PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 56 (70), 1994, 119–128?

STOCHASTIC CALCULUS ON ONE-DIMENSIONAL DIFFUSIONS

Dražen Pantić

Communicated by Zoran Ivković

Abstract. Stochastic calculus is used for complete description of the distribution type of diffusion processes with Lipschic coefficients. We give sufficient conditions for the solutions of stochastic differential equations to possess an absolutely continuous one-dimensional distribution. The probability density for stochastic differential equations with uniformly elliptic coefficients is investigated in detail. The distribution of inverse process is given too.

1. Introduction. Let $\{X_{t,x}, t \geq 0\}, x \in \mathbb{R}$, be a solution of the stochastic differential equation (SDE)

$$X_{t,x} - x = \int_0^t a(X_{s,x})\, dB_s + \int_0^t b(X_{s,x}) ds \tag{1}$$

defined on the Brownian motion $\{B_t, t \geq 0\}$ starting from zero. We assume that coefficients a and b have bounded and continuous first order derivatives ($a, b \in \mathbb{C}_b^1$). Under these conditions for $p > 1$ there exists L_p derivative $DX_{t,x}$ of $X_{t,x}$ w.r.t. initial condition x, so that, as in [**2**, §8],

$$DX_{t,x} - 1 = \int_0^t a'(X_{s,x}) DX_{s,x}\, dB_s + \int_0^t b'(X_{s,x}) DX_{s,x}\, ds. \tag{2}$$

Using the Ito formula, $DX_{t,x}$ could be explicitly computed as

$$DX_{t,x} = \exp\Big\{\int_0^t a'(X_{s,x})\, dB_s + \int_0^t (b'(X_{s,x}) - \frac{1}{2} a'(X_{s,x})^2) ds\Big\}. \tag{3}$$

The boundedness of derivatives implies that, for $p > 1$, $DX_{t,x}$ as well as its inverse $(DX_{t,x})^{-1}$ are in L_p, [**6**], while $(DX_{t,x})^{-1}$ satisfies the following SDE, [**7**, Lemma (3.7)]:

$$(DX_{t,x})^{-1} - 1 = -\int_0^t a'(X_{s,x})(DX_{s,x})^{-1}\, dB_s + \int_0^t \big(a'(X_{s,x})^2 - b'(X_{s,x})\big)(DX_{s,x})^{-1}\, ds.$$

AMS Subject Classification (1990): Primary 60 H 07, 60 H 10

Also, for the Skorohod integrable process $\mathbf{u} = \{u_s, 0 \le s \le T\}$, a directional derivative $D_u X_{t,x}$ of the functional $X_{t,x}$ could be expressed in terms of derivatives $\{DX_{t,x}, t \ge 0\}$ in the following way, [**8**],

$$D_u X_{t,x} = DX_{t,x} \int_0^t (DX_{s,x})^{-1} a(X_{s,x})\, u_s\, ds. \tag{4}$$

First we shall treat the general case. Defining process $\underline{\mathbf{u}} = \{\underline{u}_s, 0 \le s \le t\}$ as $\underline{u}_s = a(X_{s,x}) DX_{s,x}$, we have

$$D_{\underline{u}} X_{t,x} = DX_{t,x} \int_0^t a(X_{s,x})^2\, ds. \tag{5}$$

Process $\underline{\mathbf{u}}$ is Ito integrable, so for a continuously differentiable function f with compact support, it follows, by integration by parts formula, [**5**], and relation (5), that

$$\begin{aligned} \mathbb{E}\Big\{ f'(X_{t,x}) DX_{t,x} \int_0^t a(X_{s,x})^2 ds \Big\} &= \mathbb{E}\{D_{\underline{\mathbf{u}}}(f(X_{t,x}))\} \\ &= \mathbb{E}\Big\{ f(X_{t,x}) \int_0^t a(X_{s,x}) DX_{s,x}\, dB_s \Big\}. \end{aligned} \tag{6}$$

Knowing that $DX_{t,x}$ is a.s. positive, we have to find conditions for the term $\int_0^t a(X_{s,x})^2\, ds$ to be positive a.s. in order to apply next Lemma.

