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EDGE DECOMPOSITIONS OF GRAPHS WITH NO LARGE INDEPENDENT SETS

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Abstract. If the continuum hypothesis holds then every graph on ω_1 with no uncountable independent sets can be edge decomposed into the disjoint union of $\aleph_1$ subgraphs with the same property. In the absence of the continuum hypothesis this may or may not be true. Extensions to other cardinals are given.

0. Introduction

An important result in considering extensions of Ramsey's theorem to uncountable sets was given by W. Sierpiński who proved that there is a graph of cardinal $\aleph_1$ which has neither an uncountable complete graph nor an uncountable independent set. This can be regarded as a decomposition of the edges of the complete graph on $\aleph_1$ vertices into two parts such that every uncountable set of vertices spans edges in both parts. Research initiated by P. Erdős (see [**3**]) showed that under CH a decomposition even into $\aleph_1$ parts is also possible. This was the starting point of an important chapter of combinatorial set theory, the calculus of square bracket partition relations. We mention that much later S. Todorčević ([**6**]) showed that the above result is true even without CH.

In this paper we consider the question which graphs have similar decompositions. It turns out that, under CH, if X is a graph on ω_1 with no uncountable independent set, then it can be decomposed into the union of $\aleph_1$ graphs with the same property. A corollary of this is that there is no universal graph of cardinal $\aleph_1$ that omits uncountable cliques and uncountable independent sets. In the absence of CH, it is consistent that there is a graph with no such decomposition even into two parts. But the positive statement is also consistent with arbitrary values of 2^ω.

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Slightly generalizing a result of Galvin and Shelah we show that if $2^\omega < \omega_\omega$ and X is a graph of cardinal 2^ω with no independent set of cardinal 2^ω then it can be decomposed into the union of ω graphs with the same property (Theorem 8). If the axiom of constructibility holds then every regular uncountable non-weakly compact cardinal has the property described above for $\aleph_1$ (Theorem 9).

We use the standard axiomatic set theory notation. $\mathbf{R}$ denotes the set of real numbers. If S is a set, κ a cardinal, then $[S]^\kappa = \{A \subseteq S : |A| = \kappa\}$, $[S]^{<\kappa} = \{A \subseteq S : |A| < \kappa\}$. A *graph* is a set $X \subseteq [V]^2$ for some set V of *vertices*. A subset A of vertices is a *complete subgraph* or a *clique* if $[A]^2 \subseteq X$, it is an *independent set* or *empty subgraph* if $[A]^2 \cap X = \emptyset$.

If $\lambda \geq \kappa$, μ are cardinals then let $Q(\lambda, \kappa, \mu)$ denote the following statement. If X is a graph on λ which has no independent set of cardinal κ then (the edge set of) X can be written as the disjoint union of μ graphs, $X = \bigcup\{Y_\alpha : \alpha < \mu\}$ none having an independent set of cardinal κ. We emphasize that we require the strong property that every κ-subset of λ must have an edge in Y_α so for example the trivial idea of decomposing λ into μ many disjoint subsets does not work.

The negative square bracket partition relation is defined as follows. If κ, λ, μ are cardinals, then $\lambda \not\to [\kappa]^2_\mu$ means that there exists an $f : [\lambda]^2 \to \mu$ such that for every $A \in [\lambda]^\kappa$ and $\xi < \mu$ there exist $x < y$ in A such that $f(x, y) = \xi$.

Clearly, $Q(\lambda, \kappa, \mu)$ strengthens $\lambda \not\to [\kappa]^2_\mu$.

1. The main result

THEOREM 1. *If $2^\kappa = \kappa^+$ then $Q(\kappa^+, \kappa^+, \kappa^+)$ holds.*

Proof. Let X be a graph on κ^+ with no independent set of cardinal κ^+. Enumerate $[\kappa^+]^{\leq\kappa}$ as $\{B_\alpha : \alpha < \kappa^+\}$ with $B_\alpha \subseteq \alpha$. By a simple diagonal argument one can define the function $f(\beta, \alpha) < \alpha$ for $\beta < \alpha < \kappa^+$, $\{\beta, \alpha\} \in X$ such that if $\gamma < \alpha$ and $\left|\{\beta \in B_\gamma : \{\beta, \alpha\} \in X\}\right| = \kappa$ then $\{f(\beta, \alpha) : \beta \in B_\gamma, \{\beta, \alpha\} \in X\} = \alpha$. One simply has to disjointize κ many sets of cardinal κ. From this coloring we get the decomposition of X the usual way, $\{\beta, \alpha\} \in Y_\tau$ iff $f(\beta, \alpha) = \tau$.

Assume that $H \subseteq \kappa^+$, $|H| = \kappa^+$. By transfinite recursion on $\alpha < \kappa^+$ we select the increasing sequence $x_\alpha \in H$ as follows. If x_β (for $\beta < \alpha$) are selected, and $\{x_\beta : \beta < \alpha\} = B_{\tau(\alpha)}$ then let x_α be an arbitrary element of H greater than $\tau(\alpha)$. Assume, for a contradiction, that f does not assume a certain value ξ on H. Then, by our construction, for $\alpha > \xi$, x_α can only be joined into $< \kappa$ many elements of $\{x_\beta : \beta < \alpha\}$. By Lázár's set mapping theorem there is an independent set of cardinal κ^+, a contradiction to our original hypothesis on X. □

A similar argument gives the following result.

