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# FREE OBJECTS IN PRIMITIVE VARIETIES OF $n$-GROUPOIDS

**G. Čupona** and **S. Markovski**

*Dedicated to the memory of Professor Đ. Kurepa*

**Abstract**. A variety of $n$-groupoids (i.e. algebras with one $n$-ary operation $f$) is said to be a primitive $n$-variety if it is defined by a system of identities of the following form:

$$f(x_{i_1}, x_{i_2}, \dots, x_{i_n}) = f(x_{j_1}, x_{j_2}, \dots, x_{j_n}) \tag{$*$}$$

Here we give a convenient description of free objects in primitive $n$-varieties, and several properties of free objects are also established.

**1. Introduction.** Identities of the form $(*)$ are called primitive $n$-identities, where we take $n$ to be a fixed positive integer, and $i_\lambda$, $j_\mu$ are positive integers. We do not make any distinction between two equivalent identities, and that is the reason why we assume $1 \le i_\nu, j_\nu \le 2n$. A set $\Sigma$ of primitive $n$-identities is said to be complete if it contains every primitive $n$-identity which is a consequence of $\Sigma$. Everywhere in this paper we suppose that $\Sigma$ is a complete system of primitive $n$-identities, and we also take $n \ge 2$, since for $n = 1$ the only nontrivial primitive 1-identity is $f(x) = f(y)$, which gives rise to constant unars.

The main results obtained here are the construction of free $\Sigma$-objects with given basis $B$ and the following theorems, which are corollaries of the obtained construction.

THEOREM A. *A free $\Sigma$-object has a unique basis.* $\square$

THEOREM B. *Every subobject of a free $\Sigma$-object is a free $\Sigma$-object as well.* $\square$

For any identity $(*)$ we put $I = \{i_1, \dots, i_n\}$, $J = \{j_1, \dots, j_n\}$.

THEOREM C. *Assume that there is an identity $(*)$ in $\Sigma$ such that $I \cap J = \emptyset$. If $k \in \{1, 2, \dots, n-1\}$ is the largest integer such that $(*)$ is in $\Sigma$ for $I = \{1\}$ and for every $J$ with $|J| \le k$, then any free $\Sigma$-object with rank $k$ has a subobject with infinite rank.* $\square$

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Theorem D. *For every identity* $(*)$ *in* $\Sigma$ *let* $I \cap J \neq \emptyset$ *and assume that if* $(*)$ *is in* $\Sigma$ *for* $I = \{1, 2, \ldots, n\}$, *then* $|J| \geq 2$. *Then every free* $\Sigma$*-object has a subobject with infinite rank.* $\square$

Theorem E. *The word problem is solvable in any primitive* $n$*-variety.* $\square$

**2. Complete sets of primitive $n$-identities.** As we already mentioned in section 1, we assume that in $(*)$ we have $1 \leq i_\nu,\, j_\nu \leq 2n = m$ for each $\nu$. In such a way the primitive $n$-identities can be considered as transformations of the set $M = \{1, 2, \ldots, m\}$, i.e. as elements of the set $\mathcal{T} = M^m (= \{\varphi | \varphi \colon M \to M\})$. Next, in this paper we will not make any distinction between the sets $M^m$ and $M^n \times M^n$, where $M^n = \{\psi | \psi \colon \{1, 2, \ldots, n\} \to M\}$. Namely, if $\varphi \in M^m$ and $\varphi_L$, $\varphi_R \in M^n$ are defined by

$$\varphi_L(i) = \varphi(i), \qquad \varphi_R(i) = \varphi(n+i)$$

for each $i \in \{1, 2, \ldots, n\}$, then $(\varphi_L, \varphi_R)$ will be considered as another notation of $\varphi$.

We stress again that we suppose here and further on that $\Sigma$ denotes a complete set of primitive $n$-identities, where $n \geq 2$ is a given integer. By the above agreement, we also have that $\Sigma \subseteq \mathcal{T}$.