LEMMA 1. *Let X be a random variable such that there exists an integrable and almost sure positive random variable Y and a constant $C > 0$ so that for each $f \in \mathbb{C}_b^1$ we have $\mid \mathbb{E}\{f'(X) \cdot Y\} \mid \le C \sup\{\mid f(x) \mid : x \in \mathbb{R}\}$. Then the probability law generated by the random variable X is absolutely continuous with respect to the Lebesgue measure.*

Proof. Let us define positive bounded linear functional on the space of continuously differentiable functions with compact support, by

$$L(f) = \mathbb{E}\{f(X)Y\} = \mathbb{E}\{f(X)\mathbb{E}\{Y|X\}\} = \int f(x)\mathbb{E}\{Y|X = x\}G(dx),$$

where G is the probability measure generated by X. Then, we have

$$\left| \int f'(x)\mathbb{E}\{Y|X = x\}G(dx) \right| \le C \sup\{\mid f(x) \mid : x \in \mathbb{R}\}.$$

Using [**4**, §1] it follows that there exists a density g so that

$$\int f(x)\mathbb{E}\{Y|X = x\}dG(x) = \int f(x)g(x)\, dx.$$

Clearly, the previous relation could be extended to the space of bounded measurable functions, so for a Borel set B we have $\mathbb{E}\{\mathbb{I}\{X \in B\}Y\} = \int \mathbb{I}\{x \in B\}g(x)\, dx$. If B is of Lebesgue measure zero then $\mathbb{E}\{\mathbb{I}\{X \in B\}Y\} = 0$. Knowing that $Y > 0$ almost sure, one easily gets that $\mathbb{P}\{X \in B\} = 0$, i.e. the probability law generated by the random variable X is absolutely continuous with respect to the Lebesgue measure. $\square$

General case

In the next theorem we shall describe in full the first moment τ_x from which $\int_0^t a(X_{s,x})^2 ds$ is positive, as well as the behavior of the diffusion process (1) up to that moment.

THEOREM 1. *Let $\{X_{t,x}, t \geq 0\}, x \in \mathbb{R}$, be the solution of SDE (1) with coefficients in $\mathbb{C}_1^b$, and let $\{Y_{t,x}, t \geq 0\}$ be a deterministic process defined by*

$$Y_{t,x} - x = \int_0^t b(Y_{s,x})\, ds.$$

Denote $\tau_x = \inf\{t \geq 0 : \mid a(X_{t,x}) \mid > 0\}$ and $D_x = \inf\{t \geq 0 : \mid a(Y_{t,x}) \mid > 0\}$. Then, with probability one, $\tau_x = D_x$ and $X_{t,x} = Y_{t,x}$ for $t \leq D_x$.

Proof. Assume first that $0 < D_x < \infty$, and let $0 < t < D_x$. Then, $a(Y_{s,x}) = 0$ for $s \leq t$, hence $\int_0^t a(Y_{s,x})\, dB_s = 0$, and consequently

$$Y_{t,x} - x = \int_0^t a(Y_{s,x})\, dB_s + \int_0^t b(Y_{s,x}) ds.$$

Now the theorem on the existence and uniqueness of the solutions to stochastic differential equations, [**2**, §6], implies that with probability one $X \equiv Y$ on $[0, D_x]$, and hence $D_x \leq \tau_x$ a.s.

Assume now that $0 < \tau_x < \infty$, and let $0 < t < \tau_x$. Process X has almost sure continuous paths, so τ_x could be expressed also as

$$\tau_x = \inf\left\{t \geq 0 : \int_0^t a(X_{t,x})^2\, ds > 0\right\}.$$

In [**1**, Th. 6.3] it is proved that then τ_x is deterministic time, so we can conclude that for $0 < t < \tau_x$

$$\int_0^t a(X_{s,x})^2\, ds = 0$$

with probability one, hence $\int_0^t a(X_{s,x})\, dB_s = 0$ almost sure, and

$$X_{t,x} - x = \int_0^t b(X_{s,x})\, ds.$$

Now, using the theorem on the existence and uniqueness of solutions to stochastic differential equations again, one can conclude that $X \equiv Y$ on $[0, \tau_x]$ with probability one, and that $\tau_x \leq D_x$ a.s. so the proof is complete. $\square$

The following theorem describes completely the type of one-dimensional distributions of the solutions to SDE:

THEOREM 2. *Under conditions of Theorem 1 the distribution of the random variable* $X_{t,x}$ *is either degenerate or absolutely continuous, whether* $t \leq D_x$ *or* $t > D_x$.

