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ON KUREPA'S PROBLEMS IN NUMBER THEORY

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Dedicated to the memory of Prof. Đuro Kurepa

Abstract. We discuss some problems in number theory posed by Đuro Kurepa, including so called the left factorial hypothesis.

1. Introduction

Đ. Kurepa posed several problems in number theory that drew attention of many workers in number theory. Certainly, the most known of his problems is the so called left factorial hypothesis, which is still an open problem. However, Kurepa asked several other questions that are less known, but we think that they are interesting as well. The aim of this paper is to review some of these problems, and to present some of the known results concerning them.

We shall assume the following notation. We shall denote by N the set of natural numbers (nonnegative integers), N^+ denotes positive integers, while $\mathbf{Z}_n$ denotes the ring of integers modulo n. The greatest common divisor of integers a and b is denoted by (a, b). The Galois field of p elements, where p is a prime, is denoted by $\mathrm{GF}(p)$. If m and n are integers, by $\mathrm{rest}(m, n)$ we shall denote the remainder obtained from division of m by n.

2. Left factorial function

Kurepa defined in [**Ku71**] an arithmetic function $K(n)$ that he denoted by $!n$ and called it the left factorial, by

$$K(n) = !n = \sum_{i=0}^{n-1} i!, \quad n \in N^+.$$

In the same paper, Kurepa asked if

$$(!n, n!) = 2, \quad n = 2, 3, \dots \quad . \tag{KH}$$

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This conjecture, known as the left factorial hypothesis, is still an open problem in number theory. There are several results which support the truth of the hypothesis. Kurepa showed in [**Ku71**] that there are infinitely many $n \in N$ for which KH is true. Also, the conjecture is verified by use of computers (Slavić for $n < 1500$, Wagstaff for $n < 50000$, Mijajlović for $n < 310000$, and Gogić for $n < 1000000$). It is interesting that Kurepa announced the positive solution of the problem in 1992, but he never published a proof. R. Guy informed in a letter Ž. Mijajlović that R. Bond announced a proof of the conjecture too, but the proof was never published. The first mention of the left factorial function appeared in [**Ku64**], where this function is defined for infinite cardinal numbers as well.

2.1 Some equivalents to KH. There are several statements equivalent to KH. Probably the most natural one is the following assertion, which also belongs to Kurepa [**Ku71**]:

$$\forall n > 2 \;\; !n \not\equiv 0 \pmod{n}.$$

This formulation of the left hypothesis appears in [**Gu**] as problem B44, and we shall call this statement also KH. It is not difficult to see that this form of KH can be reduced to primes (see [**Ku71**]), i.e. KH is equivalent to

$$\forall p \in P \;\; p > 2 \Rightarrow (!p, p) = 1 \tag{PH}$$

where P denotes the set of all primes. Namely if PH fails, then KH fails with $p = n$. Conversely, if KH fails, then $n | !n$ for some $n > 2$. Then there is a prime $p > 2$ such that $p|n$ and $p \leq n$. If $p = n$, then PH trivially fails. If $p < n$, then

$$!n = !p + p! + \ldots + (n-1)!.$$

Now $p|n$ and $n|!n$ imply $p| \, !n$, and therefore it follows from the above relation that $p| \, !p$, contradicting PH. This establishes the equivalence of KH and PH.

If p is a prime, then it is not difficult to establish in GF(p) the following identities (see [**Mi**]):

$$!p = \sum_{k=0}^{p-1} (-1)^{k+1}/k!, \tag{2.1.1}$$

$$!p = \sum_{k=0}^{p-1} (-1)^{k}(k+1)(k+2)\ldots(p-1). \tag{2.1.2}$$

Since the identity $\binom{p-1}{k} = (-1)^k$ also holds in GF(p), by (2.1.1) and (2.1.2) the following identities are true in GF(p):

$$-!p = \sum_{k=0}^{p-1} \frac{1}{k!}\binom{p-1}{k}, \qquad !p = \sum_{k=0}^{p-1} \binom{p-1}{k}(k+1)(k+2)\ldots(p-1).$$