Every subset $\Lambda$ of $\mathcal{T}$ induces a relation $\sim_\Lambda$ on $M^n$ defined by

$$\varphi \sim_\Lambda \psi \Leftrightarrow (\varphi, \psi) \in \Lambda.$$

The following completeness theorem is a consequence of a result from [**2**]:

Proposition 2.1. *A subset* $\Lambda$ *of* $\mathcal{T}$ *is complete iff it satisfies the following conditions:*

(i) $\sim_\Lambda$ *is an equivalence relation on* $M^n$*;*

(ii) $\Lambda$ *is a left ideal in* $\mathcal{T}$, *i.e.* $\mathcal{T} \circ \Lambda \subseteq \Lambda$, *where* : *denotes the usual superposition of transformations.* $\square$

The following property (shown in [**2**]) will be used in the next section:

Proposition 2.2. *Let* $\xi$, $\eta \in \Sigma$ *be such that* $\ker \xi_R = \ker \eta_L$, *and denote by* $T(\xi, \eta)$ *the set of all elements* $\zeta \in \mathcal{T}$ *which satisfy the following conditions:* $\zeta_L = \xi_L$ *and*

$$\xi(i) = \xi(k+n), \;\; \eta(k) = \eta(j+n) \Rightarrow \zeta(i) = \zeta(j+n)$$

*for every* $i, k, j \in \{1, 2, \ldots, n\}$. *Then* $T(\xi, \eta) \neq \emptyset$ *and* $T(\xi, \eta) \subseteq \Sigma$ *(and, furthermore,* $\mathcal{T} \circ T(\xi, \eta) \subseteq \Sigma$*).* $\square$

Given any complete set $\Sigma$ of primitive $n$-identities, by $\Sigma[M]$ we denote the quotient set $M^n / \sim_\Sigma$, and if $\varphi \in M^n$, then by $[\varphi] \in \Sigma[M]$ we denote the corresponding class of equivalent elements. (Further on, we will write simply $\sim$ instead of $\sim_\Sigma$.)

For any $\mathbf{i} \in M$, let $i \in M^n$ be defined by $\mathbf{i}(\nu) = i$ for each $\nu \in \{1, 2, \ldots n\}$. We say that $\Sigma$ is with constant if $[\mathbf{1}] = [\mathbf{2}]$.

If $\varphi \in M^n$, then the set $\{\varphi(1), \dots, \varphi(n)\}$ is called the content of $\varphi$, and will be denoted by $\operatorname{cnt}(\varphi)$.

PROPOSITION 2.3. *The following conditions are equivalent:*

(i) $\Sigma$ *is with constant;*

(ii) $[\mathbf{i}] = [\mathbf{j}]$ *for any* $i, j \in M$*;*

(iii) *there exist* $\varphi$, $\eta \in M^n$ *such that* $[\varphi] = [\eta]$ *and the contents of* $\varphi$ *and* $\eta$ *are disjoint.* $\square$

If $\Sigma$ is with constant, then any element of $[\mathbf{i}]$ is called a $\Sigma$-constant; $\Sigma$ is said to be with absolute constant if $\Sigma[M]$ is a singleton. Denote by $\varepsilon$ the element of $M^n$ defined by $\varepsilon(\nu) = \nu$ for each $\nu \in \{1, 2, \dots, n\}$.

PROPOSITION 2.4. *The following conditions are equivalent:*

(i) $\Sigma$ *is with absolute constant;*

(ii) $\varphi \sim \eta$ *for any* $\varphi$, $\eta \in M^n$*;*

(iii) *there is a* $\varphi \in M^n$ *such that* $\varepsilon \sim \varphi$ *and* $\varepsilon$ *and* $\varphi$ *have disjoint contents.* $\square$

PROPOSITION 2.5. *If* $\varphi \in M^n$ *is not a* $\Sigma$*-constant, then there is an* $\eta \in [\varphi]$ *such that* $\operatorname{cnt}(\eta)$ *is a subset of* $\operatorname{cnt}(\psi)$ *for any* $\psi \in [\varphi]$.

(Then we say that $\eta$ is a minimal member of $[\varphi]$.)