THEOREM 2. *Assume that $2^\kappa = \kappa^+$. Suppose X_α are graphs on κ^+ ($\alpha < \kappa^+$) so that no X_α has an independent set of cardinal κ^+. Then there exist pairwise disjoint graphs $Y_\alpha \subseteq X_\alpha$ with the same property.* □

Corollary 3. *Assume that $2^\kappa = \kappa^+$, $X_0 \subseteq X_1 \subseteq [\kappa^+]^2$ are graphs such that X_0 has no complete subgraph of cardinal κ^+ while X_1 has no independent set of cardinal κ^+. Then there exists a graph $X_0 \subseteq Y \subseteq X_1$ with both properties.*

Proof. Apply Theorem 2 to the graphs X_0', X_1 where $X_0' = [\kappa^+]^2 - X_0$, a graph on κ^+ with no independent sets of cardinal κ^+. We get the disjoint graphs $Y_0 \subseteq X_0'$, $Y_1 \subseteq X_1$ with no independent sets of cardinal κ^+. Now set $Y = X_0 \cup Y_1$. As $Y \supseteq Y_1$ there is no independent sets of cardinal κ^+ in Y, and as $[\kappa^+]^2 - Y = X_0' - Y_1 \supseteq Y_0$ there is no complete subgraph of cardinal κ^+ in Y. $\square$

An interesting application can be given of this corollary. A *universal element* in a class of graphs is a graph which homomorphically embeds every graph in the class. (Given the graphs X, X' on V, V', respectively, a mapping $f : V \to V'$ is a *homomorphism*, if $\{x, y\} \in X$ implies $\{f(x), f(y)\} \in X'$.) We notice, that proofs of the existence of universal elements in various classes usually produce elements with the stronger property when the embeddings are required to be isomorphisms. F. Galvin and R. Laver proved that if $2^\kappa = \kappa^+$ then there is no universal graph in the class of graphs of cardinal κ^+ which do not contain complete subgraphs of cardinal κ^+. R. Schipperus asked if this is true for those graphs which omit independent sets of cardinal κ^+, as well. We show that it is, indeed, the case.

Corollary 4. *If $2^\kappa = \kappa^+$ then there is no universal graph among those of cardinal κ^+ which omit complete as well as empty graphs of cardinal κ^+.*

Proof. Let X be the alleged universal graph. By the above mentioned theorem of F. Galvin and R. Laver there is a graph Y of cardinal κ^+ which omits complete graphs of cardinal κ^+ and Y cannot be homomorphically embedded into X (see a proof in [**5**]). By the previous corollary, there is a graph $Y \subseteq Z \subseteq [\kappa^+]^2$ which omits both complete graphs and independent sets of cardinal κ^+ and clearly Z cannot be embedded into X, either. $\square$

Concluding this part we give some variants of Theorem 1. If $\mathcal{F}$ is some class of subgraphs of $[\kappa^+]^2$ we say that $X \subseteq [\kappa^+]^2$ is *$\mathcal{F}$-large* iff $X \cap F \neq \emptyset$ whenever $F \in \mathcal{F}$. $F \subseteq [\kappa^+]^2$ is a *complete bipartite (κ, κ^+) or (κ^+, κ^+) graph* iff $F = \{\{b, w\} : b \in B, w \in W\}$ for some $B, W \subseteq \kappa^+$, $|B| = \kappa$ or κ^+, $|W| = \kappa^+$. F is a *halfgraph* iff $F = \{\{b, w\} : b \in B, w \in W, b < w\}$ for some $B, W \subseteq \kappa^+$, $|B| = |W| = \kappa^+$.

Theorem 5. *If $2^\kappa = \kappa^+$ and $X \subseteq [\kappa^+]^2$ is $\mathcal{F}$-large then X is the disjoint union of κ^+ $\mathcal{F}$-large graphs, assuming that $\mathcal{F}$ is the class of the*

(a) complete graphs of cardinal κ^+;

(b) halfgraphs;

(c) complete bipartite (κ, κ^+) graphs;

(d) complete bipartite (κ^+, κ^+) graphs.

Proof. (a) is a restatement of Theorem 1. We prove (d), the other cases are simpler. Let X be an $\mathcal{F}$-large graph for the class of complete bipartite (κ^+, κ^+) graphs. Assume that $B, W \subseteq \kappa^+$, $|B| = |W| = \kappa^+$, $B \cap W = \emptyset$. We show the following Claim. It easily implies that the coloring of Theorem 1 works for $\mathcal{F}$.

