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Slobodan Aljančić memorial volume

SLOBODAN ALJANČIĆ
(1922 – 1993)

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Slobodan Aljančić, one of the most outstanding representatives of the school of clasical Analysis in Belgrade, passed away on March 19, 1993. Jovan Karamata, the founder of the school passed away on July 14, 1967, and his pupil Vojislav Avakumović died on July 19, 1990. These three scholars have finished their work and exited the scientific stage. In time, impartial scientific inquiries will pass judgment on their contributions to our science, and the significance of their work for the development of mathematics in general.

Their lives, like their work, were full of obstacles and nearly impossible difficulties to be surmounted. Conditions during World War II and in the early years of post war Yugoslavia nearly obliterated any chance of scientific pursuit. With libraries destroyed, personal scietific contact severed, and the daily struggle for survival in times of famine and suffering, there was little or no time to be concerned with science.

The war and subsequent revolution in our country completely changed the life of Slobodan Aljančić as well. During the war he began his studies at the Technical College of the University of Belgrade but in 1945 switched to Mathematics. There, Professor Jovan Karamata, who had been cut off entirely from the university and his own research for the last five years, was just beginning to reestablish contacts with his old students and to look for new ones. Slobodan Aljančić became one of his students. He soon became Karamata's assistant, but was unable to replace him at the chair for the Theory of Functions when Karamata went to Geneva. He did however continue to work in areas created and promoted by Karamata. By his contributions to some classical Real Analysis problems, and to the education of several students who became well known scientists with valuable results, Aljančić eventually became one of the principal representatives of the school of Analysis in Belgrade. The illness from which he suffered during the last twenty years of his life, and especially the last three, did not affect his strong devotion to research work. His mathematical results are valuable because of the problems selected, the method of solution, the precision of arguments and the simplicity of proofs.

During this brief look back we are going to mention two significant groups of problems which will remain associated with Aljančić's name. The first is the problem of trigonometric approximation of functions including the problem of finding estimates for the modulus of continuity of continuous functions whose Fourier series have monotone decreasing coefficients. The second important area in which Aljančić worked was the theory of regularly varying functions.

The theory of trigonometric approximations grew out of an answer to a question posed by J. Favard to Aljančić while he was in Paris during the winter semester of 1958. That response was given in a note in C.R. Acad. Sci. Paris **246** (1958), 2567–2569. In a series of papers, which drew attention of many authors including: M. Zamansky, G. Sunouchi, G. Watari and P.L. Butzer, Aljančić studied direct and inverse problems of approximation and the related saturation problems for wide classes of functions and very general summability and approximation methods. As a final result in that area we shall mention the result published in the Proc. Amer. Math. Soc. **12** (1961), 681–689.

Let $\mathbf{C}$ be the class of contiuous and periodic functions f of period 2π such that

$$\int_0^{2\pi} f(x)dx = 0$$

and let $\mathbf{T}$ be an approximation process which to every $f \in \mathbf{C}$ assigns a sequence $(T_n(f))$ of trigonometric polynomials. Let

$$\Delta_n(f) = \max_{0 \le x \le 2\pi} |f(x) - T_n(f;x)|$$

be the deviation of the polynomial $T_n(f)$ from f. If $\mathbf{M} \subset \mathbf{C}$, then the magnitude of

$$\sup_{f \in \mathbf{M}} \Delta_n(f)$$

characterizes the effectiveness of the approximation process $\mathbf{T}$ for the entire class $\mathbf{M}$. There are three different questions to be answered:

(i) given a class $\mathbf{M} \subset \mathbf{C}$, find a positive function φ, with $\varphi(n) \to 0$ as $n \to \infty$ such that

$$f \in \mathbf{M} \Rightarrow \Delta_n(f) = O(\varphi(n))$$

(a direct theorem);

(ii) given a positive function φ, with $\varphi(n) \to 0$ as $n \to \infty$, determine the class $\mathbf{M}$ such that

$$\Delta_n(f) = O(\varphi(n)) \Rightarrow f \in \mathbf{M}$$

(an inverse theorem);

(iii) for a given approximation process $\mathbf{T}$ find $\mathbf{M}$ and φ such that both the direct and the inverse theorem hold simultaneously:

$$f \in \mathbf{M} \Longleftrightarrow \Delta_n(f) = O(\varphi(n))$$

(an equivalence theorem).