Therefore, we obtain the following

THEOREM 2.1. KH *is equivalent to any of the following statements:*

1. *For all primes p,* $\mathrm{GF}(p) \models \sum_{k=0}^{p-1} (-1)^k (k+1)(k+2)\dots(p-1) \neq 0.$

2. *For all primes p,* $\sum_{k=0}^{p-1} (-1)^k (k+1)(k+2)\dots(p-1) \not\equiv 0 \pmod p.$

3. *For all primes p,* $\mathrm{GF}(p) \models \sum_{k=0}^{p-1} \binom{p-1}{k} (k+1)(k+2)\dots(p-1) \neq 0.$

4. *For all primes p,* $\sum_{k=0}^{p-1} \binom{p-1}{k} (k+1)(k+2)\dots(p-1) \not\equiv 0 \pmod p.$

5. *For all primes p,* $\mathrm{GF}(p) \models \sum_{k=0}^{p-1} \frac{1}{k!} (-1)^k ! \neq 0.$

6. *For all primes p,* $\mathrm{GF}(p) \models \sum_{k=0}^{p-1} \frac{1}{k!} \binom{p-1}{k} \neq 0.$

Here $\mathrm{GF}(p) \models \dots$ means: in $\mathrm{GF}(p)$ we have ....

The second statement in the above theorem is proved in fact also in [**StZi**, (Lemma 2.6)]. There are some other equivalences. In [**Ša**] the following equivalence to KH was proved:

$$\forall n > 2 \quad \left(\sum_{k=2}^{n-1} !k, !n \right) = 2,$$

while in [**St**] KH was proved to be equivalent to

$$\sum_{k=2}^{n} (k-1) \cdot k! \not\equiv 0 \pmod n, \quad n > 2.$$

2.2 Some formulas involving KH. There are a number of identities involving $!n$ obtained in [**St**], [**StZi**] and [**Ca**]. Stanković and Žižović (cf. [**St**] and [**StZi**]) proved the following identities (we assumed that $K(0) = 0$):

$$\sum_{i=0}^{n} K(i) = nK(n-1) + 1, \quad n \geq 1, \tag{2.2.1}$$

$$2 \sum_{i=0}^{n-1} iK(i) = K(n) + n(n-1)K(n-2), \quad n \geq 2, \tag{2.2.2}$$

$$6 \sum_{i=0}^{n-1} i^2 K(i) = (2n-1)K(n) + (2n^2 - n - 2)K(n-2) + 2 \cdot n! - 4, \quad n \geq 2. \tag{2.2.3}$$

In connection with these identities, Carlitz (cf. [**Ca**]) considered the following sums:

$$Q_m(n) = \sum_{k=0}^{n-1} k^m K(k), \quad m = 0, 1, 2, \dots ,$$
$$R_m(n) = \sum_{k=0}^{n-1} \binom{k}{m} K(k).$$

In the same paper, he proved the following generalizations of (2.2.1–3):

$$R_m(n) = \binom{n}{m+1} K(n) - K_m(n) - K_{m+1}(n), \tag{2.2.4}$$

where $\quad K_m(n) = \sum_{k=0}^{n-1} \binom{k}{m} k!,$

$$R_m(n) = \binom{n}{m+1} K(n) - \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \frac{K(n+j+1) - K(j+1)}{j+1}, \tag{2.2.5}$$

$$Q_m(n) = \sum_{k=0}^{m} k! S(m,k) R_k(n), \tag{2.2.6}$$

where $S(m,k)$ are Stirling numbers of the second kind. Let us note that by use of $s(m,k)$, i.e. Stirling numbers of the first kind, we can obtain the dual of the identity 2.2.6., that is, we can express $R_m(n)$ by $Q_m(n)$. Namely, it is well known that the matrices $||S(m,k)||$ and $||s(m,k)||$ are mutually inverse, therefore, from (2.2.6) it follows at once that

$$R_m(n) = \frac{1}{m!} \sum_{k=0}^{m} s(m,k) Q_k(n).$$

2.3. Number theoretical hypotheses related to KH. The hypothesis on the alternating factorial stated as the problem B43 in Guy's monograph [**Gu**] on unsolved problems in number theory is similar to KH (stated in [**Gu**] as problem B44). Here is the formulation of this problem:

Let $A_n = (n-1)! - (n-2)! + (n-3)! - \dots + (-1)^n \cdot 1!, \quad n = 2, 3 \dots$ *Are there infinitely many numbers* n *such that* A_n *is a prime?*

In [**Gu**] it is observed that if there is $n \in N^+$ such that $n+1$ divides A_n, then $n+1$ will divide A_m for all $m > n$, and there would be only finitely many number of prime values of the sequence A_n. Wagstaff verified this fact for $n < 46340$, while Gogić extended this result in his master thesis [**Go**] to $n < 1000000$.