Claim. *Either there is $B' \in [B]^\kappa$ such that*

$$|\{w \in W : |\{b \in B' : \{b,w\} \in X\}| = \kappa\}| = \kappa^+$$

or else there is $W' \in [W]^\kappa$ such that

$$|\{b \in B : |\{w \in W' : \{b,w\} \in X\}| = \kappa\}| = \kappa^+.$$

Proof of Claim. Enumerate B, W as $B = \{b_\alpha : \alpha < \kappa^+\}$, $W = \{w_\alpha : \alpha < \kappa^+\}$. If no B', W' as described can be selected then to every $\alpha < \kappa^+$ there is a $\beta(\alpha) \geq \alpha$ such that if $\beta \geq \beta(\alpha)$ then every w_β is joined to less than κ many b_γ $(\gamma < \alpha)$ and likewise for b_γ. Using Lázár's theorem we can get a $Z \subseteq \kappa^+$, $|Z| = \kappa^+$ such that if $\beta \neq \alpha$ are from Z then $\{b_\beta, w_\alpha\} \notin X$. Now if Z is split into the disjoint Z', Z'' of cardinal κ^+, then X has no common edge with the bipartite graph on bipartition classes $\{b_\alpha : \alpha \in Z'\}$, $\{w_\alpha : \alpha \in Z''\}$, a contradiction. □ □

2. Independence

The easy independence is contained in the following theorem.

Theorem 6. *If more than $\aleph_1$ Cohen reals are added to a model of ZFC then $Q(\omega_1, \omega_1, \omega_1)$ holds in the reulting model.*

Proof. It suffices to show that if $X \in V$ is a graph on ω_1 and P adds one Cohen real then X witnesses $Q(\omega_1, \omega_1, \omega_1)$ in V^P (i.e., either has an uncountable independent set or has a decomposition as required).

We assume that the elements of P are the functions with $\mathrm{Dom}(p) < \omega$, $\mathrm{Ran}(p) \subseteq \omega$. $p' \leq p$ iff $p' \supseteq p$, i.e., p' extends p. Notice that P is countable. The generic set $G \subseteq P$ gives a function $G' : \omega \to \omega$ the following way. $G'(n) = k$ iff there is a $p \in G$ with $p(n) = k$. For $\alpha < \omega_1$ enumerate α as $\alpha = \{\beta_n(\alpha) : n < \omega\}$. If $\gamma < \alpha$, $\{\gamma, \alpha\} \in X$, $\gamma = \beta_n(\alpha)$ color the edge $\{\gamma, \alpha\}$ as $F(\gamma, \alpha) = \beta_{G'(n)}(\alpha)$. This, indeed, is a coloring of the edges of X in V^P.

Assume that some condition q forces that X has no uncountable independent set yet Z is an uncountable set missing color ξ. There are an uncountable set $S \subseteq \omega_1$ and conditions $p_\alpha \leq q$ $(\alpha \in S)$ such that $p_\alpha \Vdash \alpha \in Z$. As P is countable we can assume that $p_\alpha = p$ $(\alpha \in S)$. Set $k = \mathrm{Dom}(p)$. As X has no uncountable independent set (in V^P so even less in V) by Lázár's set mapping theorem there is an element $\xi < \alpha \in S$ such that $\{\gamma < \alpha : \gamma \in S, \{\gamma, \alpha\} \in X\}$ is infinite. Set $\xi = \beta_m(\alpha)$. Select $\gamma < \alpha$, $\gamma \in S$ with $\gamma = \beta_n(\alpha)$, $n \geq k$. Extend p to a p' such that $\mathrm{Dom}(p') = n + 1$, $p'(n) = m$, then p' forces that the edge $\{\gamma, \alpha\}$ in Z gets color ξ, a contradiction. □

Theorem 7. *It is consistent that $Q(\omega_1, \omega_1, 2)$ fails.*

Proof. Let V be a model of CH. Let $\{A_\alpha : \alpha < \omega_2\}$ be a family of almost disjoint uncountable subsets of ω_1. We are going to define a finite support iteration of length ω_2, $P = (P_\alpha, Q_\alpha : \alpha < \omega_2)$ and in the same time the notion of nice family of conditions. We denote the support of a condition p by $\operatorname{supp}(p)$ (so it is a finite subset of ω_2). As follows, G_α is a generic subset of P_α.

Let Q_0 be the poset adding a graph X on ω_1 with finite conditions. That is, $q = (s, g) \in Q_0$ iff $s \in [\omega_1]^{<\omega}$, $g \subseteq [s]^2$, and $q' = (s', g')$ extends if $s' \supseteq s$ and $g = g' \cap [s]^2$.

When $p \in P_\alpha$ is a condition we use the notation $p(0) = (s^p, g^p)$. Assume that $\alpha < \omega_2$, P_α has already been defined, and, in V^{P_α}, $f_\alpha : X \to 2$ is a coloring of the edges of X with 0 and 1.