Proof. The first part when $t \leq D_x$ is described in Theorem 1, so let $t > D_x$. Then $\int_0^t a(X_{s,x})^2 ds$ is positive a.s. and from relation (6) and Lemma 1 we can reach our conclusion. $\square$

Uniformly elliptic case

Now, we shall treat an important special case when the coefficient a is uniformly elliptic in the sense that

$$\inf\{\mid a(x) \mid : x \in \mathbb{R}\} > 0. \tag{EL}$$

THEOREM 3. *Suppose the diffusion process* $\{X_{t,x}, t \geq 0\}$ *defined by SDE* (1) *has continuously differentiable coefficients* a *and* b *with bounded derivatives and* $\inf\{|a(x)| : x \in \mathbb{R}\} > 0$. *Then we have:*

1° *for each* $t > 0$ *and* $x \in \mathbb{R}$ *the probability law of random variable* $X_{t,x}$ *is absolutely continuous w.r.t. the Lebesgue measure with transition density* $p_t(x, \cdot)$;

2° *for any bounded, measurable function* f *and* $t > 0$ *the mapping* $x \to \mathbb{E}\{f(X_{t,x})\}$ *is continuously differentiable and*

$$\frac{\partial}{\partial x}\mathbb{E}\{f(X_{t,x})\} = t^{-1}\,\mathbb{E}\{f(X_{t,x}) \int_0^t a(X_{s,x})^{-1} DX_{s,x}\, dB_s\}; \tag{7}$$

3° *for each* $y \in \mathbb{R}$ *and* $t > 0$ *the transition density* $p_t(x,y)$ *is differentiable w.r.t. the initial condition* x *and*

$$\frac{\partial}{\partial x}p_t(x,y) = t^{-1}\,\mathbb{E}\left\{ \int_0^t a(X_{s,x})^{-1} DX_{s,x}\, dB_s | X_{t,x} = y \right\} p_t(x,y). \tag{8}$$

Proof. 1° Taking $\tilde{u}_s = a(X_{s,x})^{-1} DX_{s,x}$, one has, from (4)

$$D_{\tilde{u}} X_{t,x} = DX_{t,x}\, t. \tag{9}$$

Process $\{\tilde{u}\}$ is Ito integrable, so for a continuously differentiable function with compact support f it follows by integration by parts formula and relation (9), that

$$\mathbb{E}\{f'(X_{t,x}) DX_{t,x}\, t\} = \mathbb{E}\{D_{\tilde{u}}(f(X_{t,x}))\} = \mathbb{E}\left\{ f(X_{t,x}) \int_0^t a(X_{s,x})^{-1} DX_{s,x}\, dB_s \right\}. \tag{10}$$

Now, $DX_{t,x}$ is a.s. positive random variable and its inverse $(DX_{t,x})^{-1}$ is in L_p for each $p > 1$. Also the assumptions we made on coefficient a assure, via a Burgholder theorem, that $\int_0^t a(X_{s,x})^{-1} DX_{s,x}\, dB_s$ is in L_2, hence by Lemma 1 assertion 1° holds.

2° First, take $f \in \mathbb{C}_b^1$. Then by [**2**] and (10) one has

$$\frac{\partial}{\partial x}\mathbb{E}\{f(X_{t,x})\} = \mathbb{E}\{f'(X_{t,x})DX_{t,x})\} = t^{-1}\mathbb{E}\Big\{ f(X_{t,x}) \int_0^t a(X_{s,x})^{-1} DX_{s,x}\, dB_s\Big\}.$$

Now, for $x, y \in \mathbb{R}$ we have

$$\mathbb{E}\{f(X_{t,y})\} - \mathbb{E}\{f(X_{t,x})\} = \int_x^y \mathbb{E}\Big\{ f(X_{t,z}) \int_0^t a(X_{s,z})^{-1} DX_{s,z}\, dB_s\Big\} \frac{dz}{t}. \tag{11}$$

Both sides of (11) are continuous linear functionals on the set $\mathcal{M}_b$ of measurable and bounded functions, so we can conclude that (11) is valid for each $f \in \mathcal{M}_b$. Also, for $f \in \mathcal{M}_b$ and $t > 0$ the mapping $x \to \mathbb{E}\{f(X_{t,x})\}$ is absolutely continuous.