*Proof.* Since $A = \{\operatorname{cnt}(\xi) | \xi \in [\varphi]\}$ is a finite set, there is an $\eta \in [\varphi]$ such that $\operatorname{cnt}(\eta)$ is a minimal member in $A$. Assume that $\operatorname{cnt}(\eta)$ and $\operatorname{cnt}(\eta')$ are different minimal members in $A$. Then $\operatorname{cnt}(\eta) \cap \operatorname{cnt}(\eta') \neq \emptyset$, since $\varphi$ is not a $\Sigma$-constant. Let $i \in \operatorname{cnt}(\eta) \cap \operatorname{cnt}(\eta')$ and let $j \in \operatorname{cnt}(\eta') \backslash \operatorname{cnt}(\eta)$. Define $\zeta \in \mathcal{T}$ by $\zeta(j) = i$ and $\zeta(k) = k$ for any $k \neq j$. Then $\zeta \circ (\eta, \eta') = (\eta, \eta'') \in \Sigma$ for some $\eta'' \in M^n$ such that $\operatorname{cnt}(\eta'') = \operatorname{cnt}(\eta') \backslash \{j\}$. $\square$

Now we define the notion of the $\Sigma$-content of an element $\varphi \in M^n$, denoted by $\operatorname{cnt}_\Sigma(\varphi)$, as follows. We put $\operatorname{cnt}_\Sigma(\varphi) = \emptyset$ if $\varphi$ is a $\Sigma$-constant, and $\operatorname{cnt}_\Sigma(\varphi) = \operatorname{cnt}(\eta)$ is $\varphi$ is not a $\Sigma$-constant and $\eta$ is a minimal member of $[\varphi]$. Note that $\xi \sim \varphi$ implies $\operatorname{cnt}_\Sigma(\xi) = \operatorname{cnt}_\Sigma(\varphi)$.

PROPOSITION 2.6. *There exists a* $\varphi \in M^n$ *such that* $\operatorname{cnt}_\Sigma(\varphi)$ *is a singleton iff* $\Sigma$ *is without constant.* $\square$

$\Sigma$ is said to be essentially $k$-ary iff $|\operatorname{cnt}_\Sigma(\varepsilon)| = k$.

If $\Sigma$ is with constant, then the order of the constant of $\Sigma$ is said to be $k$ iff $\operatorname{cnt}_\Sigma(\varphi) = \emptyset$ for each $\varphi \in M^n$ such that $|\operatorname{cnt}(\varphi)| \leq k$, and $k$ is the largest such integer. Therefore we have:

PROPOSITION 2.7. *The following statements are equivalent:*

(i) $\Sigma$ *is with absolute constant;*

(ii) $\Sigma$ *is with constant of order* $n$. $\square$

**3. $\Sigma$-objects.** Let $A$ be a nonempty set and let $\Sigma$ be a complete set of primitive $n$-identities. Define a relation $\sim_{\Sigma,A}$ (shortly denoted by $\sim_A$) on the set $A^n (= \{\mathbf{a} | \mathbf{a}\colon \{1, \dots, n\} \to A\})$ as follows:

$$\mathbf{a} \sim \mathbf{b} \iff (\exists \xi \in \Sigma) \ker\xi = \ker(\mathbf{a}, \mathbf{b})$$

where $\mathbf{a}, \mathbf{b} \in A^n$ and $(\mathbf{a}, \mathbf{b}) \in A^m$ is defined as in the preceding section, i.e. $(\mathbf{a}, \mathbf{b})(i) = \mathbf{a}(i)$, $(\mathbf{a}, \mathbf{b})(i + n) = \mathbf{b}(i)$, for each $i \in \{1, 2, \dots, n\}$.

The following statement is a corollary from Proposition 1.1 (and its generalization as well):

PROPOSITION 3.1. (i) *$\sim_A$ is an equivalence relation.* (ii) *If $\mathbf{a} \sim_A \mathbf{b}$, $\mathbf{c}$ is a transformation of $A$ and $\mathbf{c} \circ (\mathbf{a}, \mathbf{b}) = (\mathbf{a}', \mathbf{b}')$, then $\mathbf{a}' \sim_A \mathbf{b}'$.* □

*Proof.* We will give only a sketch of the proof, and we will use the fact that $\Sigma$ is a complete set of identities. Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in A^n$.

(i) Then for suitably chosen $\varphi \in \mathcal{T}$ we have $\ker\varphi(\varepsilon, \varepsilon) = \ker(\mathbf{a}, \mathbf{a})$, and also if $\ker\xi = \ker(\mathbf{a}, \mathbf{b})$, then $\ker(\xi_R, \xi_L) = \ker(\mathbf{b}, \mathbf{a})$. The transitivity follows by using Proposition 2.2.