We call a family of conditions $\{p_i : i \in I\}$ in P_α a *nice family* if the following hold. Their supports form a Δ-system; $\operatorname{supp}(p_i) = a \cup b_i$. The vertex sets of the graphs in $p_i(0)$ also form a Δ-system, $s^{p_i} = s \cup s_i$, and $g^{p_i} \cap [s]^2 = g$ holds for $i \in I$. For $\beta \in a - \{0\}$, $i, j \in I$, $\varepsilon(p_i, \beta) = \varepsilon(p_j, \beta)$. For $\beta \in a - \{0\}$, $i \in I$, $p_i(\beta) = p(\beta) \cup p^i(\beta)$, where, if $\varepsilon(p_i, \beta) = 0$, then $p^i(\beta) \subseteq s_i$, if, however, $\varepsilon(p_i, \beta) = 1$, then $\bigcup\{w_\xi : \xi \in p^i(\beta)\} \subseteq s_i$.

If we set $p(0) = (s, g)$ and the so formed p is a condition then we say that $\{p_i : i \in I\}$ is a *nice family with kernel* p. It is easy to see that in this case $p_i \leq p$ for $i \in I$.

We are going to define a maximal antichain r^τ $(\tau < \tau^\alpha)$ in P_α, and $\varepsilon(\tau) < 2$ $(\tau < \tau^\alpha)$. This done, if $p \in P_\alpha$, $\beta < \alpha$, then we set $\varepsilon(p, \beta) = i$ if $p|\beta \leq r^\tau$ for some τ with $\varepsilon(\tau) = i$.

Assume that the conditions $r^{\tau'}$ $(\tau' < \tau)$ have already been selected. If no condition is incompatible with each of them then we terminate the definition and set $\tau^\alpha = \tau$. Otherwise, let r be a condition incompatible with each of them.

Assume first that there do not exist conditions $p_\xi \leq r$ and disjoint sets $\xi \in w_\xi \in [A_\alpha]^{<\omega}$ $(\xi \in B)$ for some uncountable $B \subseteq A_\alpha$ such that the following condition holds.

$(*)$ for $\xi_1 \neq \xi_2 \in B$ if $p \leq p_{\xi_1}, p_{\xi_2}$, and g^p contains $\{\xi_1, \xi_2\}$ as the only edge between w_{ξ_1} and w_{ξ_2} then $p \Vdash f_\alpha(\xi_1, \xi_2) = 1$.

Then we set $r^\tau = r$, $\varepsilon(\tau) = 0$.

Assume now that a family $\{p_\xi, w_\xi : \xi \in B\}$ as in $(*)$ exists. Fix such a family. We may as well assume that $\{p_\xi : \xi \in B\}$ is a nice family with kernel $\overline{r} \leq r$. This will be justifed by Claim 2. Finally set $\varepsilon(\tau) = 1$, $r^\tau = \overline{r}$. Also define $A'_\alpha = \{\xi : p_\xi \in G_\alpha\}$.

The objects just defined usually depend on several variables, e.g., w_ξ is in fact a $w_{\alpha,\tau,\xi}$. To keep the exposition relatively clear, whenever possible, we supress some indices.

$(p, q) \in P_{\alpha+1}$ iff either $p \in P_\alpha$ is arbitrary and $q = \emptyset$ (and in this case of course $\operatorname{supp}(p, q) = \operatorname{supp}(p)$) or else there is a $\tau < \tau^\alpha$ such that $p \leq r^\tau$. Moreover, if $\varepsilon(\tau) = 0$, we assume that $q \subseteq s^p \cap A_\alpha$ and p forces that f_α on $X \cap [q]^2$ assumes only the value 0. If, however, $\varepsilon(\tau) = 1$, we assume that $q \in [A'_\alpha]^{<\omega}$, for $\xi \in q$ $w_\xi \subseteq s^p$ and if $\xi \neq \xi'$ are in q and $\{\xi, \xi'\} \in X$ then no other edge of X goes between w_ξ and $w_{\xi'}$. Notice that by $(*)$ in this case $f_\alpha(\xi, \xi') = 1$ is forced so Q_α adds a subset with edges only in color 1. In this case, of course, we set $\operatorname{supp}(p, q) = \operatorname{supp}(p) \cup \{\alpha\}$.

The ordering on $P_{\alpha+1}$ is as follows.

$(p',q') \le (p,q)$ iff $p' \le p$ and either $q = \emptyset$ or $q' \supseteq q$ and either $\varepsilon(p,\alpha) = 0$ (and so $\varepsilon(p',\alpha) = 0$) and $q' - q \subseteq s^{p'} - s^p$ or $\varepsilon(p',\alpha) = 1$ and $\bigcup\{w_\xi : \xi \in q' - q\} \subseteq s^{p'} - s^p$.