If we define semigroup $\{P_t, t \geq 0\}$ of operators on the space $\mathcal{M}_b$ of bounded measurable functions as $P_t f(x) = \mathbb{E}\{f(X_{t,x})\}$, then Markov property of diffusion $\{X_{t,x}\}$ yields $P_{t+\delta} f(x) = \mathbb{E}\{P_\delta f(X_{t,x})\}$, for $\delta > 0$.

Substituting t with $t + \delta$ and f with $P_\delta f$, the relation (11) reads as

$$P_{t+\delta} f(y) - P_{t+\delta} f(x) = \int_x^y \mathbb{E}\{P_\delta f(X_{t,z}) \int_0^t a(X_{s,z})^{-1} DX_{s,z}\, dB_s\} \frac{dz}{t}.$$

Then, from continuity of the function $P_\delta f(x)$ it follows that for each $x \in \mathbb{R}$ there exists

$$\lim_{y\to x} \frac{P_{t+\delta} f(y) - P_{t+\delta} f(x)}{y - x},$$

and

$$\frac{\partial}{\partial x} P_{t+\delta} f(x) = t^{-1}\, \mathbb{E}\Big\{ P_\delta(X_{t,x}) \int_0^t a(X_{s,x})^{-1} DX_{s,x}\, dB_s\Big\}$$

hence $\frac{\partial}{\partial x} P_{t+\delta} f(x)$ is continuous function in x.

Now the relation (11) yields that, for each x

$$\frac{\partial}{\partial x} P_t f(x) = t^{-1}\, \mathbb{E}\Big\{ f(X_{t,x}) \int_0^t a(X_{s,x})^{-1} DX_{s,x}\, dB_s\Big\}.$$

3° Let us denote

$$W_{t,x} = t^{-1} \int_0^t a(X_{s,x})^{-1} DX_{s,x}\, dB_s. \tag{12}$$

Then (10) reads as $\mathbb{E}\{f'(X_{t,x})DX_{t,x}\} = \mathbb{E}\{f(X_{t,x})W_{t,x}\}$, or, having in mind 1°,

$$\int_{-\infty}^{\infty} f'(u)\, \mathbb{E}\{DX_{t,x} | X_{t,x} = u\} p_t(x,u)\, du = \int_{-\infty}^{\infty} f(u)\, \mathbb{E}\{W_{t,x} | X_{t,x} = u\} p_t(x,u)\, du.$$

Taking, for $\epsilon > 0$ and $y \in \mathbb{R}$,

$$f_\epsilon(x) = \int_{-\infty}^{x} e^{-(t-y)^2/2\epsilon} \frac{dt}{\sqrt{2\pi\epsilon}},$$

we get, using partial integration

$$
\begin{aligned}
\int_{-\infty}^{\infty} & e^{-(u-y)^2/2\epsilon}\, \mathbb{E}\{DX_{t,x}|X_{t,x}=u\}p_t(x,u)\, \frac{du}{\sqrt{2\pi\epsilon}} \\
&= \int_{-\infty}^{\infty}\Big(\int_{-\infty}^{u} e^{-(t-y)^2/2\epsilon}\frac{dt}{\sqrt{2\pi\epsilon}}\Big)\mathbb{E}\{W_{t,x}|X_{t,x}=u\}p_t(x,u)\, du \qquad (13)\\
&= -\int_{-\infty}^{\infty} e^{-(t-y)^2/2\epsilon}\Big(\int_{-\infty}^{t}\mathbb{E}\{W_{t,x}|X_{t,x}=u\}p_t(x,u)\, du\Big)\frac{dt}{\sqrt{2\pi\epsilon}}.
\end{aligned}
$$

From (11) it follows, via [**3**, Theorem 1.28], that $\mathbb{E}\{DX_{t,x}|X_{t,x}=y\}p_t(x,y)$ is a continuous and bounded function so letting $\epsilon \downarrow 0$ one can conclude that

$$
\begin{aligned}
\mathbb{E}\{DX_{t,x}|X_{t,x}=y\}p_t(x,y) &= -\int_{-\infty}^{y}\mathbb{E}\{W_{t,x}|X_{t,x}=u\}p_t(x,u)du \qquad (14)\\
&= -\mathbb{E}\{W_{t,x}\mathbb{I}\{X_{t,x}\le y\}\}.
\end{aligned}
$$

