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POSSIBLY EVERY REAL FUNCTION IS CONTINUOUS ON A NON-MEAGRE SET

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Abstract. We prove consistency of the following sentence: 'ZFC + every real function is continuous on a non-meagre set", answering a question of Fremlin.

0 Introduction

By Abraham, Rubin and Shelah [ARSh:153] it is consistent that every function from $\mathbb{R}$ to $\mathbb{R}$ is continuous when restricted to some uncountable set (and more ...). We may consider strengthening this statement, by demanding the subset on which the function is continuous to be large in a stronger sense. In [Fe94, Problem AR(b)], David Fremlin asked exactly this, namely, is it consistent that every function from $\mathbb{R}$ to $\mathbb{R}$ is continuous when restricted to some set which is non-meagre (i.e. not countable union of nowhere dense sets). We answer it (positively) here. We use ${}^{\omega}2$ for reals, and for $B \subseteq {}^{\omega}2$ we say $f : B \to {}^{\omega}2$ is continuous if $f(\eta_0) = \eta_1, n < \omega$ implies that for some m we have

$$(\forall \eta)[\eta_0 \restriction m \triangleleft \eta \in B \Rightarrow \eta_1 \restriction n \triangleleft f(\eta)],$$

where for sequences ν, η, $\nu \triangleleft \eta$ means "η (properly) extends ν".

The non-meagre sets here are of cardinality $\aleph_1$ and we may wonder whether we can get them of higher cardinality. From the point of view of those who asked the original question, maybe this does not add much (but I think it does make

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for much more interesting partition relations). Another interesting generalization, asked by Heinrich von Weizsäcker (see [Fe94, Problem AR(a)]), is:

($*$) is it consistent to have that every function from $\mathbb{R}$ to $\mathbb{R}$ is continuous when restricted to some non-null set?

We may ask about 2-place functions; Sierpinski colouring implies we cannot ask for one colour, but we may want the consistency of:

($**$) for every 2-place function $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ there is a non-meagre set (non-null) $A \subseteq \mathbb{R}$ and 2-place continuous functions $f_0, f_1 : A \times A \longrightarrow \mathbb{R}$ such that

$$(\forall x, y \in A)(f(x,y) = f_0(x,y) \vee f(x,y) = f_1(x,y))$$

(similarly for larger n or all n simultaneously). It is not clear to us how much this interests non-logicians, but to us it seems to be the right question and we shall deal with ($**$) and the cardinality in Rabus Shelah [RaSh:585].

We can also generalize the proof replacing $\aleph_0$ by $\mu = \mu^{<\mu}$ (as done here). On consistent partition relations see [Sh:276], [Sh:288], [Sh:546] and [RaSh:585]. Another related theorem, proved in [Sh:481], is the consistency of

($***$) if B_1 is a $2^{\aleph_0}$-c.c. Boolean algebra and B_2 is a c.c.c. Boolean algebra, then $B_1 * B_2$ is a $2^{\aleph_0}$-c.c. Boolean algebra.

Originally the proof goes through "meagre preserving" and iterations as in section 2 of [Sh:276], but the proof was cumbersome, had flaws and proper version was not manufactured. Here we present a simpler though a "degenerated" variant (i.e. we use only the forcing we have to and so we get $2^{\aleph_0} = \aleph_2$ rather than $2^{\aleph_0} = \aleph_3$).

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1 Continuity on a Non-Meagre Set

1.1 *Definition.* 1) $\mathrm{Cohen}_\mu(\alpha) = \{f : f \text{ a partial function from } \alpha \text{ to } \{0,1\} \text{ of cardinality } < \mu\}$ ordered by inclusion.

2) $\mathrm{Cohen}_\mu = \mathrm{Cohen}_\mu(\mu)$.

3) A μ-Cohen forcing P means $\mathrm{Cohen}_\mu(\alpha)$ for some ordinal α (or at least the set

$$\{p \in P : P \restriction \{q : q \geq p\} \text{ is equivalent to some } \mathrm{Cohen}_\mu(\alpha)\}$$

is a dense subset of P).

1.2 *Definition.* A set $A \subseteq {}^{\mu}2$ is μ-meagre if it is the union of $\leq \mu$ nowhere dense subsets of ${}^{\mu}2$.

1.3 Main Lemma. *Let μ be a regular cardinal such that $\mu = \mu^{<\mu}$. For $\ell < 2$ and $\alpha < \mu^+$, let $Q_{\alpha,\ell}$ be $({}^{\mu>}2, \triangleleft)$ (i.e. Cohen$_\mu$ forcing) and for $I \subseteq \mu^+ \times 2$ let P_I be the product $\prod\limits_{t \in I} Q_t$ with $(<\mu)$-support and $P = P_{\mu^+ \times 2}$. Let $\underset{\sim}{\eta}_{\alpha,\ell}$ be the $Q_{\alpha,\ell}$-name of the generic function from μ to $2 = \{0,1\}$. In*

$$V^P = V[\langle \underset{\sim}{\eta}_{\alpha,\ell} : \alpha < \mu^+, \ell < 2 \rangle]$$

let R be a forcing notion defined by:

$$R = \Big\{ (u_0, v, \bar{\nu}) \colon \begin{array}{ll} (a) & u_0 \text{ is a subset of } \mu^+ \text{ of cardinality } < \mu, \\ (b) & v \text{ is a subset of } {}^{\mu>}2 \text{ of cardinality } < \mu \\ (c) & \text{if } \rho \in v \text{ then } \ell g(\rho) \text{ is a successor ordinal and} \\ & \quad \rho(\ell g(\rho) - 1) = 1, \\ (d) & \bar{\nu} = \langle \nu_\rho : \rho \in v \rangle \text{ with } \nu_\rho \in {}^{\mu>}2, \\ (e) & \text{if } \rho_1 \triangleleft \rho_2 \text{ are from } v \text{ then } \nu_{\rho_1} \triangleleft \nu_{\rho_2} \text{ (hence they are not equal)}, \\ (f) & \text{for every } \rho \in v \text{ there is } \alpha \in u_0 \text{ such that } \rho \triangleleft \underset{\sim}{\eta}_{\alpha,0}, \\ (g) & \text{if } \alpha \in u_0, \rho \triangleleft \underset{\sim}{\eta}_{\alpha,0} \text{ and } \rho \in v \text{ then } \nu_\rho \triangleleft \underset{\sim}{\eta}_{\alpha,1}. \end{array} \Big\}.$$

<u>The order is natural</u>:
$(u_0, v, \bar{\nu}) \leq (u_0', v', \bar{\nu}')$ if and only if $u_0 \subseteq u_0'$, $v \subseteq v'$, and $\bar{\nu} = \bar{\nu}' \restriction v$ (and both are in R).