(ii) If $\ker\xi = \ker(\mathbf{a}, \mathbf{b})$ and $\mathbf{c} \circ (\mathbf{a}, \mathbf{b}) = (\mathbf{a}', \mathbf{b}')$, then there is a $\varphi \in \mathcal{T}$ such that $\ker\varphi\xi = \ker(\mathbf{a}', \mathbf{b}')$, and $\xi \in \Sigma$ implies $\varphi\xi \in \Sigma$ by Proposition 2.1. □

We denote by $\Sigma[A]$ the quotient set $A^n / \sim_A$ and by $[\mathbf{a}]$ the class of equivalent elements of $\mathbf{a} \in A^n$. (So, $[\mathbf{a}] = [\mathbf{b}]$ iff $\mathbf{a} \sim_A \mathbf{b}$.) If $A = M = \{1, 2, \dots, m\}$, then $\sim_A$ and $\sim$ have the same meaning as in section 2.

Proposition 2.2–2.6 have obvious generalizations, and we make a summary below.

(1) $|\Sigma[A]| = 1$ iff one of the following cases appears: 1.1) $|A| = 1$; 1.2) $\Sigma$ is with absolute constant; 1.3) $|A| \leq k$ and $\Sigma$ is with constant of order $k$.

(2) If $\mathbf{a} \in A^n$, then the set $\mathrm{cnt}(\mathbf{a}) = \{\mathbf{a}(1), \dots, \mathbf{a}(n)\}$ is called the content of $\mathbf{a}$. If $\Sigma$ is with constant and $|\mathrm{cnt}(\mathbf{a})| = 1$, then the class of equivalent elements $[\mathbf{a}]$ will be denoted by $\mathbf{o} (\notin A)$ and called the zero of $\Sigma[A]$. Then we also say that the $\Sigma$-content of $\mathbf{o}$ is empty, and we denote it by $\mathrm{cnt}_\Sigma(\mathbf{o}) = \emptyset$; moreover, for each $\mathbf{c} \in \mathbf{o}$ we put $\mathrm{cnt}_\Sigma(\mathbf{c}) = \emptyset$. Let $\mathbf{b} \in A^n$. If either $\Sigma$ is without constant or $[\mathbf{b}] \neq \mathbf{o}$, then in the family of sets $\{\mathrm{cnt}(\mathbf{c}) | \mathbf{c} \in [\mathbf{b}]\}$ there is the least member which will be denoted by $\mathrm{cnt}_\Sigma[\mathbf{b}]$ and called the $\Sigma$-content of $[\mathbf{b}]$; in this case we also let $\mathrm{cnt}_\Sigma(\mathbf{c}) = \mathrm{cnt}_\Sigma[\mathbf{b}]$ for each $\mathbf{c} \in [\mathbf{b}]$. And, if $\mathbf{d} \in [\mathbf{b}]$ is such that $\mathrm{cnt}(\mathbf{d}) = \mathrm{cnt}_\Sigma(\mathbf{d})$, then we say that $\mathbf{d}$ is a minimal member of $[\mathbf{b}]$. (We note that $[\mathbf{b}]$ can contain distinct minimal members.)

(3) If $\Sigma$ is with constant then $|\mathrm{cnt}_\Sigma[\mathbf{a}]| \geq 2$ for each $[a] \neq \mathbf{o}$, but if $\Sigma$ is without constant then $|\mathrm{cnt}_\Sigma(\mathbf{a})| = 1$ for every $\mathbf{a} \in A^n$ such that $|\mathrm{cnt}(\mathbf{a})| = 1$. If $\Sigma$ is essentially unary then $|\mathrm{cnt}_\Sigma(\mathbf{a})| = 1$ for every $\mathbf{a} \in A^n$.

(4) If $A \subseteq B$ then the canonical mapping from $\Sigma[A]$ into $\Sigma[B]$ is injective, and then we can assume that $\Sigma[A] \subseteq \Sigma[B]$, in the following sence: if $[\mathbf{a}] \in \Sigma[B]$ and $\mathrm{cnt}_\Sigma[\mathbf{a}] \subseteq A$, then we take $[\mathbf{a}] \in \Sigma[A]$ as well.

An algebra $(A, f)$ with $n$-ary operation $f$ (i.e. an $n$-groupoid) is called a $\Sigma$-object if it satisfies all the identities belonging to $\Sigma$.