Now, it is obvious that there exists $\lim_{y\to\pm\infty}\mathbb{E}\{DX_{t,x}|X_{t,x}=y\}p_t(x,y)$. Since $\mathbb{E}\{DX_{t,x}|X_{t,x}=\cdot\}p_t(x,\cdot)$ is non-negative, integrable function it follows that

$$\lim_{y\to\pm\infty}\mathbb{E}\{DX_{t,x}|X_{t,x}=y\}p_t(x,y)=0.$$

The relation (7) implies that

$$\mathbb{E}\{W_{t,x}\mathbb{I}\{X_{t,x}\le y\}\} = \frac{\partial}{\partial x}\mathbb{E}\{\mathbb{I}\{X_{t,x}\le y\}\} = \frac{\partial}{\partial x}\mathbb{P}\{X_{t,x}\le y\},$$

so (14) read as

$$\mathbb{E}\{DX_{t,x}|X_{t,x}=y\}p_t(x,y) = -\frac{\partial}{\partial x}\mathbb{P}\{X_{t,x}\le y\}. \qquad (15)$$

Now, since $\mathbb{P}\{X_{t,x}\le y\}$ is continuously differentiable in x, we have for $x_1<x_2$

$$
\begin{aligned}
\mathbb{P}\{X_{t,x_2}\le y\}-\mathbb{P}\{X_{t,x_1}\le y\} &= -\int_{x_1}^{x_2}\mathbb{E}\{DX_{t,z}|X_{t,z}=y\}p_t(z,y)\, dz\\
&= \int_{x_1}^{x_2}\int_{-\infty}^{y}\mathbb{E}\{W_{t,z}|X_{t,z}=u\}p_t(z,u)\, dudz,
\end{aligned}
$$

and by Fubini theorem for almost all u, $\mathbb{E}\{W_{t,\cdot}|X_{t,\cdot}=u\}p_t(\cdot,u)$ is integrable and

$$\int_{x_1}^{x_2}\int_{-\infty}^{y}\mathbb{E}\{W_{t,z}|X_{t,z}=u\}p_t(z,u)dudz = \int_{-\infty}^{y}\int_{x_1}^{x_2}\mathbb{E}\{W_{t,z}|X_{t,z}=u\}p_t(z,u)dzdu.$$

Finally, as $\mathbb{P}\{X_{t,x}\le y\}=\int_{-\infty}^{y}p_t(x,u)\, du$ we have, for each $y\in\mathbb{R}$,

$$\int_{-\infty}^{y}p_t(x_2,u)\, du - \int_{-\infty}^{y}p_t(x_1,u)\, du = \int_{-\infty}^{y}\int_{x_1}^{x_2}\mathbb{E}\{W_{t,z}|X_{t,z}=u\}p_t(z,u)\, dzdu$$

so for almost all y we have

$$p_t(x_2, y) - p_t(x_1, y) = \int_{x_1}^{x_2} \mathbb{E}\{W_{t,z} | X_{t,z} = u\} p_t(z, u) dz,$$

hence $p_t(x, y)$ is an absolutely continuous function in x and

$$\frac{\partial}{\partial x} p_t(x, y) = \mathbb{E}\{W_{t,x} | X_{t,x} = y\}\, p_t(x, y). \quad \square$$

Remark. As mentioned previously, it was proved in [**1**], through different aspects of the Malliavin calculus, that the first moment, when $\int_0^t a(X_{s,x})^2 ds$ becomes positive is deterministic. Also, from that moment the distribution of diffusion process is absolutely continuous. But that approach does not allow insight into the regularity properties of the underlying semigroup. On the other hand, Theorem 3. in points 2° and 3°, fills that gap, expressing derivatives of the expectation operator and the density of the diffusion process in terms of such a proces. Let us note that the point 1° and its proof is included here for the sake of completeness.

As an appendix to the previous theorem we shall study in detail partial derivative $\frac{\partial}{\partial x} p_t(x, y)$. Precisely, we shall show that the process

$$t^{-1} \mathbb{E}\left\{ \int_0^t a(X_{s,x})^{-1} D X_{s,x}\, dB_s | X_{t,x} \right\}$$

is a backward martingale.