Let $\underset{\sim}{\mathcal{U}}$ be an R-name such that

$$\Vdash_R \underset{\sim}{\mathcal{U}} = \bigcup \{u_0 : \text{there are } v, \bar{\nu} \text{ such that } (u_0, v, \bar{\nu}) \in \underset{\sim}{G}_R\},$$

where $\underset{\sim}{G}_R$ is the canonical R-name for the generic filter on R.
<u>Then</u>:

- (a) *In $V^{P * \underset{\sim}{R}}$, the function $\underset{\sim}{\eta}_{\alpha,0} \mapsto \underset{\sim}{\eta}_{\alpha,1}$ from $\{\underset{\sim}{\eta}_{\alpha,0} : \alpha \in \underset{\sim}{\mathcal{U}}\}$ to $\{\underset{\sim}{\eta}_{\alpha,1} : \alpha \in \underset{\sim}{\mathcal{U}}\}$ is continuous.*
- (b) *In $V^{P * \underset{\sim}{R}}$, the set $\{\underset{\sim}{\eta}_{\alpha,0} : \alpha \in \underset{\sim}{\mathcal{U}}\}$ is a non-μ-meagre subset of ${}^{\mu}2$, in fact everywhere non-meagre (i.e. its intersection with any non-empty open set is non-meagre).*
- (c) *$P * \underset{\sim}{R}$ is a μ-Cohen forcing, moreover $P_{\alpha \times 2} \lessdot P$ (i.e. $P_{\alpha\times 2}$ is a complete suborder of P) and $P * \underset{\sim}{R}/P_{\alpha \times 2}$ is a μ-Cohen forcing for every $\alpha < \mu^+$, $\alpha \geq \mu$.*

1.3A *Remark.* We can get slightly more than continuity for $\mu > \aleph_0$ (a slight change of the forcing notion), namely, for some club $W \subseteq \mu$, for every $i \in W$ we have that $\underset{\sim}{\eta}_{\alpha,0} \restriction i$ determines $\underset{\sim}{\eta}_{\alpha,1} \restriction i$ (and so can get this in the theorem).

Proof. For $\alpha \leq \mu^+, \alpha \geq \mu$ let

$$A'_\alpha =: \Big\{(p,r) : \begin{array}{ll} (\alpha) & p \in P_{\alpha \times 2} \text{ and} \\ & \quad p = \langle \eta^p_{\gamma,\ell} : (\gamma,\ell) \in u^p \times 2\rangle \text{ where } \eta^p_{\gamma,\ell} \in {}^{\mu>}2, u^p \in [\alpha]^{<\mu}, \\ (\beta) & r = (u_0, v, \bar\nu) \text{ satisfies clauses } (a),(b),(c),(d),(e) \\ & \quad \text{of the definition of } R \text{ and} \\ & \quad (f)' \quad \text{for every } \rho \in v \text{ there is } \beta \in u^p \text{ such that} \\ & \qquad\qquad \beta \in u_0 \text{ and } \rho \trianglelefteq \eta^p_{\beta,0}, \\ & \quad (g)' \quad \text{if } \beta \in u_0 \text{ and } \rho \in v \text{ and } \rho \trianglelefteq \eta^p_{\beta,0}, \\ & \qquad\qquad \text{then } \nu_\rho \trianglelefteq \eta^p_{\beta,1}, \\ (\gamma) & u_0 \subseteq u^p. \Big\} \end{array}$$

and partially order A'_α in the natural way.

Also let

$$A_\alpha = \Big\{(p,r) : \begin{array}{ll} \multicolumn{2}{l}{p, r \text{ satisfy clauses } (\alpha),(\beta),(\gamma) \text{ above and}} \\ (\delta) & \text{if } \beta \in u_0 \text{ and } \rho \in v \text{ then } \neg(\eta^p_{\beta,0} \vartriangleleft \rho), \\ (\varepsilon) & \text{for every } \beta \in u_0 \text{ there is } \gamma \in u_0 \cap \mu \text{ such that} \\ & \qquad (\eta^p_{\beta,0}, \eta^p_{\beta,1}) = (\eta^p_{\gamma,0}, \eta^p_{\gamma,1})\Big\}. \end{array}$$

Note that in A'_α any increasing sequence of length $< \mu$ has a least upper bound, i.e. if a sequence $\langle q_i : i < \delta\rangle \subseteq A'_\alpha$ is increasing, $\delta < \mu$ then there is a condition $q \in A'_\alpha$ such that $q_i \leq q$ for all $i < \delta$ and if $\bigwedge\limits_{i<\delta} q_i \leq q' \in A'_\alpha$ then $q \leq q'$.

Now:

(A) A_α is a dense subset of A'_α. If $(p,r) \in A_\alpha$ and $\mu \leq \alpha \leq \mu^+$ then $(p,r) \in P * \underset{\sim}{R}$ (i.e. $p \Vdash_P$ "$r \in \underset{\sim}{R}$").

[Why? The second phrase should be clear. Trivially $A_\alpha \subseteq A'_\alpha$. Suppose $(p,r) \in A'_\alpha$ and we shall find $(p',r) \in A_\alpha$ such that $(p,r) \leq (p',r)$. Let $\beta^* =: \sup(u^p \cap \mu) + 1$ and $\varepsilon = \sup\{\ell g(\rho) + 1 : \rho \in v\}$ and let $\bar{\mathbf{0}}_\varepsilon$ be the sequence of zeroes of length ε and let $u^p = \{\gamma_\zeta : \zeta < |u^p|\}$ be an enumeration.

Let $u^{p'} = u^p \cup \{\beta^* + \zeta : \zeta < |u^p|\}$ and

$$\eta^{p'}_{\gamma,\ell} = \begin{cases} \eta^p_{\gamma,\ell}{}^\frown \bar{\mathbf{0}}_\varepsilon & \underline{\text{if}} \quad \gamma \in u^p, \ell = 0, \\ \eta^p_{\gamma,\ell} & \underline{\text{if}} \quad \gamma \in u^p, \ell = 1, \\ \eta^p_{\gamma_\zeta,\ell}{}^\frown \bar{\mathbf{0}}_\varepsilon & \underline{\text{if}} \quad \gamma = \beta^* + \zeta, \ell = 0, \\ \eta^p_{\gamma_\zeta,\ell} & \underline{\text{if}} \quad \gamma = \beta^* + \zeta, \ell = 1. \end{cases}$$

Note: if $\rho \in v$ then $\rho \trianglelefteq \eta{}^\frown \bar{\mathbf{0}}_\varepsilon \Leftrightarrow \rho \trianglelefteq \eta$ (because by clause (c) in the definition of R, the last element in ρ is one).]

(B) A_{μ^+} is a dense subset of $P * \underset{\sim}{R}$.

[Why? Since P is μ-closed, for each $(p, \underset{\sim}{r}) \in P * \underset{\sim}{R}$ we can find $p_1 \in P$ and r such that $p \le p_1 \in P$ and $p_1 \Vdash_P$ "$\underset{\sim}{r} = r$". Next we choose p_2 such that $p_1 \le p_2 \in P$ and clauses (α)–(ε) in the definition of A_{μ^+} hold. So $(p, \underset{\sim}{r}) \le (p_2, r)$ (in $P * \underset{\sim}{R}$) and $(p_2, r) \in A_{\mu^+}$.]