Suppose now that for a given approximation process $\mathbf{T}$, and a positive function φ, with $\varphi(n) \to 0$ as $n \to \infty$, there is a class $\mathbf{M} \subset \mathbf{C}$ such that an equivalence theorem holds for $\mathbf{T}, M$ and φ and

$$\Delta_n(f) = o(\varphi(n)) \Rightarrow f = 0.$$

We say then φ is the order of best approximation of functions in $\mathbf{M}$ by the method $\mathbf{T}$ and that $\mathbf{M}$ is its class of saturation. A theorem of this type is called a saturation theorem for the method $\mathbf{T}$.

Problems of this type can be best illustrated by considering continuous functions which have a bounded derivative of order r, or the more general class $W^r(0)$ of functions which can be expressed in terms of Weyl's integral of order r. If we denote by ${}^2\Lambda_\alpha(0 \le \alpha \le 1)$ the Zygmund class of functions defined by the symmetric modulus of continity of order 2 and by $R_n^\lambda(f)$ the Riesz typical mean of f defined by

$$R_n^\lambda(f,x) = \sum_{k+1}^{n-1} \left(1 - \frac{k^\lambda}{n^\lambda}\right) (a_k[f]\cos(kx) + b_k[f]\sin(kx))$$

where $a_k[f]$ and $b_k[f]$ are the Fouruer coefficients of f, then the Aljančić's result mentioned erlier can be formulated as follows:

Let $k = 0, 1, 2, \ldots$ and $0 < \alpha < 1$. Then

(i) $f - R_n^\lambda(f) = o(n^{-\lambda}) \Rightarrow f \equiv 0$

(ii) $f \in W^\lambda(0) \Leftrightarrow f - R_n^\lambda(f) = o(n^{-\lambda})$

(iii) $f^{(k)} \in {}^2\Lambda_\alpha,\ k + \alpha < \lambda \Leftrightarrow f - R_n^\lambda(f) = o(n^{-k-\lambda})$

Part (i) is a saturation theorem for the Riesz typical means $R_n^\lambda(f)$. This theorem contains a number of results obtained earlier by A. Zygmund, B.Sz. Nagy, M. Zamansky, G. Sunouchi and C. Watary. Later, it was extended to multidimensional typical means and to other approximation process based on more general expansions.

Realizing the importance of classes of functions defined by various moduli of continuity, Aljančić became interested in finding estimates for these moduli in terms of certain asymptotic properties of the coefficients of the Fourier series of the function f. First results of this type were obtained by G.G. Lorenz (Math. Zeitschrift **51** (1948), 135–149. Aljančić, either alone or in cooperation with other mathematicians, published several papers dealing with various aspects of this problem. His most general result was published in the Proceedings of the American Mathematcal Society **17** (1966), 287–294:

If $f \in \mathbf{L}^p(1 < p < \infty)$ and his Fourier series

$$\sum_{k=1}^{\infty} \mu_k[f]\cos(kx) \quad \text{or} \quad \sum_{k=1}^{\infty} \mu_k[f]\sin(kx)$$

has monotone decreasing coefficients

$$\mu_k[f] \ge \mu_{k+1}[f] \to 0, \quad (k \to \infty)$$

and if for a fixed p $(1 < p < \infty)$

$$(*) \qquad \sum_{k=1}^{\infty} k^{p-2}\mu_k^p[f] < \infty$$

then for the $\mathbf{L}^p$ modulus of continuity of f

$$\omega_p(f;h) = \sup_{|t|\le h} \|f(\cdot + t) - f(\cdot)\|_p$$

the following estimate holds:

$$\omega_p\left(f;\frac{1}{n}\right) \le \frac{A_p}{n}\left(\sum_{k=1}^{n} k^{2p-2}\mu_k^p[f]\right)^{1/p} + B_p\left(\sum_{k=n}^{\infty} k^{p-2}\mu_k^p[f]\right)^{1/p}$$

To underline the importance of this result, let us mention that condition $(*)$ is both necessary and sufficient for the validity of Aljančić estimate.

Also apparent in these two works is Aljančić's style in Mathematics – general results with a minimum of hypotheses and precise yet simple proofs. His train of thought is easy to follow because all of his arguments are clear and without superfluous details. Therefore his papers will remain permanent models and prominent examples of work which rank with the those of the great masters of classical Analysis.