In his paper [**Ku74**], Kurepa asked several question concerning KH. He introduced there the statement $H_4(s)$ in the following way:

$$(n \geq 2 \wedge s \geq 1) \Rightarrow (K(n), K(n+s)) = 2, \quad n, s \in N^+. \tag{$H_4(s)$}$$

Then Kurepa asked ([**Ku74**], Problem 2.9.) if KH implies $H_4(s)$ for all $s \in N^+$. We note that this implication does not hold since, for example:

$$\begin{aligned}
K(7) &= 874 = 2\cdot 19\cdot 23\\
K(12) &= 43954714 = 2\cdot 19\cdot 31\cdot 37313\\
K(16) &= 1401602636314 = 2\cdot 19\cdot 41\cdot 491\cdot 1832213\\
K(25) &= 6474780714695678449403140 = 2\cdot 41\cdot 103\cdot 2875688099\cdot 26658285041.
\end{aligned}$$

The same examples also show that the strong left factorial hypothesis does not hold, as Kurepa formulated it in [**Ku74**]:

The numbers $K(n)/2$, $n = 2, 3, \ldots$ are pairwise relatively prime.

In the same paper, Kurepa introduced the sequence of sets

$$A(r) = \{n \in N^+ \mid r < n, K(n) \equiv r \pmod{n}\}.$$

He asked there for a description of these sets, and in particular is there any r for which $A(r)$ is finite. He also asked if $A(3) = \emptyset$. We note here that $467 \in A(3)$.

Kurepa asked in [**Ku71**] if $!n$ is square-free, with the only exception $!3 = 2^2$. This hypothesis, which we shall call KH2, is verified in [**Mi**] for $n \le 40$ by finding prime decompositions of $!n$ for $n < 40$. There is a simple connection between KH and KH2. Namely, if p is a prime and $n \ge p$, then $p^2 | !n$ implies $p | !n$, and so $p | !p$. Hence we obtain

PROPOSITION 2.3.1 KH *implies that for any $m > 1$ there are at most finitely many n such that $m^2 | !n$.*

2.4. Computational verification of KH. There are simple recurrent formulas for the remainder of $!n$ divided by n. Using these formulas it easy to check KH and to perform the related computation. Let r_n be the sequence defined by $r_n = \mathrm{rest}(!n, n)$, $n \in N^+$. The following proposition enables one to design an algorithm for computing the values of r_n (cf. [**Mi**, Lemma 2.1-3]):

PROPOSITION 2.4.1 *Let q be a prime, and let the finite sequences s_i, t_i, v_i be defined in* GF(q) *in the following way:*

1. $s_{q-1} = 0, \quad s_i = 1 + i s_{i+1}, \qquad i = q-2, q-3, \ldots, 1.$
2. $t_1 = 0, \quad t_i = (-1)^i + i t_{i-1}, \quad i = 2, 3, \ldots, q-1.$
3. $v_1 = 0, \quad v_i = 1 - i v_{i-1}, \qquad i = 2, 3, \ldots, q-1.$

Then $r_q = s_1 = t_{q-1} = v_{q-1}$.

Observe that s_q is defined by the regressive induction. Using these formulas it is easy to develop a simple computer program for verifying KH by computing r_n. Let KH(x) denote the truth of the left factorial hypothesis for all positive integers $n \le x$. Mijajlović [**Mi**] verified KH(311009) and Gogić [**Go**] extended it to all $n < 1000000$.