If $\mu \le \alpha \le \beta \le \mu^+$ and $(p, r) \in A_\beta$ then we let $(p, r) \restriction \alpha = (p \restriction \alpha, r \restriction \alpha)$ where $p \restriction \alpha = \langle \eta^p_{\gamma,\ell} : (\gamma, \ell) \in (u^p \cap \alpha) \times 2 \rangle$ and $r \restriction \alpha = (u^r_0 \cap \alpha, v^r, \bar{\nu}^r)$. It is easy to check that:

(C) If $\mu \le \alpha \le \beta \le \mu^+$ and $(p, r) \in A_\beta$ then $(p, r) \restriction \alpha \in A_\beta$ and $(p, r) \restriction \alpha \in A_\alpha$ (note that clause $(f)'$ of the definition of A_α follows from clause (ε) in the definition of A_α).

(D) If $\mu \le \alpha \le \beta \le \mu^+$ and $(p_1, r_1) \in A_\beta$ and $(p_1, r_1) \restriction \alpha \le (p_2, r_2) \in A_\alpha$ then $(p_1, r_1), (p_2, r_2)$ are compatible (in A_β); moreover, they have the natural least upper bound

$$\left(p_2 \cup p^*, (u_0^{r_1} \cup u_0^{r_2}, v^{r_2}, \bar{\nu}^{r_2})\right)$$

in A'_β, where $p^* \in P_{\beta \times 2}$ is such that $u^{p^*} = u^{p_1} \setminus \alpha$, and for $\gamma \in u^{p^*}$

$$\eta^{p^*}_{\gamma,\ell} = \begin{cases} \eta^{p_1}_{\gamma,\ell} & \text{if} \quad \text{either } \gamma \notin u_0^{r_1} \text{ or } \ell = 0, \\ \eta^{p_1}_{\gamma,1} \cup \bigcup \{\nu^{r_2}_\rho : \rho \trianglelefteq \eta^{p_1}_{\gamma,0}\} & \text{if} \quad \gamma \in u_0^{r_1} \text{ and } \ell = 1. \end{cases}$$

Moreover:

$(D)^+$ $\underline{\textit{if}}$ $\langle (p_{1,i}, r_{1,i}) : i \le \delta \rangle$ is an increasing continuous sequence of conditions in A'_β such that $\delta < \mu$ and $(p_{1,i}, r_{1,i}) \in A_\beta$ for each non-limit $i \le \delta$ and $(p_{1,i}, r_{1,i}) \restriction \alpha \le (p_2, r_2)$ for $i < \delta$

$\underline{\textit{then}}$ $(p_{1,\delta}, r_{1,\delta}), (p_2, r_2)$ are compatible in A'_β (and $(p_{1,\delta}, r_{1,\delta}) \restriction \alpha \le (p_2, r_2)$).

Hence

$(D)^{++}$ if $\mu \le \alpha \le \beta \le \mu^+$ then $A_\alpha \subseteq A_\beta$, moreover $A_\alpha \lessdot A_\beta$ (by (D)) and the quotient A_β / A_α is μ-closed (i.e. increasing chains of length $< \mu$ have upper bound).

Remark. For better understanding note that if Q_0 is adding a μ-Souslin tree, $\underset{\sim}{Q}_1$ is the forcing determined by this tree, then the composition $Q_0 * \underset{\sim}{Q}_1$ is not even μ-closed. The point is that increasing sequences in $Q_0 * \underset{\sim}{Q}_1$ of length $< \mu$ have an upper bound, but not necessarily least upper bound.

Now (note the support)

(E) the sequence $\langle A_\alpha : \alpha \in [\mu, \mu^+] \rangle$ is increasing, and for limit $\alpha \in (\mu, \mu^+)$, A_α is the inverse limit of $\langle A_\beta : \beta < \alpha \rangle$ if $\mathrm{cf}(\alpha) < \mu$, and direct limit of it if $\mathrm{cf}(\alpha) \geq \mu$. Hence $\mu \leq \alpha < \beta \leq \mu^+ \Rightarrow A_\beta / A_\alpha$ is a μ-Cohen forcing (remember that atomless μ-closed forcing notion of size μ is μ-Cohen).

It is also clear that

(F) $P_{\alpha \times 2} \lessdot P_{\beta \times 2}$ if $\alpha < \beta \leq \mu^+$.

Lastly note that

(G) for $\alpha \in [\mu, \mu^+), A_\alpha / P_{\alpha \times 2}$ is μ-closed and hence it is μ-Cohen.

From the desirable conclusions, we have gotten clause (c) (by combining clauses (E) and (G)).

(H) If $(p_1, r_1) \in A_{\mu^+}$, $\beta^* \in \mu^+ \setminus u^{p_1}$ and $\nu^* \in {}^{\mu >}2$ then there is $(p_2, r_2) \in A_{\mu^+}$ such that

$$(p_1, r_1) \leq (p_2, r_2), \quad \beta^* \in u_0^{r_2} \quad \text{and} \quad \nu^* \triangleleft \eta^{p_2}_{\beta^*, 0}.$$

[Why? Let $\beta^* \in \mu^+ \setminus u^{p_1}$, $\beta^{**} \in \mu \setminus (u^{p_1} \cup \{\beta^*\})$ and let

$$\nu^{\otimes} = \bigcup \{ \nu^{r_1}_\rho : \rho \trianglelefteq \nu^* \text{ and } \rho \in v^{r_1} \} \quad \text{and} \quad \gamma = \sup\{ \ell g(\nu) + 1 : \nu \in v^{r_1} \cup \{\nu^*\} \}.$$

Now define (p_2, r_2) as follows:

$$u^{p_2} = u^{p_1} \cup \{\beta^*, \beta^{**}\}$$

$$\eta^{p_2}_{\beta,\ell} = \begin{cases} \eta^{p_1}_{\beta,\ell} & \text{if} \quad \beta \in u^{p_1}, \ell < 2, \\ \nu^* {}^\frown \bar{\mathbf{0}}_\gamma & \text{if} \quad \beta \in \{\beta^*, \beta^{**}\}, \ell = 0, \\ \nu^{\otimes} & \text{if} \quad \beta \in \{\beta^*, \beta^{**}\}, \ell = 1, \end{cases}$$

and $r_2 = (u_0^{r_2}, v^{r_2}, \bar{\nu}^{r_2})$ is defined by $u_0^{r_2} = u_0^{r_1} \cup \{\beta^*, \beta^{**}\}$, $\quad v^{r_2} = v^{r_1}$, $\quad \bar{\nu}^{r_1} = \bar{\nu}^{r_2}$. Easily (p_2, r_2) is as required.]

