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CONTINUOUS TIME PROBABILITY LOGIC

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Abstract. Continuous time probability logic $L^t_{\mathcal{A}P}$ is a logic appropriate for the study of space with a family of continuous time probability measures. We prove the completeness theorem for the logic $L^t_{\mathcal{A}P}$ for both continuous and uniformly continuous cases. Also, we prove the finite compactness theorem for universal conjunctive formulas of $L^t_{\mathcal{A}P}$.

Let $\mathcal{A}$ be a countable admissible set and $\omega \in \mathcal{A}$. We consider the reals of $\mathcal{A}$ to be the Dedekind cuts of $\mathbb{Q}$ in $\mathcal{A}$. The interval $[0,1]$ usually represents time.

The logic $L^t_{\mathcal{A}P}$ has probability quantifiers $P^t,\ t \in \mathbb{Q} \cap [0,1]$ corresponding to probability measure at time t. A model $\langle \mathfrak{A}, \mu^t \rangle_{t\in[0,1]}$ consists of a classical structure $\mathfrak{A} = \langle A, R_i, c_j \rangle$ without operations and a family of (uniformly) continuous time probability measures on the space, i.e. for each $\varepsilon > 0$ there is a $\delta > 0$ such that for each $s, t \in [0,1]$ and each measurable $B \subseteq [0,1]$, if $|s-t| < \delta$, then $|\mu^t(B) - \mu^s(B)| < \varepsilon$. The quantifiers are interpreted as for the standard probability logic $L_{\mathcal{A}P}$ [**3**].

Example. Let μ be a Lebesgue measure on $[0,1]$ and $\mu^t(B) = \int_B f_t \, d\mu$, where $B \subseteq [0,1]$, $f_t(x) = 2(1+t)x$ for $x \in [0, \frac{1}{2})$ and $f_t(x) = 2(1-3t)x + 4t$ for $x \in [\frac{1}{2}, 1]$. It is easy to see that $\{\mu^t : t \in [0,1]\}$ is a family of (uniformly) continuous time probability measures on $[0,1]$.

Let $\vec{x}$ be a finite sequence of variables and let $\varphi(\vec{x})$ be a formula of $L^t_{\mathcal{A}P}$. We will use the abbreviation $|P^t(\varphi(\vec{x})) - P^s(\varphi(\vec{x}))| < \varepsilon$ for the sentence

$$\bigwedge_{q\in\mathbb{Q}\cap[0,1]} \Big(\big((P^t\vec{x} \geq q)\varphi(\vec{x}) \to (P^s\vec{x} \geq q-\varepsilon)\varphi(\vec{x})\big) \wedge \big((P^s\vec{x} \geq q)\varphi(\vec{x}) \to (P^t\vec{x} \geq q-\varepsilon)\varphi(\vec{x})\big)\Big),$$

where $\varepsilon \in \mathbb{Q}^+$. Axioms and rules of inference for the logic $L^t_{\mathcal{A}P}$ are those of $L_{\mathcal{A}P}$ as listed in [**2**] with the axiom B_4 from [**3**], with remark that each probability

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quantifier P^t, $t \in \mathbb{Q} \cap [0,1]$ can play the role of P, together with the following axiom for continuous and uniformly continuous case respectively:

Axiom of continuity

$$\bigwedge_{\varepsilon \in \mathbb{Q}^+} \bigvee_{\delta \in \mathbb{Q}^+} \bigwedge_{\substack{s,t \in \mathbb{Q}\cap[0,1] \\ |s-t|<\delta}} |P^t(\varphi(\vec{x})) - P^s(\varphi(\vec{x}))| < \varepsilon \ ; \tag{C}$$

Axiom of uniform continuity

$$\bigwedge_{\varepsilon \in \mathbb{Q}^+} \bigvee_{\delta \in \mathbb{Q}^+} \bigwedge_{n} \bigwedge_{\varphi \in \Phi_n} \bigwedge_{\substack{s,t \in \mathbb{Q}\cap[0,1] \\ |s-t|<\delta}} |P^t(\varphi(\vec{x})) - P^s(\varphi(\vec{x}))| < \varepsilon \ , \tag{UC}$$

where $\Phi = \bigcup_n \Phi_n$, $\Phi_n = \{ \varphi \in \Phi : \varphi$ is a formula with n free variables$\}$ and $\Phi, \Phi_n \in \mathcal{A}$.