As $\{X_{t,x}, t \geq 0\}$ is a Markov process, for an $f \in \mathcal{M}_b$ and $s < t$ we have

$$\mathbb{E}_x \{f(X_t)\} = \mathbb{E}_x \{\mathbb{E}\{f(X_t) | \mathcal{F}_s\}\} = \mathbb{E}_x \{\mathbb{E}_{X_s} \{f(X_{t-s})\}\},$$

where $\{\mathcal{F}_t, t \geq 0\}$ is a standard filtration generated by $\{B_t, t \geq 0\}$, and $\mathbb{E}_x$ is the expectation operator generated by the initial value $X_0 = x$. Now, 2° of Theorem 3 yields

$$\begin{aligned}
\frac{\partial}{\partial x} \mathbb{E}_x \{f(X_t)\} &= \mathbb{E}_x \{f(X_t) W_t\} = \mathbb{E}_x \left\{ f(X_t) \mathbb{E}\{W_t | X_t\} \right\} \text{ as well as} \\
\frac{\partial}{\partial x} \mathbb{E}_x \{f(X_t)\} &= \frac{\partial}{\partial x} \mathbb{E}_x \left\{ \mathbb{E}_{X_s} \{f(X_{t-s})\} \right\} = \mathbb{E}_x \left\{ \mathbb{E}_{X_s} \{f(X_{t-s})\}\, W_s \right\} \\
&= \mathbb{E}_x \left\{ \mathbb{E}_{X_s} \{f(X_{t-s})\}\, \mathbb{E}\{W_s | X_s\} \right\} = \mathbb{E}_x \left\{ f(X_t)\, \mathbb{E}\{W_s | X_s\} \right\},
\end{aligned}$$

hence $\mathbb{E}\{W_{t,x} | X_{t,x}\} = \mathbb{E}\{\mathbb{E}\{W_{s,x} | X_{s,x}\} | X_{t,x}\}$ for $s < t$. If we put $\tilde{W}_{t,x} = \mathbb{E}\{W_{t,x} | X_{t,x}\}$, then the previous relation could be rewritten as

$$\tilde{W}_{t,x} = \mathbb{E}\{\tilde{W}_{s,x} | X_{s,x}\}, \ s < t.$$

Let us define the backward flow of σ-algebras as

$$\overleftarrow{\mathcal{F}}_{t,x} = \sigma\{X_{u,x}, u \geq t\}, \ t > 0, \ x \in \mathbb{R},$$

and let $\phi(x_1,\dots,x_n)$ be n-dimensional bounded Borel function. For $s < t \le t_1 < \cdots < t_n$ we have

$$\begin{aligned}
\mathbb{E}_x\{\tilde W_{s,x}\phi(X_{t_1,x},\dots,X_{t_n,x})\} &= \mathbb{E}_x\left\{\,\mathbb{E}\{\tilde W_{s,x}\,\phi(X_{t_1,x},\dots,X_{t_n,x})|\mathcal{F}_t\}\,\right\}\\
&= \mathbb{E}_x\left\{\,\tilde W_{s,x}\,\mathbb{E}\{\phi(X_{t_1,x},\dots,X_{t_n,x})|\mathcal{F}_t\}\,\right\}\\
&= \mathbb{E}_x\left\{\,\tilde W_{s,x}\,\mathbb{E}_{X_{t,x}}\{\phi(X_{t_1-t},\dots,X_{t_n-t})\}\,\right\}\\
&= \mathbb{E}_x\left\{\,\mathbb{E}\{\tilde W_{s,x}|X_{t,x}\}\,\mathbb{E}_{X_{t,x}}\{\phi(X_{t_1-t},\dots,X_{t_n-t})\}\,\right\}\\
&= \mathbb{E}_x\left\{\,\mathbb{E}\{\tilde W_{s,x}|X_{t,x}\}\,\phi(X_{t_1,x},\dots,X_{t_n,x})\,\right\},
\end{aligned}$$

hence $\mathbb{E}\{\tilde W_{s,x}|\overleftarrow{\mathcal{F}}_{t,x}\} = \mathbb{E}\{\tilde W_{s,x}|X_{t,x}\} = \tilde W_{t,x}$, i.e. the process

$$\tilde W_{t,x} = t^{-1}\mathbb{E}\left\{\int_0^t a(X_{s,x})^{-1}DX_{s,x}\,dB_s|X_{t,x}\right\}$$

is a backward martingale, relative to the flow of σ-algebras

$$\overleftarrow{\mathcal{F}}_{t,x} = \sigma\{X_{u,x}, u \ge t\},\ t > 0,\ x \in \mathbb{R}.$$