(I) If $(p_1, r_1) \in A_{\mu^+}$, $\beta^* \in u_0^{r_1}$ then there is $(p_2, r_2) \in A_{\mu^+}$ such that

$$(p_1, r_1) \leq (p_2, r_2), \quad \eta^{p_2}_{\beta^*, 0} \in v^{r_2} \quad \text{and} \quad \eta^{p_2}_{\beta^*, 1} = (\nu_{\eta^{p_2}_{\beta^*, 0}})^{r_2}.$$

[Why? By clause (A) above it is enough to find such (p_2, r_2) in A'_β. Take $\beta^{**} \in u_0^{r_1} \cap \mu$ such that $(\eta^{p_1}_{\beta^{**},0}, \eta^{p_1}_{\beta^{**},1}) = (\eta^{p_1}_{\beta^*,0}, \eta^{p_1}_{\beta^*,1})$, and let

$$\gamma = \sup\{\ell g(\nu) + 1 : \nu \in v^{r_1} \cup \{\eta^{p_1}_{\beta,\ell} : \beta \in u^{p_1}, \ell < 2\}\}.$$

Define condition (p_2, r_2) by: $\quad u^{p_2} = u^{p_1}$, and

$$\eta^{p_2}_{\beta,\ell} = \begin{cases} \eta^{p_1}_{\beta^*,0}\,\hat{}\,\bar{\mathbf{0}}_\gamma\,\hat{}\,\langle 1\rangle & \text{if} \quad \beta \in \{\beta^*, \beta^{**}\}, \ell = 0, \\ \eta^{p_1}_{\beta,1}\,\hat{}\,\langle 1\rangle & \text{if} \quad \beta \in \{\beta^*, \beta^{**}\}, \ell = 1, \\ \eta^{p_1}_{\beta,\ell}\,\hat{}\,\bar{\mathbf{0}}_{2\gamma+1} & \text{if} \quad \beta \in u^{p_1} \setminus \{\beta^*, \beta^{**}\}, \ell < 2, \end{cases}$$

$$u_0^{r_2} = u_0^{r_1}, \quad v^{r_2} = v^{r_1} \cup \{\eta^{p_1}_{\beta^*,0}\,\hat{}\,\bar{\mathbf{0}}_\gamma\,\hat{}\,\langle 1\rangle\}, \quad \text{and}$$

$$\nu^{r_2}_\rho = \begin{cases} \nu^{r_1}_\rho & \text{if} \quad \rho \in v_1^{r_1}, \\ \eta^{p_1}_{\beta^*,1}\,\hat{}\,\langle 1\rangle & \text{if} \quad \rho = \eta^{p_1}_{\beta^*,0}\,\hat{}\,\bar{\mathbf{0}}_\gamma\,\hat{}\,\langle 1\rangle. \end{cases}$$

It is easy to check that $(p_2, r_2) \in A_{\mu^+}$ is as required.]

Due to clause (I) above we immediately get assertion (a) of 1.3, i.e. the function $\eta \mapsto \bigcup\{\nu^r_\rho : \rho \lhd \eta,\ \rho \in \bigcup\limits_{r \in G_{\underset{\sim}{R}}} v^r\}$ is trivially continuous and its range is contained in ${}^{\mu\geq}2$, but clause (I) implies that for each $\alpha \in \underset{\sim}{\mathcal{U}}$ the image of $\underset{\sim}{\eta}_{\alpha,0}$ is $\underset{\sim}{\eta}_{\alpha,1} \in {}^{\mu}2$ (and not a proper initial segment of it).

For clause (b) assume that

$(*)$ $\nu^* \in {}^{\mu>}2$ and $\langle \underset{\sim}{B}_i : i < \mu\rangle$ is a sequence of $(P * \underset{\sim}{R})$-names, $\Vdash_{P*\underset{\sim}{R}}$ "$\underset{\sim}{B}_i \subseteq {}^{\mu}2$ is nowhere dense and (for simplicity) with no isolated points".

It suffices to prove

$$\Vdash_{P*\underset{\sim}{R}} \text{“}\{\underset{\sim}{\eta}_{\alpha,0} : \alpha \in \underset{\sim}{\mathcal{U}} \text{ and } \nu^* \lhd \underset{\sim}{\eta}_{\alpha,0}\} \not\subseteq \bigcup_{i<\mu} \underset{\sim}{B}_i\text{”}.$$

Let $\underset{\sim}{T}_i = \{\eta \restriction \gamma : \eta \in \underset{\sim}{B}_i \text{ and } \gamma < \mu\}$, so $\Vdash_{P*\underset{\sim}{R}}$ "$\underset{\sim}{T}_i \subseteq {}^{\mu>}2$ is a nowhere dense subtree and $(\forall \eta \in {}^{\mu}2)(\eta \in \underset{\sim}{B}_i \Rightarrow \bigwedge\limits_{\gamma<\mu} \eta \restriction \gamma \in \underset{\sim}{T}_i)$", where "$T \subseteq {}^{\mu>}2$ is a nowhere dense subtree" means: $\langle\rangle \in T$, $(\forall \eta \in T)(\forall \gamma < \ell g(\eta))(\eta \restriction \gamma \in T)$ and for every $\eta \in T$ and $\gamma < \mu$ there are two $\lhd$-incomparable sequences ν_0, ν_1 such that

$$\eta \lhd \nu_0 \in T \;\;\&\;\; \eta \lhd \nu_1 \in {}^{\mu>}2 \setminus T \;\;\&\;\; \ell g(\nu_0) \geq \gamma \;\;\&\;\; \ell g(\nu_1) \geq \gamma.$$

We can find $\alpha < \mu^+$ such that $\alpha > \mu$ and for every $i < \mu$, $\underset{\sim}{T}_i$ is an A_α-name (remember $\mu^{<\mu} = \mu$). Let $(p_0, r_0) \in P * \underset{\sim}{R}$. By the statement (B) above there is

(p_1, r_1) such that $(p_0, r_0) \leq (p_1, r_1) \in A_{\mu^+}$. We can easily find (p_2, r_2) such that $(p_1, r_1) \leq (p_2, r_2) \in A_{\mu^+}$ and $\beta^* \in (\alpha, \mu^+)$ such that $\beta^* \in u_1^{r_2}$ and $\nu^* \lhd \eta^{p_2}_{\beta^*,0} \in v_1^{r_2}$, $(\nu_{\eta^{p_2}_{\beta^*,0}})^{r_2} = \eta^{p_2}_{\beta^*,1}$ (apply clauses (H) and (I)).