Probably the most important part of Aljančić's work in Mathematics is his contribution to the development of the theory of regularly varying functions, created by Jovan Karamata. The usefulness of that theory can be seen by its applications in various branches of analysis and particularly by its role in probability theory. Today there exist several monographs in which it is presented in detail (E. Seneta, N.H. Bingham, G.M. Goldie, J.L. Teugels). The theory of regularly varying functions is one of the few theories which originated in Belgrade. Foreign authors give credit also to Karamata's students, including: Avakumović, Aljančić, Bojanić, Baisansky, Tomić and their own students for the theory's full development and his group became known outside as the Belgrade School of Analysis. Their published works are too numerous to be mentioned here in detail.

In the fall of 1952, while discussing the behavior of trigonometric series with Aljančić, we found that a large number of such problems can be reduced to the problem of asymptotic behavior of a class of definite integrals whose essential part is a slowly varying function. Since Karamata was in Geneva at that time making final plans to leave Belgrade, we encountered every difficulty that is experienced by people who not only try to began research work in a new area, but who are just starting their scientific careers without help from senior research workers. Our published paper, as a result of that intensive research, (Publ. Inst. Math. (Beograd), **7** (1954), 81–94 contains some of the basic results regarding slowly varying functions and their immediate consequences which, though simple, were previously unknown. After Karamata's use of regularly varying functions in theorems of Tauberian and Abelian type in 1931, there were no new results which would require the use of regularly varying functions. Feller's book, Probability Theory, appeared much later. Consequently, these results of Aljančić and the others from the Belgrade circle completed the foundation of the theory of regularly varying functions.

In this group of papers, which served as the basis for the formulation of general theorems of the theory of regularly varying functions, one should mention the paper of Aljančić and Karamata published in 1956 in the Recueil des Travaux de l'Academie Serbe des Sciences. Starting from the well known formula

$$\int_0^\infty \frac{f(ax)-f(bx)}{x}dx = (\lambda - \Lambda)\log\left(\frac{b}{a}\right), \quad a,b>0$$

for the evaluation of Cauchy–Frullani integral, which is valid if the limits

$$\lambda = \lim_{x\to 0} f(x) \;\text{ and }\; \Lambda = \lim_{x\to 0} f(x)$$

exists, they give a new necessary and sufficient condition for a function to be regularly varying function and they show that Frullani integral exists if and only if the function

$$\exp\left(\int_0^x \frac{f(t)}{t}dt\right)$$

is a regularly varying function in the sense of Karamata.

Aljančić has contributed significantly also to the theory of O-RV functions introduced by Avakumović in 1936. In 1977 Aljančić and Aranđelović published a paper in Publications de l'Institut Mathématique Belgrade, in which they studied in detail the class of O-RV functions. They showed that the class of O-RV functions can be defined by the requirement that the limit

$$\lim_{x\to\infty}\sup\frac{K(tx)}{K(x)} = r(t)$$

should be finite for every $t > 0$. From this definition follows the analog of the uniform convergence property and other results obtained earlier by Avakumović in Rad Jugoslav. Akad. Znan. Umjet. **254** (1936), 167–186. The asymptotic behavior of $r(t)$ as $t \to 0$ or $t \to \infty$ is obtained assuming that $r(t)$ satisfies the condition $r(st) \le r(s)r(t)$ for all small or large values of s and t.

As all of Aljančić work shows, he wrote papers only when he had something new to say, and he always said it with elegance. His results show harmony between hypotheses and clearly written proofs. Therefore these results are not only cited, but they also inspire new investigations.

Aljančić was a man of exceptional character and behavior. He treated everyone with care and respect, whether they were older—his professors, or younger—his students. Many mathematics students said that they would not have continued their studies without his encouragement and support during the initial classes and recitations. He never complained to his friends when he had difficulties or when he was ill. Such things did not affect his relationship with others. That is how he remained to the end of his life. His character had something that inspired a real closeness among the people around him.

The two of us will always remember with greatest affection the hours we spent together during 1950's in the Mathematical Institute and in the Academy's club. Our joint work was the first part of a long journey into scientific research. Every wonderful quality Aljančić possessed as person, he projected into our midst and our work. His own exemplary behavior insured that there would never be anything but cooperation between all of us.

Now, after almost half a century, when many of our dear friends have passed away, we will always remember them with gratitude and sadness, and we will especially remember Slobodan Aljančić.