Claim 0. *The ordering on $P_{\alpha+1}$ is transitive.*

Proof. Assume that $(p'',q'') \le (p',q') \le (p,q)$. Clearly, $p'' \le p' \le p$. If $q = \emptyset$, the statement is obvious. If not, $q'' \supseteq q' \supseteq q$ and $p \le r^\tau$ for some τ. If $\varepsilon(\tau) = 0$ then $q'' - q' \subseteq s^{p''} - s^{p'}$, $q' - q \subseteq s^{p'} - s^p$, and so $q'' - q \subseteq s^{p''} - s^p$. If $\varepsilon(\tau) = 1$, a similar argument works. $\square$

Claim 1. *Assume that $\{p', p''\} \subseteq P_\alpha$ form a nice family. Let p be their coordinatewise union, i.e., $s^p = s^{p'} \cup s^{p''}$, $g^p = g^{p'} \cup g^{p''}$, and $p(\beta) = p'(\beta) \cup p''(\beta)$ for $0 < \beta < \alpha$. Then p is a condition, extends p', p'', which are, therefore, compatible.*

Proof. By transfinite induction on $\beta \le \alpha$ we show that $p|\beta$ is a condition. The cases $\beta = 1$ or limit are clear. Assume that $\beta = \gamma + 1$. If $p'(\gamma)$ or $p''(\gamma) = \emptyset$ then the statement is again obvious. If neither $p'(\gamma)$ nor $p''(\gamma) = \emptyset$ then, as $p'|\gamma$ and $p''|\gamma$ are compatible, they must extend the same r^τ (for some $\tau < \tau^\gamma$).

If $\varepsilon(\tau) = 0$ $p|\gamma$ forces that $p'(\gamma)$ and $p''(\gamma)$ contain only color 0 edges and, as $g^p = g^{p'} \cup g^{p''}$, this is true for $p(\gamma)$. If $\varepsilon(\tau) = 1$, a similar argument works. $\square$

Claim 2. *If $p_\xi \le p$ are conditions in P_α for $\xi \in B$ where $|B| = \aleph_1$, then there are $\overline{p}_\xi \le p_\xi$, $\overline{p}_\xi \le \overline{p}$ for some $\overline{p} \le p$ ($\xi \in B'$, $|B'| = \aleph_1$) such that $\{\overline{p}_\xi : \xi \in B'\}$ is a nice family with kernel $\overline{p}$. Therefore, P_α is ccc, and $\tau^\alpha < \omega_1$.*

Proof. Using the Δ-system lemma we can assume that the supports form a Δ-system, $\mathrm{supp}(p_\xi) = a \cup b_\xi$, and $s^{p_\xi} = s \cup s_\xi$ with $g^{p_\xi} \cap [s]^2 = g$ for some g. Using the pigeon-hole principle we can also assume that for $\beta \in a - \{0\}$, $\xi \in B'$ (for some uncountable $B' \subseteq B$) $\varepsilon(p_\xi, \beta)$ is independent of ξ. With the removal of finitely many further elements of B' we can also insure that $p_\xi(\beta) \subseteq s_\xi$ resp. $\bigcup\{w_\eta : \eta \in p_\xi(\beta)\} \subseteq s_\xi$ holds. This way we get an uncountable nice subfamily $\{p_\xi : \xi \in B'\}$. To conclude, set $\xi_0 = \min(B')$, $\overline{p} = p_{\xi_0}$, and let $\overline{p}_\xi$ be the coordinatewise union of p_ξ and p_{ξ_0} (see Claim 1.). Now $\{\overline{p}_\xi : \xi \in B - \{\xi_0\}\}$ is a nice family with kernel $\overline{p}$. $\square$

Claim 3. *Assume that $r^\tau \in G_\alpha$, $\varepsilon(\tau)=1$. Then A'_α is uncountable.*

Proof. Let $\{p_\xi : \xi \in B\}$ be the nice family as in the definition of $P_{\alpha+1}$ with kernel r^τ. If $p \le r^\tau$ is a condition then all but finitely many p_ξ ($\xi \in B$) are compatible with p so if $\mu < \omega_1$ is arbitrary, then we can find a $\nu > \mu$ such that p and p_ν are compatible, and their common lower bound will establish $\nu \in A'_\alpha$. $\square$

We now define $K_\alpha = \bigcup\{q : (p,q) \in G_{\alpha+1}\}$, the set added by Q_α with edges only in one color.

Claim 4. $1 \Vdash |K_\alpha| = \omega_1$.

Proof. Assume that $(p,q) \in P_{\alpha+1}$.

First, assume that $p \le r^\tau$ for some τ with $\varepsilon(\tau) = 0$. Select $\nu > \mu$, $\nu \in A_\alpha$ such that $\nu \notin s^p$. Define p', q' as follows. $s^{p'} = s^p \cup \{\nu\}$, $g^{p'} = g^p$, $p'(\beta) = p(\beta)$ for $0 < \beta < \alpha$. $q' = q \cup \{\nu\}$. It is easy to see that (p', q') is a condition, extends (p, q) and forces that K_α has points beyond μ.

Assume now that $p \le r^\tau$ for some τ with $\varepsilon(\tau) = 1$. Let p_ξ, w_ξ $(\xi \in B)$ be the objects in the definition of $P_{\alpha+1}$.

