see that $\mathrm{Ric}\,(X,Y) = 0$ for any vector fields $X$ and $Y$. Moreover $S = 0$. Therefore, by (2.11) we get that $M$ is locally flat;

(5) $\Rightarrow$ (6): it folows from (2.11).

*Remark* 4.2. The curvature of a Riemannian manifold is said to be harmonic if the diveregence of its curvature tensor is zero. It is well known that a Riemannian manifold has harmonic curvature, if the Ricci operator $Q$ satisfies

$$(\nabla_X Q)Y = (\nabla_Y Q)X$$

for any vector fields $X$, $Y$ (e.g., see [**1**]). Theorem 4.4 is also valid for a 3-dimensional almost cosymplectic manifold $M^3$. At first the equivalences (1)–(5) are also valid for a 3-dimensional almost cosymplectic manifold. Here, put that (7) $M^3$ has a harmonic curvature. Then, from (4.7) we have $\mathrm{Ric}\,(X,Y) = 0$ for any vector fields $X$, $Y$. Thus (1) $\Rightarrow$ (7). By (4.8) we get (7) $\Rightarrow$ (1). Moreover, from (2.12) we obtain (7) $\Longleftrightarrow$ (6). Therefore for an almost cosymplectic manifold $M^3$ the equivalences (1)–(7) hold good.

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