PROPOSITION 3.2. *An $n$-groupoid $(A, f)$ is a $\Sigma$-object iff*

$$\mathbf{a} \sim_A \mathbf{b} \Rightarrow f(\mathbf{a}) = f(\mathbf{b})$$

*for every $\mathbf{a}, \mathbf{b} \in A^n$.* $\square$

Denote by $\mathrm{nat}(\sim_A)$ the natural mapping $\mathbf{a} \mapsto [\mathbf{a}]$ from $A^n$ into $\Sigma[A]$. Then by Proposition 3.2 we have:

PROPOSITION 3.3. *An $n$-groupoid $(A, f)$ is a $\Sigma$-object iff there is a uniquie mapping $\underline{f}: \Sigma[A] \to A$ such that $\underline{f} \circ \mathrm{nat}(\sim_A) = f$. (Certainly, the existence of such a mapping $\underline{f}$ implies its uniqueness.)* $\square$

Now we have a more convenient alternative definition of a $\Sigma$-object. Namely, if $\underline{f}$ is a mapping from $\Sigma[A]$ into $A$, then the pair $(A, \underline{f})$ is called a $\Sigma$-object with carrier $A$ and operation $\underline{f}$. Futher on, by a $\Sigma$-object we will understand the kind of structure we have just defined. Thus, for subobjects and homomorphisms we have the following characterizations:

PROPOSITION 3.4. *If $\mathbf{A} = (A, \underline{f})$ is a $\Sigma$-object and $C \subseteq A$, then $C$ is a subobject of $\mathbf{A}$ iff $\underline{f}(\Sigma[C]) \subseteq C$.* $\square$

Thus, any subobject of a $\Sigma$-object is a $\Sigma$-object too.

PROPOSITION 3.5. *Let $\mathbf{A} = (A, \underline{f})$ and $\mathbf{B} = (B, \underline{g})$ be $\Sigma$-objects, and let $h: A \to B$ be a mapping. Then $h$ induces a unique mapping $\underline{h}: \Sigma[A] \to \Sigma[B]$ such that $\underline{h} \circ \mathrm{nat}(\sim_A) = \mathrm{nat}(\sim_B) \circ h$, and $h$ is a homomorphism from $\mathbf{A}$ into $\mathbf{B}$ iff $h \circ \underline{f} = \underline{g} \circ \underline{h}$.* $\square$

(We note that $h: A \to B$ induces a mapping $h^{(n)}: A^n \to B^n$ such that $[\mathbf{a}] = [\mathbf{b}]$ in $\Sigma[A]$ implies $[h^{(n)}(\mathbf{a})] = [h^{(n)}(\mathbf{b})]$ in $\Sigma[B]$, and then $\underline{h}([\mathbf{a}]) = [h^{(n)}(\mathbf{a})]$ for each $\mathbf{a} \in A^n$.)

The notion of a partial $\Sigma$-object can be defined as follows. Let $A$ be a nonempty set, $\mathcal{D}$ a subset of $\Sigma[A]$ and $\underline{f}$ a mapping from $\mathcal{D}$ into $A$. Then we say that the triple $(A, \mathcal{D}, \underline{f})$ is a partial $\Sigma$-object. It can be easily seen that this definition is compatible with Evans' definition of partial algebras in a variety of algebras (see [**3**], where the words "incomplete" and "a class of algebras $\mathcal{V}$" are used instead of "partial" and "a variety $\mathcal{V}$"). Furthermore, if $(A, \mathcal{D}, \underline{f})$ is a given partial $\Sigma$-object and $q$ a fixed element of $A$, then if we define $\underline{g}: \Sigma[A] \to A$ by

$$\underline{g}([\mathbf{a}]) = \begin{cases} f([\mathbf{a}]), & \text{if } [\mathbf{a}] \in \mathcal{D} \\ q, & \text{if } [\mathbf{a}] \in \Sigma[A] \backslash \mathcal{D}' \end{cases}$$

then $(A, \underline{g})$ is a $\Sigma$-object which is an extension of $(A, \mathcal{D}, \underline{f})$. Now we can apply the well known Evans' result [**3**, p. 68] "if $\mathcal{V}$ is a class of algebras having the property that any incomplete $\mathcal{V}$-algebra can be embedded in a $\mathcal{V}$-algebra, then the word problem can be solved for this class" to obtain the proof of Theorem E of section 1.