It suffices to prove that for each $i < \mu$, $(p_2, r_2) \Vdash$ "$\underset{\sim}{\eta}_{\beta^*,0} \notin \underset{\sim}{B}_i$". So let $i < \mu$ and $(p_2, r_2) \leq (p_3, r_3) \in A_{\mu^+}$. Since $(p_3, r_3) \restriction \alpha \in A_\alpha$, $\underset{\sim}{T}_i$ is an A_α-name and $\Vdash_{A_\alpha}$ "$\underset{\sim}{T}_i \subseteq {}^{\mu>}2$ is nowhere dense", there is a condition $(p_4, r_4) \in A_\alpha$ such that $(p_3, r_3) \restriction \alpha \leq (p_4, r_4)$ and for some ν^{**} we have

$$\eta^{p_3}_{\beta^*,0} \lhd \nu^{**} \in {}^{\mu>}2 \quad \text{ and } \quad (p_4, r_4) \Vdash_{A_\alpha} \text{“}\nu^{**} \notin \underset{\sim}{T}_i\text{”}.$$

Lastly we define $(p_5, r_5) \in A'_{\mu^+}$ such that (p_5, r_5) is above (p_3, r_3) and above (p_4, r_4) and $(p_5, r_5) \Vdash$"$v^{**} \lhd \eta_{\beta^*,0}$". So let

$$u^{p_5} = u^{p_3} \cup u^{p_4},$$

$$\eta^{p_5}_{\beta,\ell} = \begin{cases} \eta^{p_4}_{\beta,\ell} & \text{if} \quad \beta \in u^{p_4}, \\ \eta^{p_3}_{\beta,\ell} & \text{if} \quad \beta \in u^{p_3} \setminus (u^{p_4} \cup \{\beta^*\}), \\ \nu^{**} & \text{if} \quad \beta = \beta^*, \ell = 0, \\ \bigcup\{\nu^{r_4}_\rho : \rho \in v^{r_4} \text{ and } \rho \trianglelefteq \nu^{**}\} & \text{if} \quad \beta = \beta^*, \ell = 1, \end{cases}$$

$$r_5 = (u_0^{r_4} \cup u_0^{r_3}, v^{r_4}, \bar{\nu}^{r_4}).$$

By clause (A) there is (p_6, r_6) such that $(p_5, r_5) \leq (p_6, r_6) \in A_{\mu^+}$. This clearly suffices. Thus we finish proving clause (b) of the conclusion of 1.3.

Together we get all the required conclusions. $\square_{1.3}$

1.4 Theorem. *Assume that* $\mu = \mu^{<\mu}, 2^\mu = \mu^+, 2^{\mu^+} = \mu^{+2}$ *and* S *is a stationary subset of*

$$S^{\mu^{+2}}_{\mu^+} =: \{\delta < \mu^{+2} : cf(\delta) = \mu^+\}.$$

Suppose that $\diamondsuit_S$ *holds.*

Then there is a μ*-closed,* μ^+*-c.c. forcing notion* P *of cardinality* μ^{+2} *such that, in* V^P*,*

- $(*)$ *for every function* $f : {}^\mu 2 \to {}^\mu 2$*, there is a non* μ*-meagre set* $A \subseteq {}^\mu 2$ *such that* $f \restriction A$ *is continuous.*

Proof. Without loss of generality we may also assume that $S^{\mu^{+2}}_{\mu^+} \setminus S$ is stationary.

For $\alpha \leq \mu^{+2}$ let $\mathfrak{K}_\alpha$ be the family of sequences $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \alpha\rangle$ such that

- (a) $\bar{Q}$ is iteration with support $< \mu$ and P_α is the limit,

(b) for $i \in \alpha \cap \mu^{+2} \setminus S$, $\underset{\sim}{Q}_i = Q_i$ is $({}^{\mu>}2, \triangleleft)$ (i.e. a μ-Cohen forcing notion) and we denote by $\underset{\sim}{\eta}^{\bar{Q}}_{i+1}$ the name of a Cohen generic element of ${}^{\mu}2$ adjoined by Q_i,

(c) if $\beta_1 < \beta_2 \leq \alpha$, $\beta_2 < \mu^{+2}$, $\beta_1 \notin S$ then P_{β_2}/P_{β_1} and $P_{\beta_2}/(P_{\beta_1} * Q_{\beta_1})$ are μ-Cohen forcing notions (and for clarity we fix representation of P_{β_2} as

$$P_{\beta_1} \times Q_{\beta_1} \times \prod \{Q^{\beta_1,\beta_2}_j : j < j(\beta_1, \beta_2)\}$$

with support $< \mu$),

(d) we may assume that each $\underset{\sim}{Q}_i$ (for $i < \alpha$) is μ-closed, μ^+-cc and of cardinality $\leq \mu^+$.

Note that by clause (c), each P_β is a μ-closed forcing notion satisfying the μ^+-cc. Let $\mathfrak{K} = \bigcup\limits_{\alpha < \mu^{+2}} \mathfrak{K}_\alpha$.

There is a natural ordering of $\mathfrak{K}$:

$$\bar{Q}^1 \leq \bar{Q}^2 \qquad \text{if and only if} \qquad \bar{Q}^1 = \bar{Q}^2 \restriction \ell g(\bar{Q}^1).$$

This partial order is μ^{+2}-complete and every (strictly) increasing chain of length μ^{+2} has a limit in $\mathfrak{K}_{\mu^{+2}}$.

Note that by clause (c) we have that

($\boxtimes$) if $\bar{Q} \in \mathfrak{K}_\alpha, \beta < \alpha, \beta \notin S$ and $\Vdash_{P_\beta}$ "$\underset{\sim}{B} \subseteq {}^{\mu}2$ is not μ-meagre"

then $\Vdash_{P_\alpha}$ "$\underset{\sim}{B} \subseteq {}^{\mu}2$ is not μ-meagre".

Because of this, and by $\diamondsuit_S$, it is enough to show that

($\bigotimes$) <u>if</u> $\bar{Q} \in \mathfrak{K}_{\mu^{+2}}$, $P = P_{\mu^{+2}} = \lim(\bar{Q})$, and $\underset{\sim}{f}$ is a P-name of a function from ${}^{\mu}2$ to ${}^{\mu}2$ <u>then</u> for some club E of μ^{+2}, for every $\delta \in S \cap E$ we have

(α) $i < \delta \Rightarrow \underset{\sim}{f}(\underset{\sim}{\eta}^{\bar{Q}}_{i+1})$ is a P_δ-name,

(β) there is a $\bar{Q}^{\delta+1} \in \mathfrak{K}_{\delta+1}$ such that $\bar{Q}^{\delta+1} \restriction \delta = \bar{Q} \restriction \delta$ and $\Vdash_{\lim(\bar{Q}^{\delta+1})}$ "for some $A \subseteq \delta, |A| = \mu^+$, we have that

(i) $\{\underset{\sim}{\eta}^{\bar{Q}}_{i+1} : i \in A\}$ is not meagre,

(ii) the function $\{(\underset{\sim}{\eta}^{\bar{Q}}_{i+1}, \underset{\sim}{f}(\underset{\sim}{\eta}^{\bar{Q}}_{i+1})) : i \in A\}$ is continuous".