In the practice we usually prefer uniformly continuity case. We need the following three sorts of auxiliary models.

Definition. (i) A weak model for $L^t_{\mathcal{A}P}$ is a structure

$$\mathfrak{A} = \langle A, R_i^{\mathfrak{A}}, c_j^{\mathfrak{A}}, \mu_n^t \rangle_{n\in\mathbb{N}, t\in[0,1]}$$

such that each $\{ \mu_n^t : t \in [0,1] \}$ is a family of (uniformly) continuous time finitely additive probability measures on A^n with each singleton measurable and the set $\{ \vec{c} \in A^n : \mathfrak{A} \models \varphi[\vec{a}, \vec{c}] \}$, is μ_n^t-measurable for each $\varphi(\vec{x}, \vec{y}) \in L^t_{\mathcal{A}P}$ and $\vec{a} \in A^m$.

(ii) A middle model for $L^t_{\mathcal{A}P}$ is a weak model $\langle \mathfrak{A}, \mu_n^t \rangle$ such that the following is true: for each $\varepsilon > 0$, there is a $\delta > 0$ such that for each formula $\varphi(\vec{x}, \vec{y})$, for each $\vec{a} \in A^m$ and $s, t \in [0,1]$, if $|s-t| < \delta$, then

$$|\mu_n^t \{ \vec{c} \in A^n : \mathfrak{A} \models \varphi[\vec{a}, \vec{c}] \} - \mu_n^s \{ \vec{c} \in A^n : \mathfrak{A} \models \varphi[\vec{a}, \vec{c}] \}| < \varepsilon \ ,$$

i.e. the axiom of continuity holds uniformly in the formula φ (given ε, the same δ works for all φ).

(iii) A graded model for $L^t_{\mathcal{A}P}$ is a middle model $\langle \mathfrak{A}, \mu_n^t \rangle_{n\in\mathbb{N}, t\in[0,1]}$ such that: each $\{ \mu_n^t : t \in [0,1] \}$ is a family of (uniformly) continuous time countable additive probability measures on A^n, each n-placed relation R_i is μ_n^t-measurable and identity relation is μ_2^t-measurable, $\mu_n^t \times \mu_m^t \subseteq \mu_{n+m}^t$, each μ_n^t is preserved under permutations of $\{1, 2, \ldots, n\}$ and each $\{ \mu_n^t : n \in \mathbb{N} \}$ has the Fubini property (see [**3**]).

The proof of the completeness theorem for the continuous case of $L^t_{\mathcal{A}P}$ makes use of the Loeb–Hoover–Keisler construction (see [**2**] and [**3**]). The only remark is that the family $\{ \mu^t : t \in \mathbb{Q} \cap [0,1] \}$ of probability measures can be extended to the family $\{ \mu^t : t \in [0,1] \}$ of continuous time probability measures by $\mu^t(B) = \lim_{n\to\infty} \mu^{t_n}(B)$, where $\{ t_n \}$ is a sequence of rationals from $[0,1]$ which convergences to $t \in [0,1]$.

THEOREM. (Completeness theorem for continuous case of $L^t_{\mathcal{A}P}$) *A sentence φ of $L^t_{\mathcal{A}P}$ is consistent (axiom (C)) if and only if φ has a continuous time probability model.*

The proof of the completeness theorem for the uniformly continuous case of $L^t_{\mathcal{A}P}$ makes use of the weak–middle–strong model construction (see [**5**]).