By simple modification of the above formulas one can obtain in the ring $\mathbf{Z}_{p^2}$, where p is a prime, the following recurrent formulas:

$$\begin{aligned} s_{n-1} &= n, \\ s_i &= 1 + i s_{i+1}, \quad i = n-2, n-3, \ldots, 1, \end{aligned} \tag{2.4.1}$$

so that $\mathrm{rest}(!n, p^2) = s_1$. Thus using 2.4.1, and assuming KH, in [**Mi**] it was proved: if $m^2 | !n$ then $m \geq 1227$.

By inspection, we see that the total number of arithmetical operations used in the verification of KH(x) is

$$A(x) = \sum_{p \leq x} 4p, \tag{2.4.2}$$

where p in the sum runs over primes.

Using the prime number theorem in the form

$$\pi(x) = \sum_{p \leq x} 1 = \frac{x}{\ln x} + O\left(\frac{x}{\ln^2 x}\right),$$

and integration by parts, we obtain

$$A(x) = \frac{2x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).$$

Therefore, the growth of the number of arithmetical operations used in the verification of KH(x) is

$$a(k, x) = \frac{A(kx)}{A(x)} \sim k^2 \quad \text{as } x \to \infty.$$

This means, as it was explained in [**Mi**], that the efficiency in the verification of KH(x) by use of parallel computers with k parallel processors is $\sqrt{k}$.

2.5 Left factorial function in complex domain. The gamma-function $\Gamma(z)$ is defined by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \quad (\mathrm{Re}\, z > 0), \tag{2.5.1}$$

and for other values of the complex variable z by analytic continuation, furnished by the functional equation

$$z\Gamma(z) = \Gamma(z+1). \tag{2.5.2}$$

Since $\Gamma(n+1) = n!$ for $n \in N$, it follows that

$$K(n) = \sum_{i=0}^{n-1} \Gamma(i+1) = \int_0^\infty e^{-t} \sum_{i=0}^{n-1} t^i dt = \int_0^\infty e^{-t} \frac{t^n - 1}{t-1} dt \quad (n \in N^+).$$

Hence for Re$z > 0$ it makes sense to define

$$K(z) = \int_0^\infty e^{-t} \frac{t^z - 1}{t-1} dt, \tag{2.5.3}$$

and since one easily obtains

$$K(z)=K(z+1)-\Gamma(z+1), \tag{2.5.4}$$

then (2.5.4) provides analytic continuation of $K(z)$ to the whole complex plane. In particular, since $K(1)=\Gamma(1)=1$, it follows that $K(0)=0$. Kurepa [**Ku71**] defined $K(z)$ for arbitrary complex z by (2.5.3) and (2.5.4). In [**Ku73**] he established that $K(z)$ is a meromorphic function having only simple poles at the points $z = -1,-3,-4,-5,\ldots$. The residue of $K(z)$ at $z=-1$ equals to -1, and at $z=-n$ $(n=3,4,5,\ldots)$ it equals $\sum_{k=2}^{n-1}(-1)^{k-1}/k!$. This follows from (2.5.2), (2.5.4) and the fact that $\Gamma(z)$ is a meromorphic function with residues $(-1)^n/n!$ at simple poles $z=-n$ $(n\in N)$. Kurepa [**Ku73**] also studied the zeros of $K(z)$, and showed the asymptotic relations

$$\lim_{x\to\infty}\frac{K(x)}{\Gamma(x)}=1,\qquad \lim_{x\to\infty}\frac{K(x)}{\Gamma(x+1)}=0,$$

of which the second is a corollary of the first in view of (2.5.2). Further results on $K(z)$ as a function of the complex variable z were obtained by Slavi c [**Sl**]. His main result is that

$$K(z)=-\frac{\pi}{e}\mathrm{cotg}\pi z+\frac{1}{e}\left(\sum_{n=1}^{\infty}\frac{1}{n!n}+C\right)+\sum_{n=0}^{\infty}\Gamma(z-n)$$

holds for all complex z, where $C=-\int_0^\infty e^{-x}\ln x\,dx=0.577215\ldots$ is Euler's constant.