(Note: $\underset{\sim}{\eta}^{\bar{Q}}_{i+1}$ is a P_δ-name (as $i+1 < \delta$) and also $\underset{\sim}{f}(\underset{\sim}{\eta}^{\bar{Q}}_{i+1})$ is a P_δ-name (by clause (α))).

Proof of $(\bigotimes)$. For each $\delta \in S^{\mu^{+2}}_{\mu^+} \setminus S$, $\underset{\sim}{f}(\underset{\sim}{\eta}^{\bar{Q}}_{\delta+1})$ is a $P_{\mu^{+2}}$-name of a member of ${}^{\mu}2$, so for some $\gamma_\delta > \delta + 1$, it is a P_{γ_δ}-name and we can demand that γ_δ is a successor ordinal. As

$$P_{\gamma_\delta} = P_\delta \times Q_\delta \times \prod_{j<j(\delta,\gamma_\delta)} Q_j^{\delta,\gamma_\delta}$$

(with support $< \mu$), we find a set $w_\delta \subseteq j(\delta, \gamma_\delta)$ of cardinality $\leq \mu$ such that $\underset{\sim}{f}(\underset{\sim}{\eta}^{\bar{Q}}_{\delta+1})$ is a $P_\delta \times Q_\delta \times \prod\limits_{j\in w_\delta} Q_j^{\delta,\gamma_\delta}$-name. Also as $\mathrm{cf}(\delta) = \mu^+$ there is $\beta_\delta < \delta$ such that $\underset{\sim}{f}(\underset{\sim}{\eta}^{\bar{Q}}_{\delta+1})$ is a $P_{\beta_\delta} \times Q_\delta \times \prod\limits_{j\in w_\delta} Q_j^{\delta,\gamma_\delta}$-name and we can demand that β_δ is a successor ordinal. By Fodor's lemma (as $2^\mu = \mu^+$) for some stationary $S_1 \subseteq S^{\mu^{+2}}_{\mu^+} \setminus S$ (and $\zeta < \mu^+$, $\beta^* < \mu^{+2}$) we have that

(α) $\delta \in S_1 \Rightarrow \beta_\delta = \beta^*$
(β) $\delta \in S_1 \Rightarrow \ \mathrm{otp}(w_\delta) = \zeta$
(γ) for $\delta_1, \delta_2 \in S_1$ we demand $f_{\delta_2,\delta_1}(\underset{\sim}{f}(\underset{\sim}{\eta}^{\bar{Q}}_{\delta_1+1})) = \underset{\sim}{f}(\underset{\sim}{\eta}^{\bar{Q}}_{\delta_2+1})$, where f_{δ_2,δ_1} is the natural isomorphism

$$\text{from} \quad P_{\beta^*} \times Q_{\delta_1} \times \prod_{j\in w_{\delta_1}} Q_j^{\delta_1,\gamma_{\delta_1}} \quad \text{onto} \quad P_{\beta^*} \times Q_{\delta_2} \times \prod_{j\in w_{\delta_2}} Q_j^{\delta_2,\gamma_{\delta_2}}.$$

Clearly $Q^\delta = \prod\limits_{j\in w_\delta} Q_j^{\delta,\gamma_\delta}$ is isomorphic to Cohen forcing.

Lastly, let

$$E = \left\{\delta < \mu^{+2} : \delta = \sup(S_1 \cap \delta) \text{ and } \delta_1 \in S_1 \cap \delta \Rightarrow \gamma_{\delta_1} < \delta\right\}.$$

Clearly the set E is a club of μ^{+2}. We are going to show that for each $\delta^* \in E \cap S$ there is $\bar{Q}^{\delta^*+1}$ as required in $(\bigotimes)$.

Let $\delta^* \in E \cap S$. We can find an increasing continuous sequence $\langle \beta_\varepsilon : \varepsilon < \mu^+ \rangle$ with limit δ^* and a sequence $\langle \delta_\varepsilon : \varepsilon < \mu^+ \rangle$ such that $\beta_0 = \beta^*$, β_ε a successor ordinal if ε is a non-limit, $\beta_\varepsilon \notin S$, $\delta_\varepsilon \in S_1$, and $\beta_\varepsilon < \delta_\varepsilon < \gamma_{\delta_\varepsilon} < \beta_{\varepsilon+1}$. Let $\underset{\sim}{\eta}^*_{\varepsilon,0}$ and $\underset{\sim}{\eta}^*_{\varepsilon,1}$ be P_δ / P_{β_0}-names such that $\underset{\sim}{\eta}^*_{\varepsilon,0}$ is $\underset{\sim}{\eta}^{\bar{Q}}_{\delta_\varepsilon+1}$ and $\underset{\sim}{\eta}^*_{\varepsilon,1}$ is a name for the Cohen subset of μ added by Q^{δ_ε}.

Now let

$$R^0_{\delta^*} =: \prod [\{Q_{\delta_\varepsilon} : \varepsilon < \mu^+\} \cup \{Q^{\delta_\varepsilon} : \varepsilon < \mu^+\}]$$

and note that $R^0_{\delta^*} \lessdot P_{\delta^*}/P_{\beta^*}$. The forcing notion $R^0_{\delta^*}$ is naturally isomorphic to P from 1.3 with $\underset{\sim}{\eta}^*_{\varepsilon,\ell}$ corresponding to $\underset{\sim}{\eta}_{\varepsilon,\ell}$. Moreover the quotient $P_{\delta^*}/(P_{\beta^*} \times R^0_{\delta^*})$

is a μ-Cohen forcing notion. [Why? For each $\varepsilon < \mu^+$, $\delta_\varepsilon, \gamma_{\delta_\varepsilon}, \beta_\varepsilon \notin S$ and hence we may write $P_{\beta_{\varepsilon+1}}$ as

$$P_{\beta_\varepsilon} \times Q_{\beta_\varepsilon} \times \prod_{j<j(\beta_\varepsilon,\delta_\varepsilon)} Q_j^{\beta_\varepsilon,\delta_\varepsilon} \times Q_{\delta_\varepsilon} \times \prod_{j<j(\delta_\varepsilon,\gamma_{\delta_\varepsilon})} Q_j^{\delta_\varepsilon,\gamma_{\delta_\varepsilon}} \times Q_{\gamma_{\delta_\varepsilon}} \times \prod_{j<j(\gamma_{\delta_\varepsilon},\beta_{\varepsilon+1})} Q_j^{\gamma_{\delta_\varepsilon},\beta_{\varepsilon+1}}.$$

But

$$\prod_{j<j(\delta_\varepsilon,\gamma_{\delta_\varepsilon})} Q_j^{\delta_\varepsilon,\gamma_{\delta_\varepsilon}} = Q^{\delta_\varepsilon} \times \prod_{j\in j(\delta_\varepsilon,\gamma_{\delta_\varepsilon})\setminus w_{\delta_\varepsilon}} Q_j^{\delta_\varepsilon,\gamma_{\delta_\varepsilon}}.$$

So we may represent $P_{\beta_{\varepsilon+1}}$ as

$$P_{\beta_{\varepsilon+1}} = P_{\beta_\varepsilon} \times Q_{\delta_\varepsilon} \times Q^{\delta_\varepsilon} \times R^*_\varepsilon,$$

where R^*_ε is a product (with support $< \mu$) of μ-Cohen forcing notions. Now, since the sequence $\langle \beta_\varepsilon : \varepsilon < \mu^+\rangle$ is increasing and continuous with limit δ^* we may write

$$P_{\delta^*} = P_{\beta^*} \times \prod_{\varepsilon<\mu^+} Q_{\delta_\varepsilon} \times \prod_{\varepsilon<\mu^+} Q^{\delta_\varepsilon} \times \prod_{\varepsilon<\mu^+} R^*_\varepsilon$$

(all products with $< \mu$ support).]