Again, if $\mu < \omega_1$ is given, then for $\xi \in B$ large enough, the following (p', q') forces that K_α has points outside μ. $s^{p'} = s^p \cup w_\xi$. $g^{p'} = g^p \cup g^{p_\xi}$, $p'(\beta) = p(\beta) \cup p_\xi(\beta)$ (for $0 < \beta < \alpha$), $q' = q \cup \{\xi\}$. $\square$

Claim 5. *In V^P, X does not contain an uncountable independent set.*

Proof. Assume that $1 \Vdash U$ is an uncountable independent set. For uncountably many x, say for $x \in S$, there is a condition p^x forcing $x \in U$. We can as well assume that $\{p^x : x \in S\}$ is a nice family. Let $\{\alpha_1, \dots, \alpha_m\}$ be the common part of the Δ-system $\{\mathrm{supp}(p^x) : x \in S\}$. As $\{A_\alpha : \alpha < \omega_2\}$ is an almost disjoint family, we can also assume (by shrinking S if needed) that for $1 \le i \le m$ either $S \cap A_{\alpha_i} = \emptyset$ or $S \subseteq A_{\alpha_i}$ and that the latter can hold for at most one i. Assume that there is, indeed, such an i, as the other case is easier. We can again assume that

$$p^x(\alpha_i) = \{\eta_1, \dots, \eta_t, \xi_1^x, \dots, \xi_n^x\}$$

form a Δ-system. By Claim 2, P_{α_i} is ccc, so we can assume that $p^x|\alpha_i \le r^\tau$ holds for the same τ $(x \in S)$.

Assume first that $\varepsilon(\tau) = 0$. We can also assume that for some j, $1 \le j \le n$, $x = \xi_j^x$ holds for $x \in S$.

As $\varepsilon(\tau) = 0$ the conditions $p^x|\alpha_i$ $(x \in S)$ and the sets $w_x = \{\xi_1^x, \dots, \xi_n^x\}$ may not be witnesses for $(*)$ so there exist $x_1 \ne x_2$ in S such that for some $p \le p^{x_1}|\alpha_i$, $p^{x_2}|\alpha_i$, $p(0)$ contains $\{x_1, x_2\}$ as the only edge between w_{x_1} and w_{x_2} and $p \Vdash f_{\alpha_i}(x_1, x_2) = 0$.

Let q be the following condition. $\mathrm{supp}(q) = \mathrm{supp}(p^{x_1}) \cup \mathrm{supp}(p^{x_2})$, $q|\alpha_i = p$, $q(\beta) = p^{x_1}(\beta) \cup p^{x_2}(\beta)$ for $\beta \ge \alpha_i$. q really is a condition as for $\beta = \alpha_i$, $q(\beta)$ has only color 0 edges, and for $\beta > \alpha_i$ the extra edge, $\{x_1, x_2\}$ does not appear to cause problems. $q \le p^{x_1}$, p^{x_2} can easily be seen and q forces that $\{x_1, x_2\} \in X$, a contradiction.

Assume now that $\varepsilon(\tau) = 1$. Then, there is a collection $\{w_\xi : \xi \in B\}$ of finite sets as in the definition of P_{α_i+1}. Again, (by shrinking S if needed) we can assume that $x \in w_{\xi_j^x}$ holds for the same j $(x \in S)$ and either for every $x \in S$ $x = \xi_j^x$ or for every $x \in S$ $x \ne \xi_j^x$. If now $x \ne x'$ are in S, then $p \le p^x$, $p^{x'}$ is a condition, where $\mathrm{supp}(p) = \mathrm{supp}(p^x) \cup \mathrm{supp}(p^{x'})$, $s^p = s^{p^x} \cup s^{p^{x'}}$, $g^p = g^{p^x} \cup g^{p^{x'}} \cup \big\{\{x, x'\}\big\}$, and $p(\beta) = p^x(\beta) \cup p^{x'}(\beta)$ otherwise. (Namely, p forces that if $\{\xi_j^x, \xi_{j'}^{x'}\}$ is an edge in X then it gets color 1 by f_{α_i}.) p then forces that $\{x, x'\} \in X$, a contradiction. $\square$

3. Other cardinals

We prove a result on graphs of size 2^ω which extends a result of Galvin and Shelah [**4**]. We basically use the idea of that paper but can only prove the result for $2^\omega < \omega_\omega$.

Theorem 8. *If $2^\omega < \omega_\omega$ then $Q(2^\omega, 2^\omega, \omega)$ holds.*

Proof. By Theorem 1 we can assume that $2^\omega > \omega_1$. Set $\mathbf{R} = \{r(\alpha) : \alpha < 2^\omega\}$, $[2^\omega]^{\aleph_0} = \{A_\alpha : \alpha < 2^\omega\}$, one-to-one enumerations. It suffices to show that $Q(2^\omega, 2^\omega, 2)$ holds so let X be a graph on 2^ω with no independent set of cardinal 2^ω, we are going to decompose its edges into two classes. For $\beta < \alpha$ set

$$E^+(\beta,\alpha) = \bigl\{\gamma \in A_\beta : \{\gamma,\alpha\} \in X, r(\gamma) > r(\alpha)\bigr\};$$

$$E^-(\beta,\alpha) = \bigl\{\gamma \in A_\beta : \{\gamma,\alpha\} \in X, r(\gamma) < r(\alpha)\bigr\}.$$