**4. A construction of free $\Sigma$-objects.** Here we will give a construction of free $\Sigma$-objects with basis $B$, where $B$ is a given nonempty set. Let $(B_p|p \geq 0)$ be a sequence of sets defined inductively as follows:

$$B_0 = B, \qquad B_{p+1} = B_p \cup \Sigma[B_p],$$

and let

$$F(\Sigma, B) = \bigcup(B_p|p \geq 0).$$

(We will write simply $F$ instead of $F(\Sigma, B)$, when $\Sigma$ and $B$ are known.) By induction on $p$ one can easily prove that $\Sigma[F] = F\backslash B$.

If $u \in F$ and if $p$ is the least number such that $u \in B_p$, then we say that $p$ is the hierarchy of $u$ and write $\chi(u) = p$. It is clear that if $\Sigma$ is with constant, then $\chi(\mathbf{o}) = 1$.

Proposition 4.1. *Let $u \in F$ and let $u$ not be a constant. Then $\chi(u) = p+1$ iff $\mathrm{cnt}_\Sigma(u) = \{v_1, v_2, \dots, v_k\}$ is such that $\chi(v_i) \leq p$ for each $i$ and $\chi(v_j) = p$ for some $j$ $(i, j \in \{1, 2, \dots, k\})$.* $\square$

Define an operation $\underline{f}\colon \Sigma[F] \to F$ by $\underline{f}(u) = u$ for each $u \in \Sigma[F]$. Then we have:

Proposition 4.2. *$(F, \underline{f})$ is a $\Sigma$-object generated by the set $B$.* $\square$

Let $(C, \underline{g})$ be an arbitrary $\Sigma$-object and let $h\colon B \to C$ be a mapping. Put $h_0 = h$ and suppose that $h_r\colon B_r \to C$ is a well defined mapping for each $r \leq p$ in such a way that $h_r$ is an extension of $h_{r-1}$, and if $r > 0$, $\chi(u) = r$, then $h_r(u) = \underline{g} \circ \underline{h}_{r-1}(u)$, where $\underline{h}_{r-1}\colon \Sigma[B_{r-1}] \to \Sigma[C]$ is defined as in Proposition 3.5. Now define $h_{p+1}\colon B_{p+1} \to C$ to be the extension of $h_p$ such that $h_{p+1}(u) = \underline{g} \circ \underline{h}_p(u)$ for each $u$ with $\chi(u) = p+1$. (Note that if $\chi(u) = p+1$, then $u \in \Sigma[B_p]$, and thus $\underline{h}_p(u) \in \Sigma[C]$ is well defined by Proposition 3.5.) In such a way we have defined a chain of mappings $(h_p|p \geq 0)$, and its union $\overline{h} = \bigcup(h_p|p \geq 0)$ is an extension of $h$ and a homomorphism from $(F, \underline{f})$ into $(C, \underline{g})$ as well. Thus we have the following

Theorem 4.3. *If $B$ is a nonempty set, then $(F, \underline{f})$ is a free object with basis $B$.* $\square$

The preceding construction of free $\Sigma$-objects is somewhat obscure, but in some cases it can be considerably simplified.

*Example* 4.4. If $\Sigma$ is with constant and $a, b \in B$, then we have $[a^n] = [b^n] = \mathbf{o}$, where $\mathbf{o}$ is the zero of $F$. (Here, and later on, $a^n\colon i \mapsto a$ for each $a \in A$, $i \in \{1, \dots, n\}$.) Clearly, $\mathbf{o} \in B_1\backslash B$ and if $\Sigma$ is with absolute constant, then $F = B \cup \{\mathbf{o}\}$ and $\underline{f}(u) = \mathbf{o}$ for each $u \in \Sigma[B \cup \{\mathbf{o}\}]$. Therefore, if $\Sigma$ is with absolute constant, then every constant $n$-groupoid is freely generated by the set of elements distinct from the constant (i.e. $\mathbf{o}$). We have the same result if $\Sigma$ is with constant, of order $k$ and $|B| < k$. (Moreover, if $\Sigma$ is with constant, then any one-element groupoid can be considered as free $\Sigma$-object with empty basis.) $\square$