Let $\underset{\sim}{Q}_{\delta^*}$ be $R[\langle \underset{\sim}{\eta}^*_{\varepsilon,0}, \underset{\sim}{\eta}^*_{\varepsilon,1} : \varepsilon < \mu^+\rangle]$ (from 1.3), $\bar{Q}^{\delta^*+1} = (\bar{Q} \restriction \delta^*)\hat{\ }\langle (P_{\delta^*}, \underset{\sim}{Q}_{\delta^*})\rangle$. To check that $\bar{Q}^{\delta^*+1}$ is as desired we use 1.3. Thus P_{δ^*}/P_β (for $\beta < \delta$, $\beta \notin S$) is μ-Cohen by 1.3(c) and because, as said above, $P_{\delta^*}/(P_{\beta^*} \times R^0_{\delta^*})$ is μ-Cohen. It follows from 1.3(a) that the function $\underset{\sim}{\eta}^*_{\varepsilon,0} \mapsto \underset{\sim}{\eta}^*_{\varepsilon,1}$ (for $\varepsilon \in \underset{\sim}{\mathcal{U}}$) is continuous and hence, as the names $\underset{\sim}{f}(\underset{\sim}{\eta}^{\bar{Q}}_{\delta_\varepsilon+1})$ are isomorphic, the function $\underset{\sim}{\eta}^{\bar{Q}}_{\delta_\varepsilon+1} \mapsto \underset{\sim}{f}(\underset{\sim}{\eta}^{\bar{Q}}_{\delta_\varepsilon+1})$ (for $\varepsilon \in \underset{\sim}{\mathcal{U}}$) is continuous (it is obviously Borel, which actually suffices, but in fact, as the $\underset{\sim}{\eta}^*_{\varepsilon,0}$ are Cohen over the definition of $\underset{\sim}{f}$ it is continuous). Finally, by 1.3(b),

$$\Vdash_{P_{\delta+1}} \text{“}\{\underset{\sim}{\eta}^{\bar{Q}}_{\delta_\varepsilon+1} : \varepsilon \in \underset{\sim}{\mathcal{U}}\} \text{ is not meagre”}.$$

This finishes the proof. $\square_{1.4}$

1.5 *Remark.* 1. Alternatively, one may force the iteration instead of using $\diamondsuit_S$.

2. The forcing notion from 1.4 is constructed like “λ-free not free (Boolean) algebra of cardinality λ”, with *free* interpreted as *μ-Cohen.*

2 Final remarks

Haim Judah asked us whether adding Cohen reals is enough.

2.1 OBSERVATION. *If P is the forcing notion for adding $2^{\aleph_0}$ Cohen real, then in V^P there is $f : \mathbb{R} \to \mathbb{R}$ which is not continuous on any uncountable set.*

Proof. Let $\lambda = 2^{\aleph_0}$, $P = \{f : f \text{ is a finite function from } \lambda \text{ to } \{0,1\}\}$. For any set $w \subseteq \lambda$ let $P_w = \{f \in P : \mathrm{Dom}(f) \subseteq w\}$. It is known that $P_w \lessdot P = P_\lambda$ and for any P-name $\underset{\sim}{\eta}$ of a real (i.e. $\Vdash_P$ "$\underset{\sim}{\eta} \in {}^\omega 2$") for some countable $w \subseteq \lambda$, $\underset{\sim}{\eta}$ is a P_w-name. For every ordinal $\alpha < \lambda$ let $\underset{\sim}{r}_\alpha \in {}^\omega 2$ be defined by:

$$\underset{\sim}{r}_\alpha(n) = \ell \text{ if and only if} \quad \text{for some } f \in \underset{\sim}{G}_P,\ f(\omega\alpha + n) = \ell.$$

Clearly $\underset{\sim}{r}_\alpha$ is a Cohen real. Let $\langle \underset{\sim}{s}_\zeta : \zeta < \lambda \rangle$ list all P-names of reals, so for some countable $w_\zeta \subseteq \lambda$, $\underset{\sim}{s}_\zeta$ is a P_{w_ζ}-name. Without loss of generality

$$\alpha + \omega = \beta + \omega \ \ \& \ \ \alpha \in w_\zeta \qquad \Rightarrow \qquad \beta \in w_\zeta.$$

We now define by induction on $\zeta < \lambda$ an ordinal $\alpha_\zeta < \lambda$ as follows:

α_ζ is the first ordinal α such that

$$\alpha \notin \{\alpha_\varepsilon : \varepsilon < \zeta\} \cup \{\beta : \omega + k\beta \in w_\zeta \text{ for some } k \in \omega\}.$$

Let $G \subseteq P$ be a generic filter over V. In $V[G]$ we define a function $F : {}^\omega 2 \to {}^\omega 2$ as follows:

$$\text{for } s \in {}^\omega 2 \quad \text{let} \quad \gamma(s) = \min\{\zeta : s = \underset{\sim}{s}_\zeta[G]\} \quad \text{and} \quad F(s) = \underset{\sim}{r}_{\alpha_{\gamma(s)}}.$$

Suppose now that, in $V[G]$, a set $A \subseteq {}^\omega 2$ is uncountable and $F \restriction A$ is continuous. Thus we have a Borel function $F' : {}^\omega 2 \longrightarrow {}^\omega 2$ such that $F \restriction A = F' \restriction A$. The function F' is coded by a single real $s \in {}^\omega 2$, $\gamma(s) = \zeta$.