Let $g : \mathbf{R} \to \mathbf{R}$ be a one-to-one function such that $g(r(\alpha))$ is neither the infimum nor the supremum of $\bigl\{g(r(\gamma)) : \gamma \in E^+(\beta,\alpha)\bigr\}$ (for $\beta < \alpha$) and likewise for $E^-(\beta,\alpha)$. We now determine the decomposition of X into two classes, $X = Y_0 \cup Y_1$. If $\{\beta,\alpha\} \in X$ then put $\{\beta,\alpha\} \in Y_0$ if either $r(\beta) < r(\alpha)$ and $g(r(\beta)) < g(r(\alpha))$ or $r(\beta) > r(\alpha)$ and $g(r(\beta)) > g(r(\alpha))$. In the other case, set $\{\beta,\alpha\} \in Y_1$.

Assume, towards a contradiction, that some $B \subseteq 2^\omega$, $|B| = 2^\omega$ misses the edges of, say, Y_1. Set $S = \{\alpha < 2^\omega : \mathrm{cf}(\alpha) = \omega_1\}$, $x(\alpha) = \min(B - \alpha)$ for $\alpha \in S$.

Claim 1. *For stationary many $\alpha \in S$, for every $\varepsilon > 0$ there exists a $\beta \in \alpha \cap B$ such that $\{\beta, x(\alpha)\} \in X$ and $\bigl|g\bigl(r(\beta)\bigr) - g\bigl(r(x(\alpha))\bigr)\bigr| < \varepsilon$.*

Proof. If not, there is a closed unbounded set C such that for $\alpha \in S \cap C$ there is an $\varepsilon > 0$ that no $\beta \in \alpha \cap B$ can be found as prescribed. We can assume that $x(\alpha)$ is one-to-one on $S \cap C$. As $\mathrm{cf}(2^\omega) > \omega$ for 2^ω many $x(\alpha)$ the same ε applies. By further shrinking, for a set B' of cardinal 2^ω it is even true that $g\bigl(r(x(\alpha))\bigr) \in I$ $(x(\alpha) \in B')$ with a certain interval I of length ε. By our assumption B' is not independent in X so there are $\beta < x(\alpha)$ in B', $\{\beta, x(\alpha)\} \in X$, such that $\bigl|g\bigl(r(\beta)\bigr) - g\bigl(r(x(\alpha))\bigr)\bigr| < \varepsilon$, a contradiction. $\square$

Claim 2. *There exist $A \in [B]^{\aleph_0}$, $B' \subseteq B$, $|B'| = 2^\omega$, such that for every $\tau \in B'$ and $\varepsilon > 0$ there is a $\beta \in A$ such that $\{\beta,\tau\} \in X$ and $\bigl|g\bigl(r(\beta)\bigr) - g\bigl(r(\tau)\bigr)\bigr| < \varepsilon$.*

Proof. By Claim 1 there is a stationary set $S' \subseteq S$ such that for $\alpha \in S$ and $t = 1, 2, \ldots$ there exist $\beta_t\bigl(x(\alpha)\bigr) < \alpha$ with

$$\bigl|g\bigl(r(x(\alpha)\bigr) - g\bigl(r(\beta_t(x(\alpha)))\bigr)\bigr| < 1/t.$$

As $\mathrm{cf}(\alpha) > \omega$ for $\alpha \in S'$ by Fodor's theorem $\sup\{\beta_t(x(\alpha)) : 1 \le t < \omega\}$ is bounded for a stationary subset $S'' \subseteq S'$.

We get therefore that there is a $B' \subseteq B$ of cardinal 2^ω and an A with set $|A| < 2^\omega$ such that $H_\alpha \subseteq A$ for $\alpha \in B'$ where $H_\alpha = \{\beta_t(\alpha) : 1 \le t < \omega\}$. Set

$2^\omega = \omega_n$, notice that $1 < n < \omega$ by our assumptions. We show that if $0 \le i < n-1$ and there is a set A of cardinal ω_{i+1} as above for a certain B' then there is one of cardinal ω_i (with a different B'). This clearly suffices for the proof of the Claim. Enumerate A as $\{\delta_\xi : \xi < \omega_{i+1}\}$. For every $\alpha \in B'$ there is some $\zeta(\alpha) < \omega_{i+1}$ that $H_\alpha \subseteq \{\delta_\xi : \xi < \zeta(\alpha)\}$. For a set $B'' \subseteq B'$ of cardinal 2^ω and some $\zeta < 2^\omega$ $\zeta(\alpha) = \zeta$ holds so $\{\delta_\xi : \xi < \zeta\}$ is as required. □

To conclude the proof of the Theorem let $A = A_\alpha$ be as in Claim 2 and let $\tau \in B$ have $\tau > \alpha$. For $t = 1, 2, , \ldots$ there exist $\beta_t \in A$ with $\{\beta_t, \tau\} \in X$ and $\big|g\big(r(\beta_t)\big) - g\big(r(\tau)\big)\big| < 1/t$. Either for infinitely many t $r(\beta_t) < r(\tau)$ or for infinitely many $r(\tau) < r(\beta_t)$ holds. Assume the former. By our assumption on Y_0 in this case $g(r(\beta_t)) < g(r(\tau))$ holds for these values. Remember that also $g(r(\tau)) - 1/t < g(r(\beta_t))$ holds. So $g(r(\tau)) = \sup\{g(r(\gamma)) : \gamma \in E^-(\alpha, \tau)\}$ a contradiction. □

Theorem 9. (V=L) *If $\kappa > \omega$ is a non-weakly compact, regular cardinal, then $Q(\kappa, \kappa, \kappa)$ holds.*

We need a lemma which is a special case of Theorem 2 in [**2**]. For the reader's convenience we provide a proof.