Formula (2.5.3) is useful for many purposes. For example, for $p\ge 3$ it gives

$$!p=K(p)=\int_0^\infty e^{-t}\frac{((t-1)+1)^p-1}{t-1}dt\equiv\int_0^\infty e^{-t}(t-1)^{p-1}dt \pmod p, \tag{2.5.5}$$

since when we expand $((t-1)+1)^p$ by the binomial theorem we can use that $\binom{p}{k}$ $(k=1,\ldots,p-1)$ is divisible by p. But

$$\begin{aligned}\int_0^\infty e^{-t}(t-1)^{p-1}dt&=\int_0^1 e^{-t}(t-1)^{p-1}dt+\frac{1}{e}\int_0^\infty e^{-u}u^{p-1}du\\&=\int_0^1 e^{-t}(t-1)^{p-1}dt+(p-1)!/e.\end{aligned} \tag{2.5.6}$$

Since the first integral in (2.5.6) is a natural number and

$$0<\int_0^1 e^{-t}(t-1)^{p-1}dt\le 1-\frac{1}{e},$$

it follows from (2.5.5) and (2.5.6) that

$$!p\equiv\left[\frac{(p-1)!}{e}\right]+1 \pmod p, \tag{2.5.7}$$

where $[x]$ denotes the integer part of x. Therefore, from (2.5.7) we can obtain, in view of PH, another equivalent of KH, namely

$$[(p-1)!/e] \not\equiv -1 \pmod p \quad \text{for} \quad p > 2.$$

In connection with (2.5.7) one can define $R(p)$ to be the least nonnegative residue of $!p \pmod p$. The evaluation of $R(p)$ is rather involved, but perhaps one could try to evaluate the summatory function of $R(p)$. The following problem seems to be of interest: does there exist a constant $C > 0$ such that

$$\sum_{p \le x} R(p) \sim \frac{Cx^2}{\ln x} \quad (x \to \infty)?$$

3. Other hypotheses

Kurepa presented several problems in number theory at the Problem Session of the 5th Balkan Mathematical Congress, held in Belgrade, 1974. These problems are published as a supplement to [**Ku74**]. The first problem on this list concern the set

$$P(n) = \{x \in N^+ : \{x-2n, x, x+2n\} \subseteq P\}, \quad n \in N^+ \tag{3.1}$$

where P is the set of prime numbers. Kurepa asked what are the properties of $P(n)$, and in particular:

P1. Is $P(1) = \{5\}$? P2. Is there some $n \in N^+$ such that $P(n) = \emptyset$?

We note the following properties of the sequence $P(n)$. The set $P(n)$ is related to a part of Problem A6 in [**Gu**]. Namely, as noted there, it is not known whether there are infinitely many sets of three consecutive primes in an arithmetic progression, but S. Chowla has shown [**Ch**] this without the restriction to consecutive primes. Thus, as $x-2n$, x, $x+2n$ is an arithmetic progression, we have

$$\bigcup_{n \in N^+} P(n) \quad \text{is infinite.} \tag{3.2}$$

Further, assume $n = 3k+1$, $k \in N$. Then $x - 2n \equiv x + 1 \pmod 3$, and $x + 2n \equiv x + 2 \pmod 3$, thus 3 divides $(x-2n)x(x+2n)$. If $x \in P(n)$ then $x - 2n = 3$, i.e. $x = 2n+3$. Hence $P(n) = \emptyset$, or $P(n)$ is an one-element set, i.e. $P(n) = \{6k+5\}$, where $6k+5, 12k+7 \in P$. For example, $P(1)$, $P(4)$, $P(7)$, $P(10)$ are one-element sets, while $P(13) = \emptyset$, and this answers questions P1 and P2.

By (3.2), without any restriction on n, there are infinitely many n such that $P(n)$ is an one-element set. However, we may ask if there are infinitely many $k \in N$ such that $P(n)$ is an one-element set, where $n = 3k+1$. We already observed that this is the case iff $6k+5, 12k+7 \in P$. We do not know the right answer, and obviously this question is related to the twin primes conjecture, and to the conjectures 5 and 4 in [**Sh**], which in turn would imply that there are infinitely many Mersenne primes. On the other hand, as the functions $6k+5$ and $12k+7$ are linearly independent, from the Bateman–Horn conjecture [**BaHo**] it would follow that the number of $k \le m$ such that $6k+5, 12k+7 \in P$ is asymptotic to $C\int_2^m (\log x)^{-2}dx$, where C is a positive constant. In particular, it would follow that there are infinitely many $n \in 3N+1$ such that $P(n)$ is an one-element set.