Recalling the definition of $\tilde W_{t,x}$ and 3° of the Theorem 3, we deduce

$$\int_{-\infty}^{\infty}\Big|\frac{\partial}{\partial x}p_t(x,y)\Big|dy = \mathbb{E}\{|\tilde W_{t,x}|\} < \infty.$$

Moreover, for $s < t$

$$\begin{aligned}
\int_{-\infty}^{\infty}\Big|\frac{\partial}{\partial x}p_t(x,y)\Big|dy = \mathbb{E}\{|\tilde W_{t,x}|\} &= \mathbb{E}\{\ |\mathbb{E}\{\tilde W_{s,x}|\overleftarrow{\mathcal{F}}_{t,x}\}|\ \}\\
&\le \mathbb{E}\{\ \mathbb{E}\{|\tilde W_{s,x}||\overleftarrow{\mathcal{F}}_{t,x}\}\ \} = \mathbb{E}\{\ |\tilde W_{s,x}|\ \}\\
&= \int_{-\infty}^{\infty}\Big|\frac{\partial}{\partial x}p_s(x,y)\Big|dy.
\end{aligned}$$

We have proved the following Corollary:

Corollary 1. *For each $t > 0$ and $x \in \mathbb{R}$ the process*

$$\tilde W_{t,x} = t^{-1}\mathbb{E}\left\{\int_0^t a(X_{s,x})^{-1}DX_{s,x}\,dB_s|X_{t,x}\right\}$$

is a backward martingale relative to the flow of σ-algebras

$$\overleftarrow{\mathcal{F}}_{t,x} = \sigma\{X_{u,x}, u \ge t\},\ t > 0,\ x \in \mathbb{R}.$$

Consequently, the function $\int_{-\infty}^{\infty}\Big|\frac{\partial}{\partial x}p_t(x,y)\Big|dy$ is finite and decreasing with t. □

Point cut as spatial homeomorphysm

It is well known [**7**, Theorem 2.20] that, for each $t > 0$, the mapping $x \to X_{t,x}$ is a.s. homeomorphysm of $\mathbb{R}$ onto $\mathbb{R}$, hence we can define the inverse random transformation $X_t^{-1}(\cdot) : \mathbb{R} \to \mathbb{R}$ so that $X_t^{-1}(X_{t,x}) = x, \quad x \in \mathbb{R}$.

In the next theorem we shall investigate the distribution of the random variable $X_t^{-1}(y)$, i.e. the random function $X_t^{-1}(\cdot)$ applied to the point y.

THEOREM 4. *Suppose the diffusion process* $\{X_{t,x}, t \geq 0\}$ *defined by SDE* (1) *has continuously differentiable coefficients* a *and* b *with bounded derivatives and* $\inf\{|a(x)| : x \in \mathbb{R}\} > 0$. *Then:*

1° *for each* $t > 0$ *and* $y \in \mathbb{R}$ *function* $\mathbb{E}\{DX_{t,\cdot}|X_{t,\cdot} = y\}p_t(\cdot, y)$ *is integrable and*
$$\int_{-\infty}^{+\infty} \mathbb{E}\{DX_{t,x}|X_{t,x} = y\}p_t(x,y)\,dx = 1; \tag{16}$$

2° *for each* $t > 0$ *and* $x, y \in \mathbb{R}$
$$\mathbb{P}\{X_t^{-1}(y) \leq x\} = \int_{-\infty}^{x} \mathbb{E}\{DX_{t,z}|X_{t,z} = y\}p_t(z,y)\,dz; \tag{17}$$

3° *the probability density*
$$\rho_t(y,x) = \mathbb{E}\{DX_{t,x}|X_{t,x} = y\}\,p_t(x,y) \tag{18}$$
of the random variable $X_t^{-1}(y)$ *satisfies Chapman-Kolmogorov equations:*
$$\rho_{t+s}(y,x) = \int_{-\infty}^{+\infty} \rho_s(y,z)\,\rho_t(z,x)\,dz \tag{19}$$
for $t, s > 0$ *and* $x, y \in \mathbb{R}$.