THEOREM. (Middle completeness theorem for $L^t_{\mathcal{A}P}$) *A sentence φ of $L^t_{\mathcal{A}P}$ is consistent if and only if it has a middle model in which each theorem of $L^t_{\mathcal{A}P}$ is true.*

Sketch of the proof. Let $K = L \cup C$ be the language introduced in the Henkin construction of the weak model of φ, where C is a set of new constant symbols and $C \in \mathcal{A}$. Let M be the language introduced in Rašković [**5**], with additional predicate $Q(r)$ with meaning $r \in \mathbb{Q} \cap [0,1]$, and with predicates $\mu^t(X,r)$, $t \in \mathbb{Q} \cap [0,1]$ which play the role of $\mu_i(X,r)$, $i = 1,2$ from [**5**]. Sentences of the theory T of $M_{\mathcal{A}}$ are those of Rašković [**5**], except for some differences in notation, together with the following:

(1) Diagram of $\mathbb{Q} \cap [0,1]$;

(2) Axiom of uniformly continuity:

$$(\forall\varepsilon > 0)(\exists\delta > 0)(\forall X)(\forall s)(\forall t)((Q(s) \wedge Q(t) \wedge |s-t| < \delta) \rightarrow |\mu^t(X) - \mu^s(X)| < \varepsilon)\ ,$$

instead of the axiom of absolute continuity from [**5**].

By the Barwise compactness theorem, T has a standard model $\mathfrak{B}$, which can be transformed to a middle model $\mathfrak{A}$ of φ (see [**5**]). $\square$

Each middle model in which all theorems of $L^t_{\mathcal{A}P}$ hold is elementarily equivalent to a (strong) uniformly continuous time probability model. The only remark is that uniform continuity in the middle model implies the uniform continuity of internal sets in the nonstandard superstructure. From that we get uniform continuity for all Loeb measurable sets, because these can be approximated by internal ones (see [**4**]). As before, the family $\{\,\mu^t : t \in \mathbb{Q} \cap [0,1]\,\}$ of probability measures from the weak–middle–strong model construction can be extended to the family $\{\,\mu^t : t \in [0,1]\,\}$ of uniformly continuous time probability measures. This completes the proof of the following theorem.

THEOREM. (Completeness theorem for uniformly continuous case of $L^t_{\mathcal{A}P}$) *A sentence φ of $L^t_{\mathcal{A}P}$ is consistent (axiom (UC)) if and only if φ has a uniformly continuous time probability model.*

Finally, we are looking at a part of $L^t_{\mathcal{A}P}$ satisfying the finite compactness property, because $L^t_{\mathcal{A}P}$ cannot satisfy the full compactness property (see [**3**]).

Definition. The set of universal conjunctive formulas of $L^t_{\mathcal{A}P}$ is the least set containing all quantifier-free formulas and closed under arbitrary $\wedge$, finite $\vee$, and the quantifiers $(P^t\vec{x} \geq r)$, $t \in \mathbb{Q} \cap [0,1]$.

Now we can state the finite compactness theorem for $L^t_{\mathcal{A}P}$ logic. This result cannot be extended to (strong) model of $L^t_{\mathcal{A}P}$ (see [**3**]).

THEOREM. (Finite compactness theorem for $L^t_{\mathcal{A}P}$) *Let T be a set of universal conjunctive sentences of $L^t_{\mathcal{A}P}$. If every finite subset of T has a graded model, then T has a graded model.*

Proof. Let $\mathfrak{A}_\Psi$ be a middle model for a finite subset $\Psi \subseteq T$. Take an ultraproduct ${}^*\mathfrak{A} = \prod_D \mathfrak{A}_\Psi$ such that, for each $\varphi \in T$, almost every $\mathfrak{A}_\Psi$ satisfies φ. Form a graded structure $\widehat{\mathfrak{A}}$ from ${}^*\mathfrak{A}$ by Loeb construction (see [**3**]). Then every universal conjunctive formula true in almost all $\mathfrak{A}_\Psi$ holds in $\widehat{\mathfrak{A}}$ too. The condition of uniform continuity of probability measures is expressed by the first-order sentence:

$$(\forall\varepsilon > 0)(\exists\delta > 0)(\forall X)(\forall s)(\forall t)((Q(s) \wedge Q(t) \wedge |s-t| < \delta) \to |\mu^t(X) - \mu^s(X)| < \varepsilon) \ .$$

So by the ordinary Łos' theorem and Loeb construction this sentence holds both in ${}^*\mathfrak{A}$ and in $\widehat{\mathfrak{A}}$. $\square$

Remark. The probability logic $L_{\mathcal{A}P}$ from [**3**] can be obtained as a special case of continuous time probability logic $L^t_{\mathcal{A}P}$ by taking $\mu^t = \mu^s = \mu$ for all $s, t \in [0,1]$.

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