Let $p_0 \in G$, $\underset{\sim}{A}, \underset{\sim}{F}, \underset{\sim}{F}', \underset{\sim}{s}$ and $\zeta < \lambda$ be such that

$$p_0 \Vdash_P \text{"}\underset{\sim}{A}, \underset{\sim}{F}, \underset{\sim}{F}', \underset{\sim}{s} = \underset{\sim}{s}_\zeta, \zeta = \gamma(\underset{\sim}{s}) \text{ are as above".}$$

The set $\{\xi < \lambda : [\omega\alpha_\xi, \omega\alpha_\xi + \omega) \cap w_\zeta \neq \emptyset\}$ is countable, $p \Vdash_P$"$\underset{\sim}{A}$ is uncountable", so we find $\xi < \lambda$ and $p_1 \geq p_0$, $p_1 \in G$ such that

$$p_1 \Vdash_P \text{"}\underset{\sim}{s}_\xi \in \underset{\sim}{A} \text{ and } \gamma(\underset{\sim}{s}_\xi) = \xi\text{"}$$

and $[\omega\alpha_\xi, \omega\alpha_\xi + \omega) \cap w_\zeta = \emptyset$. Now

$$p_1 \Vdash_P \text{"}\underset{\sim}{F}'(\underset{\sim}{s}_\xi) = \underset{\sim}{F}(\underset{\sim}{s}_\xi) = \underset{\sim}{r}_{\alpha_\xi}\text{".}$$

But we can compute $\underset{\sim}{F}'(\underset{\sim}{s}_\xi)$ in $V[G \cap P_{w_\zeta \cup w_\xi}]$ and (by the choice of ξ) $\underset{\sim}{r}_{\alpha_\xi}$ is Cohen over $V[G \cap P_{w_\zeta \cup w_\xi}]$ (remember the choice of α_ξ's), which is a contradiction. $\square_{2.1}$

2.2 *Remark.* Instead of forcing it suffices to assume the existence of a Luzin set in a strong sense.

* * *

We call $A \subseteq {}^\omega 2$ a Luzin set if it is uncountable and non-meagre and any uncountable $B \subseteq A$ is non-meagre.

2.3 *Definition.* A forcing notion Q is Luzin-preserving if it satisfies the c.c.c. and

- $(*)$ for any Luzin set $A \subseteq {}^\omega 2$ and a sequence $\langle p_\eta : \eta \in A\rangle$ of members of Q, we have that
- $(**)$ for some $\eta \in A$
  $p_\eta \Vdash_Q$ "for some $\rho \in {}^{\omega >}2$,
  $(\forall \nu)[\rho \trianglelefteq \nu \in {}^{\omega >}2 \Rightarrow$ for uncountably many $\eta_1 \in A, \nu \triangleleft \eta_1$ & $p_\eta \in \underset{\sim}{G}_Q]$".

2.4 Claim. 0) *Assume there are Luzin sets, then in Definition 2.3 "it satisfies the c.c.c." follows from $(*)$ of 2.3.*
1) *A forcing notion Q is Luzin-preserving iff it satisfies the c.c.c. and for any Luzin $A \subseteq {}^\omega 2$ we have, $\Vdash_Q$ "A is Luzin".*
2) *Being Luzin-preserving is preserved by composition.*
3) *If $\langle P_i : i < \delta\rangle$ is an increasing continuous sequence of Luzin-preserving forcing notions then $\bigcup_{i<\delta} P_i$ is a Luzin-preserving forcing notions.*
4) *The statement "a forcing notion P is Luzin preserving" depend on P only up to equivalence of forcing notions*
5) *In Definition 2.3 we can weaken the demand to:*

$(*)'$ *for some $\eta \in A$ there is q such that $p_\eta \le q \in Q$ and $q \Vdash_Q$ "...".*

Proof. 0) Should be clear.
1) $\Rightarrow$ (the "only if" implication).
By definition, Q satisfies the c.c.c. Assume A is Luzin but $p \Vdash$ "A is not Luzin". Then for some Q-names $\underset{\sim}{\eta}_i$ (for $i < \omega_1$), we have

$$p \Vdash \text{“}\underset{\sim}{\eta}_i \in A \text{ for } i < \omega_1,\ \bigwedge_{i<j} \underset{\sim}{\eta}_i \neq \underset{\sim}{\eta}_j \text{ and } \{\underset{\sim}{\eta}_i : i < \omega_1\} \text{ is not meagre”.}$$

Thus changing the $\underset{\sim}{\eta}_i$'s we can get

$$(\bigoplus)\quad p \Vdash \text{“}\underset{\sim}{\eta}_i \in A \text{ for } i < \omega_1,\ \bigwedge_{i<j} \underset{\sim}{\eta}_i \neq \underset{\sim}{\eta}_j \text{ and } \{\underset{\sim}{\eta}_i : i < \omega_1\} \text{ is nowhere dense”.}$$

Let $p \le p_i \in Q$, $p_i \Vdash$ "$\underset{\sim}{\eta}_i = \nu_i$" so $\nu_i \in \left({}^{\omega}2\right)^V$ and necessarily $\nu_i \in A$. For no ν the set $\{i : \nu_i = \nu\}$ is uncountable as then Q fails the c.c.c. So without loss of generality $i \ne j \Rightarrow \nu_i \ne \nu_j$. Let $A' = \{\nu_i : i < \omega_1\}$. It is an uncountable subset of A and hence it is Luzin. Let $p_\nu = p_i$ if $\nu = \nu_i$. Apply $(*)$ of Definition 2.3 to $\langle p_\nu : \nu \in A' \rangle$ and get contradiction to $(\bigoplus)$ above.

$\Leftarrow$ (the "if" implication). Of course Q satisfies the c.c.c. Let $\langle p_\eta : \eta \in A \rangle$ be given, $A \subseteq {}^{\omega}2$ a Luzin set and $p_\eta \in Q$ (for $\eta \in A$) and we should show that $(**)$ of Definition 2.3 holds. Then $\underset{\sim}{A} = \{\eta \in A : p_\eta \in \underset{\sim}{G}_Q\}$ is a Q-name of a subset of A. It is known that for all but countably many $\eta \in A$, we have

$$p_\eta \Vdash \text{"}\underset{\sim}{A} \subseteq A \text{ is uncountable"}$$

so that (by the assumption of the implication we are proving)

$$p_\eta \Vdash \text{"}\underset{\sim}{A} \text{ is non-meagre, hence nowhere meagre above some } \underset{\sim}{\nu}_\eta \in {}^{\omega >}2\text{"}.$$

We have really finished, but we can elaborate. As we can replace A by $A \setminus B$ for any countable $B \subseteq A$. Without loss of generality this holds for every $\eta \in A$. Now for $\eta \in A$ and $p'_\eta, p_\eta \le p'_\eta \in P$ there are ν^*_η and q_η, $p'_\eta \le q_\eta \in Q$ such that $q_\eta \Vdash_Q$ "$\underset{\sim}{\nu}_\eta = \nu^*_\eta$". So for all $\eta \in A$ we have

$$q_\eta \Vdash \text{"}\{\nu \in A : p_\nu \in \underset{\sim}{G}_Q\} \text{ is uncountable and everywhere non-meagre above } \nu^*_\eta\text{"}.$$

As this holds for any p'_η, $p_\eta \le p'_\eta \in P$ we are done.

2) Follow by 2.4(1) (and as c.c.c. is preserved).

3) Like the preservation of c.c.c.

4) Easy (e.g. use 2.4(1)).

5) Left to the reader. $\square_{2.4}$

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