Proof. 1° As for each $t > 0$, $X_{t,\cdot}$ is a.s. homeomorphysm of $\mathbb{R}$ onto $\mathbb{R}$ and $DX_{t,x}$ partial derivative in $L_2(\Omega)$ sense of the random variable $X_{t,x}$ w.r.t. x, is a.s. positive, we have $\lim_{x\to\pm\infty} X_{t,x} = \pm\infty$, and from (15) we have, for $x_1 < x_2$
$$\mathbb{P}\{X_{t,x_2} \leq y\} - \mathbb{P}\{X_{t,x_1} \leq y\} = -\int_{x_1}^{x_2} \mathbb{E}\{DX_{t,z}|X_{t,z} = y\}p_t(z,y)\,dz. \tag{20}$$
Now, letting $x_1 \to -\infty$ and $x_2 \to +\infty$ one can conclude that
$$\int_{-\infty}^{+\infty} \mathbb{E}\{DX_{t,z}|X_{t,z} = y\}p_t(z,y)\,dz = 1,$$
what was to be proved in 1°.

2° From the fact that a.s. $X_{t,x}$ is a homeomorphysm and that $X_t^{-1}(y)$ is its inverse, we have the following equivalence $X_t^{-1}(y) \leq x \Leftrightarrow X_{t,x} \geq y$, so, using (20), with $x_1 = -\infty$ and $x_2 = x$, we get:
$$\begin{aligned}\mathbb{P}\{X_t^{-1}(y) \leq x\} &= \mathbb{P}\{X_{t,x} \geq y\} = 1 - \mathbb{P}\{X_{t,x} < y\} = \\ &= \int_{-\infty}^{x} \mathbb{E}\{DX_{t,z}|X_{t,z} = y\}p_t(z,y)dz.\end{aligned}$$

3° Let us denote by $\mathbb{P}_x\{\cdot\}$ the probability measure generated by the Markov process $\{X_{t,x}, t \geq 0\}$ starting from x. Then for $t, s > 0$ the Markov property yields

$$\mathbb{P}\{X_{t+s,x} \leq y\} = \mathbb{P}_x\{X_{t+s} \leq y\} = \int_{-\infty}^{+\infty} \mathbb{P}_z\{X_s \leq y\} p_t(x,z)\, dz = \\ = \mathbb{E}\{\mathbb{P}_{X_{t,x}}\{X_s \leq y\}\}.$$

Using the relation (15) we have

$$\begin{aligned}
\rho_{t+s}(y,x) &= -\frac{\partial}{\partial x}\mathbb{P}\{X_{t+s,x} \leq y\} = -\frac{\partial}{\partial x}\mathbb{E}\big\{\mathbb{P}_{X_{t,x}}\{X_s \leq y\}\big\} \\
&= -\mathbb{E}\big\{\frac{\partial}{\partial x}\mathbb{P}_{X_{t,x}}\{X_s \leq y\}\, DX_{t,x}\big\} \\
&= -\mathbb{E}\big\{-\mathbb{E}_{X_{t,x}}\{DX_s | X_s = y\} p_s(X_{t,x}, y)\, \mathbb{E}\{D_{t,x} | X_{t,x}\}\big\} \\
&= \int_{-\infty}^{+\infty} \mathbb{E}_z\{DX_s | X_s = y\} p_s(z,y)\, \mathbb{E}\{D_{t,x} | X_{t,x} = z\} p_t(x,z)\, dz \\
&= \int_{-\infty}^{+\infty} \mathbb{E}\{DX_{s,z} | X_{s,z} = y\} p_s(z,y)\, \mathbb{E}\{D_{t,x} | X_{t,x} = z\} p_t(x,z)\, dz \\
&= \int_{-\infty}^{+\infty} \rho_s(y,z)\, \rho_t(z,x)\, dz,
\end{aligned}$$

and the theorem is completely proved. □

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Matematički fakultet
Studentski trg 16
11000 Beograd, p.p. 550
Yugoslavia
E-address: epantidr@ubbg.etf.bg.ac.yu

(Received 25 04 1994)
(Revised 22 12 1994)