If $n = 3k+2$, where $k \in N$, we have a similar conclusion, i.e. 3 divides $(x-2n)x(x+2n)$, and so if $x \in P(n)$ then $x = 2n+3$, thus $P(n) = \{6k+7\}$, where $6k+7, 12k+11 \in P$. We have also a similar discussion as in the case $n = 3k+1$.

Finally, if $n = 3k$, $k \in N$, then the problem whether $P(n)$ is infinite reduces to the question whether there are infinitely many prime triplets $x-6k$, x, $x+6k$. This question is related to Problem A9 in [**Gu**] and according to the discussion supplemented to the problem, it is likely that there are infinitely many such triplets.

The second problem Kurepa stated in his list concerns the sequence $s_n = p_n^2 - p_{n-1} - p_{n+1}$, where p_n is the n-th prime. Kurepa asked what could be the sign of the elements of this sequence. We note the following:

LEMMA. $p_n^2 > p_{n-1} + p_n + p_{n+1}$ if $p_n \ge 5$.

Proof. If $p_n \ge 5$, then $p_n^2 - p_{n-1} - p_{n+1} \ge 5p_n - 2p_{n+1} = p_n + 2(2p_n - p_{n+1})$. By Bertrand's postulate, which says that for every positive integers m there is a prime in the interval $[m, 2m]$, we have $p_{n+1} < 2p_n$, so

$$p_n^2 - p_{n-1} - p_{n+1} > p_n, \quad \text{i.e.} \quad p_n^2 > p_{n-1} + p_n + p_{n+1}. \qquad \diamondsuit$$

By the above lemma, we see that for all $p_n \ge 3$, $p_n^2 - p_{n-1} - p_{n+1} > 0$. Actually we can show that

$$p_n^2 > \sum_{k=1}^{n+1} p_k \quad (n \ge n_0). \tag{3.3}$$

Namely, from the prime number theorem it follows that

$$p_n = n(\ln n + O(\ln\ln n)). \tag{3.4}$$

Using (3.4) it follows that the left-hand side of (3.3) is asymptotic to $n^2 \ln^2 n$, while the right-hand side is

$$\sum_{k=1}^{n+1} k \ln k + O(n^2(\ln\ln n)^2) = \frac{n^2}{2}\ln n + O(n^2(\ln\ln n)^2).$$

In the third problem of his list Kurepa considered the sequence defined by $\pi_n = p_n^2 - p_{n-1}p_{n+1}$, and asked what could be the sign of members of this sequence, and how often they take the same sign. First, let us note that obviously $\pi_n < 0$ or $\pi_n > 0$. Further, this question is related to Problem A14 in [**Gu**]. Namely, Erdős and Straus call the prime p_n *good* if $p_n^2 > p_{n-i}p_{n+i}$ for all $1 \le i \le n-1$. Pomerance [**Po**] proved that there are infinitely many good primes, and therefore there are infinitely many n such that $\pi_n > 0$. Pomerance also proved that

$$\limsup_{n\to\infty}(p_n^2 - M(n)) = +\infty, \quad \text{where} \quad M(n) = \max_{0<i<n} p_{n-i}p_{n+i}.$$

Now suppose that p_{n-1} and $p_n = p_{n-1} + 2$ are twin primes. Then $p_{n+1} \geq p_n + 6$, thus (if $p_n \geq 3$)

$$p_n^2 - p_{n-1}p_{n+1} \leq p_{n-1}^2 + 4p_{n-1} + 4 - p_{n-1}(p_{n-1} + 6) = -2p_{n-1} + 4 < 0.$$

Hence, from the twin prime conjecture it would follow that there are infinitely many $n \in N^+$ such that $\pi_n < 0$. For some further inequalities involving p_n we refer the reader to the monograph of Mitrinović and Popadić [**MiPo**].

The last problem in number theory (Problem 4) from the Kurepa's list concerns the left factorial hypothesis, and we discussed it already in the previous section.

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