Lemma 10. *Assume that α is an ordinal, $S \subseteq \alpha \cup \{\alpha\}$ is a set of infinite cardinals, nonstationary in every point $\le \alpha$. Assume that $A_\gamma \subseteq \gamma$, $|A_\gamma| = \gamma$ for $\gamma \in S$. Then there exist disjoint $A'_\gamma \subseteq A_\gamma$ with $|A'_\gamma| = \gamma$ ($\gamma \in S$).*

Proof. By induction on α. The cases when α is not a cardinal or α is a successor cardinal are trivial. Assume that α is a limit cardinal. We can assume without loss of generality that α is in S.

We first consider the case when $\mathrm{cf}(\alpha) > \omega$. Let $C \subseteq \alpha$ be a closed unbounded set of cardinals, $C \cap S = \emptyset$ such that $|A_\alpha \cap \gamma| = \gamma$ holds for every $\gamma \in C$. If $\delta < \delta'$ are successive points in C then by the inductive hypothesis the sets

$$\big\{A_\gamma - \delta : \delta < \gamma < \delta', \gamma \in S\big\} \cup \big\{A_\alpha \cap [\delta, \delta')\big\}$$

can be disjointized say, by $A'_\gamma \subseteq A_\gamma - \delta$ for $\delta < \gamma < \delta', \gamma \in S$, and by $A^\delta_\alpha \subseteq A_\alpha \cap [\delta, \delta')$. We can now take A'_γ as above for $\gamma \in S \cap \alpha$ and $A'_\alpha = \bigcup\{A^\delta_\alpha : \delta \in C\}$.

If $\mathrm{cf}(\alpha) = \omega$ we modify the argument as follows. Select a sequence of cardinals δ_n ($n < \omega$) converging to α such that $\delta_0 = 0$ and

$$|A_\alpha \cap [\delta_n, \delta_{n+1})| \ge \delta_n.$$

Select $B_n \subseteq A_\alpha \cap [\delta_n, \delta_{n+1})$ of cardinal δ_n. By our inductive hypothesis we can disjointize the system

$$\{A_\gamma - B_n - \delta_n : \delta_n < \gamma \le \delta_{n+1}\}$$

by $A'_\gamma \subseteq A_\gamma$, $|A'_\gamma| = \gamma$. We can set $A'_\alpha = \bigcup\{B_n : n < \omega\}$. □

Proof of Theorem 9. Theorem 1 allows us to assume that κ is inaccessible. By Jensen's fine structure theory [**1**] there is a stationary set $S \subseteq \kappa$ nonstationary in every ordinal $\alpha < \kappa$. We can assume that S consists of cardinals. By $\diamondsuit_S$ there is a sequence $\{Z_\alpha : \alpha \in S\}$ $(Z_\alpha \subseteq \alpha)$ such that for every $Z \subseteq \kappa$ the set $\{\alpha \in S : Z_\alpha = Z \cap \alpha\}$ is stationary.

Assume that X is a graph on κ with no independent sets of cardinal κ. For $\alpha < \kappa$, using Lemma 10, we can define the values $f(\gamma, \alpha) < \alpha$ $(\gamma < \alpha, \{\gamma, \alpha\} \in X)$, such that for $\beta \in S$, $\beta \leq \alpha$ it is true that $\{f(\gamma, \alpha) : \gamma \in Z_\beta, \{\gamma, \alpha\} \in X\} = \beta$ whenever $|\{\gamma \in Z_\beta, \{\gamma, \alpha\} \in X\}| = \beta$.

Assume that $Z \in [\kappa]^\kappa$ and f misses the value $\xi < \kappa$ on $X \cap [Z]^2$. There is a stationary $S' \subseteq S$ such that for $\alpha \in S'$ then $\xi < \alpha$ and $Z_\alpha = Z \cap \alpha$. Set $x(\alpha) = \min(Z - \alpha)$. By the way f was defined $d(\alpha) = \big|\{\gamma < \alpha : \{\gamma, x(\alpha)\} \in X\}\big| < \alpha$ for $\alpha \in S'$. Notice that we can replace "$\gamma < \alpha$" by "$\gamma < x(\alpha)$". By Fodor's lemma $d(\alpha) < \tau$ on an $S'' \in [\kappa]^\kappa$ and by Lázár's set mapping theorem there is an independent set of cardinal κ in X, a contradiction. $\square$

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