*Example* 4.5. Assume that $\Sigma$ is essentially unar, i.e. for each $\varphi \in M^n$ there is an $i \in \{1, 2, \dots, n\}$ such that $(\varphi, \mathbf{j}) \in \Sigma$ for $j = \varphi(i)$. Then the class of $\Sigma$-objects can be viewed as the class of unars. Namely, if $(G,h)$ is a unar and if we define a mapping $\underline{g}\colon \Sigma[G] \to G$ by $\underline{g}(\mathbf{a}) = h(\mathbf{a}(i))$, then we get a $\Sigma$-object $(G,\underline{g})$, and any $\Sigma$-object can be obtained in such a manner. Moreover, $(G,\underline{g})$ is a free $\Sigma$-object with basis $B$ iff $(G,h)$ is a free unar with basis $B$. $\square$

We note that a subunar of a finitely generated free unar is finitely generated too, and thus Example 4.5 shows that the assumptions of Theorem $D$ are essential.

*Example* 4.6. Let $n = 3$ and let $\mathcal{V}$ be a variety defined by the identities

$$f(x,x,x) = f(x,x,y) = f(y,y,y), \quad f(x,y,z) = f(y,x,z) = f(x,z,y).$$

If $B = \{b\}$, $\mathbf{o} \neq b$ and if we put $G = \{\mathbf{o}, b\}$ and $g(u,v,w) = \mathbf{o}$ for each $u,v,w \in G$, then $(G,g)$ is a free object in $\mathcal{V}$ with basis $B$ of rank 1. Now, take $B = \{b,c\}$, $b \neq c$ and $\mathbf{o} \notin B$, and define the sets $B_p$ inductively by

$$B_0 = B \cup \{\mathbf{o}\}, \quad B_{p+1} = B_p \cup \{\{u,v,w\} | u \neq v \neq w \neq u, \ \ u,v,w \in B_p\}$$

Let $H = \bigcup(B_p | p \geq p)$ and let

$$h(u,v,w) = \begin{cases} \{u,v,w\}, & \text{if } u \neq v \neq w \neq u. \\ \mathbf{o}, & \text{otherwise} \end{cases}$$

Then $\mathbf{H} = (H,h)$ is a free object in $\mathcal{V}$ with basis $B$. The subset $D$ of $H$, where $D = \{d_i | i \geq 0\}$ and the elements $d_i$ are defined inductively by $d_0 = \{\mathbf{o}, b, c\}$, $d_{p+1} = \{\mathbf{o}, b, d_p\}$ is a basis of infinite rank of the subobject $\mathbf{L}$ of $\mathbf{H}$ generated by $D$. $\square$

*Example* 4.7. There exist exactly 6 nonequivalent primitive 2-identities: $xy = xy$, $xy = yx$, $xy = xx$, $xy = yy$, $xx = yy$, $xy = zw$. (Here a usual notation of identities is used.) One can form 7 primitive 2-varieties, 6 of them being defined by a single identity of the above ones, and $\mathcal{V} = \operatorname{Var}(\{xy = yx, xx = yy\})$. In the variety $\mathcal{V}$ we can describe a free object with nonempty basis $B$ by $F = \bigcup(B_p | p \geq 0)$, where $B_0 = B$, $B_1 = B \cup \{\mathbf{o}\} \cup \{\{u,v\} | u,v \in B, u \neq v\}$, $\mathbf{o} \notin B$, and $B_{p+1} = B_p \cup \{\{u,v\} | u,v \in B_p, u \neq v\}$ when $p \geq 1$. $\square$

**5. Some properties of free $\Sigma$-objects.** Here we will give proofs of Theorems $A$, $B$, $C$ and $D$ of section 1. Although one can prove these theorems by using an induction on hierarchy, we will rather use the ideas involved in [**1**].

Assume that $\mathbf{G} = (G,\underline{g})$ is a $\Sigma$-object. An element $a \in G$ is said to be prime in $\mathbf{G}$ if $a \neq \underline{g}([\mathbf{b}])$ for any $[\mathbf{b}] \in \Sigma[G]$. If $\Sigma$ is with constant, then each element of $\mathbf{G}$ is said to be an improper divisor of the zero $\mathbf{o} \in \Sigma[G]$. If $c \in G$ is nonzero and nonprime element, then there is a $[\mathbf{b}] \in \Sigma[G]$ such that $c = \underline{g}([\mathbf{b}])$, and let $\mathbf{a}$ be a minimal member of $[\mathbf{b}]$. Then each element $d \in \operatorname{cnt}(\mathbf{a}) = \operatorname{cnt}_\Sigma[\mathbf{a}]$ is called a proper divisor of $c$. A sequence (finite or infinite) of elements $a_1, a_2, \ldots$ of $G$ is said to be a divisor chain in $\mathbf{G}$ iff for every $i > 1$ $a_i$ is a proper divisor of $a_{i-1}$.

Now we have another characterization of free $\Sigma$-objects:

THEOREM 5.1. *A $\Sigma$-object $\mathbf{H} = (H, \underline{h})$ is a free $\Sigma$-object with a nonempty basis $B \subseteq H$ iff the following conditions hold:*

(i) *$B$ is the set of prime elements in $\mathbf{H}$.*

(ii) *If $c \in H$ is nonprime, then there is a unique $[\mathbf{b}] \in \Sigma[B]$ such that $c = \underline{h}([\mathbf{b}])$.*

(iii) *Every divisor chain in $\mathbf{H}$ is finite.*

*Proof.* It is clear that $(F, \underline{f})$ satisfies (i), (ii) and (iii).

Conversely, if $\mathbf{H}$ satisfies (i), (ii) and (iii), then it is easy to show by induction on hierarchy that there is an isomorphism $g: (F, \underline{f}) \to (H, \underline{h})$ such that $g(b) = b$ for each $b \in B$. $\square$

Now, Theorem A is a direct consequence of Theorem 5.1, for the set of prime elements of a free $\Sigma$-object is its unique basis. (We should emphasize here that we do not need Theorem 5.1 to prove Theorem A, since it follows directly from the definition of primitive $n$-identities.)

Assume that $\mathbf{G}$ is a subobject of $(F, \underline{f})$. The set of prime elements in $\mathbf{G}$ (considered as a $\Sigma$-object) is empty only if $\Sigma$ is with zero and $G = \{\mathbf{o}\}$, and then $\mathbf{G}$ is free with an empty basis. If the set $C$ of prime elements in $\mathbf{G}$ is nonempty, then $C$ is a basis of $\mathbf{G}$, since conditions (ii) and (iii) of Theorem 5.1 are hereditary. This completes the proof of Theorem B.

Now, let $\Sigma$ be with constant of order $k < n$, and let $B = \{a_1, a_2, \dots, a_k\}$. Then $B_1 = B \cup \{\mathbf{o}\}$ and $\mathrm{cnt}_\Sigma(a_1 a_2 \dots a_k \mathbf{o}^{n-k}) = \{a_1, a_2, \dots, a_k, \mathbf{o}\}$. Consider the subset $C = \{c_1, c_2, \dots, c_p, \dots\}$ of $F$, where $c_1 = [a_1 \dots a_k \mathbf{o}^{n-k}]$, $c_{p+1} = [a_1 \dots a_k c_p^{n-k}]$. Let $Q$ be the subobject of $(F, \underline{f})$ generated by $C$. Clearly, $C$ is the set of prime elements in $Q$. (Namely, $c_p$ is a divisor of $c_{p+1}$ in $F$, but this does not hold in $Q$.) This completes the proof of Theorem C, since the conditions for $\Sigma$ stated in Theorem C show that $\Sigma$ is with constant of order $k$.

It remains to show Theorem D. First we note that the assumption in this Theorem can be expressed by $|\mathrm{cnt}_\Sigma(\varepsilon)| = k \geq 2$. Take $\varphi$ to be a minimal member in $[\varepsilon]$, and $i \in \mathrm{cnt}_\Sigma(\varphi)$. Let $B$ be a nonempty set, $b \in B$ and define a sequence $a_1, a_2, \dots, a_n$ by $a_1 = b$, $a_{i+1} = [a_i^n]$ for $0 < i < n$, and an infinite sequence $c_1, c_2, \dots, c_p, \dots$ by $c_1 = a_n$, $\quad c_{p+1} = [a_1 a_2 \dots a_{i-1} c_p a_{i+1} \dots a_n]$. Then $a_i \neq a_j$ for $i \neq j$ and $c_r \neq c_s$ for $r \neq s$. This implies that $C = \{c_r | r \geq 1\}$ is an infinite basis of the subobject $Q$ of $(F, \underline{f})$ generated by $C$.

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Prirodno-matematički fakultet, Institit za informatika, 91000 Skopje, Makedonija

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