PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 58 (72), 1995, 13–20
Slobodan Aljančić memorial volume

QUASIASYMPTOTICS AND $\mathcal{S}$-ASYMPTOTICS IN $\mathcal{S}'$ AND $\mathcal{D}'$

S. Pilipović

Dedicated to the memory of Academician Slobodan Aljančić

Abstract. We prove that the quasiasymptotics (resp. S-asymptotics) in $\mathcal{D}'$ implies the quasiasymptotics (resp. S-asymptotics) in $\mathcal{S}'$ under some natural assumptions.

1. Introduction

Regularly varying functions of Karamata [**4**] naturally appear in the theory of generalized asymptotic behaviour of distributions. Vladimirov, Drožžinov and Zavialov [**11**] have introduced the notion of quasiasymptotic behaviour of tempered distributions in connection with the Laplace transform and proved that the functions for comparison must be regularly varying ones. Pilipović and Stanković [**8**] introduced the notion of S-asymptotic behaviour of distributions and proved that the functions for comparison are of the form $\rho(e^x)$ where ρ is a regularly varying function.

Note that the properties of regularly varying functions proved by Professor Aljančić and his collaborators, especially in [**1**], are used very much in the theory of asymptotic behaviour of distributions.

We will give in this article assertions concerning the relations between the quasiasymptotics in the sense of $\mathcal{D}'$ and $\mathcal{S}'$ convergences. We will answer the question whether the quasiasymtotis (resp. $\mathcal{S}$–asimptotics) in $\mathcal{D}'$ implies the quasiasimptotics (resp. $\mathcal{S}$–asimptotics) in $\mathcal{S}'$? Such partial results appear in several papers but they are not justified in all the details.

2. Notions

The Schwartz spaces of test functions and distributions on the real line $\mathbf{R}$ are denoted by $\mathcal{D}$ and $\mathcal{D}'$; $\mathcal{S}$ is the space of rapidly decreasing functions, its dual $\mathcal{S}'$

AMS Subject Classification (1991): Primary 46F10

is the space of tempered distributions and $\mathcal{S}'_+$ is the subspace of $\mathcal{S}'$ with elements supported by $[0,\infty)$.

Let K be a compact set of the real line. Recall [**9**],

$$\mathcal{D}_K = \bigcap_{p\in\mathbf{N}_0} \mathcal{D}^p_K \quad \text{and} \quad \mathcal{D}'_K = \bigcup_{p\in\mathbf{N}_0} \mathcal{D}'^p_K$$

have topological meaning. This implies that a sequence f_n in $\mathcal{D}'_K$ converges to $f \in \mathcal{D}'_K$ iff it belongs to some $\mathcal{D}'^{p_0}_K$, and in the dual norm of $\mathcal{D}'^{p_0}_K$ converges to $f \in \mathcal{D}'^{p_0}_K$.

The Fourier transformation $\mathcal{F}$ of $\mathcal{S}$ and $\mathcal{S}'$ is defined as usual (cf. [**9**] or [**10**]).

The space of exponentially growing distributions were introduced and studied by Hasumi [**3**] and many others after him. Recall [**3**], $\mathcal{K}_1$ is equal to the space of smooth functions ϕ on $\mathbf{R}$ for which all the norms

$$\sup_{\alpha\le k, x\in\mathbf{R}} \{e^{k|x|}|\phi^{(\alpha)}(x)|\},\ k\in\mathbf{N}_0,$$

are finite. The topological properties of this space are the same as for $\mathcal{S}'$. We only note that $\mathcal{K}'_1$ consists of distribution of the form

$$f = \sum_{\alpha=0}^{m} (e^{k|x|}\ f_\alpha)^{(\alpha)} \tag{1}$$

for some $m,k\in\mathbf{N}$ and some continuous bounded functions f_α, $\alpha = 1,\dots,m$, where ${}^{(\alpha)}$ is the distributional derivative.

A real valued continuous function L defined on $(0,a)$ (resp. (a,∞)), $a>0$, is called slowly varying at zero (resp. at infinity) (cf. [**2**]) if for every $\lambda>0$,

$$\lim_{x\to 0^+} \frac{L(\lambda x)}{L(x)} = 1 \qquad \left(\text{resp.}\quad \lim_{x\to\infty} \frac{L(\lambda x)}{L(x)} = 1\right).$$

We refer to [**2**] and [**1**] for the properties of slowly varying functions. Let $\alpha\in\mathbf{R}$. Recall [**9**],

$$f_{\alpha+1}(x) = \begin{cases} \frac{H(x)x^\alpha}{\Gamma(\alpha+1)}, & \alpha > -1 \\ f^{(n)}_{\alpha+n+1}(x), & \alpha\le -1, \end{cases} \quad x\in\mathbf{R},$$

where n is the smallest integer for which $\alpha+n>-1$ and H is Heaviside's function.

Definition 1. Let $f\in\mathcal{D}'$, L be a function which is slowly varying at zero (resp. ∞) and $\nu\in\mathbf{R}$. If for every $\varphi\in\mathcal{D}$ there exists the limit

$$\lim_{\varepsilon\to 0^+} \left\langle \frac{f(\varepsilon x)}{\varepsilon^\nu L(\varepsilon)}, \varphi(x)\right\rangle \qquad \left(\text{resp.}\quad \lim_{k\to\infty} \left\langle \frac{f(kx)}{k^\nu L(k)}, \varphi(x)\right\rangle\right) \tag{2}$$

which is different from zero for some φ, then it is said that f has the quasiasymptotics at 0 (resp. ∞) in $\mathcal{D}'$ with respect to $\varepsilon^\nu L(\varepsilon)$ (resp. $k^\nu L(k)$).

If in the limits given above the spaces $\mathcal{D}$ and $\mathcal{D}'$ are substituted by $\mathcal{S}$ and $\mathcal{S}'$ respectively, then it is said that f has the quasiasymptotics at 0 (resp. at ∞) in $\mathcal{S}'$ with respect to $\varepsilon^\nu L(\varepsilon)$ (resp. $k^\nu L(k)$).

Banach–Steinhaus theorem implies that the existance of the limit in (2) when $\varepsilon \to 0^+$ (resp. $k \to \infty$) implies that there exists $g \in \mathcal{D}'$, $g \neq 0$, such that

$$\lim_{\varepsilon\to 0^+} \left\langle \frac{f(\varepsilon x)}{\varepsilon^\nu L(\varepsilon)}, \varphi(x) \right\rangle = \langle g(x), \varphi(x) \rangle, \quad \varphi \in \mathcal{D}$$

$$\Big(\text{resp.} \quad \lim_{k\to\infty} \left\langle \frac{f(kx)}{k^\nu L(k)}, \varphi(x) \right\rangle = \langle g(x), \varphi(x) \rangle, \quad \varphi \in \mathcal{D}\Big).$$

It is easy to prove [**11**, p. 73] that the existence of limit (2) in $\mathcal{D}'$ implies that g is homogenous with the order of homogenity $\nu \in \mathbf{R}$. Thus,

$$g(x) = C_+ f_{\nu+1}(x) + C_- f_{\nu+1}(-x),\ x \in \mathbf{R},\ (C_+, C_-) \neq (0,0).$$

3. Relations between quasiasymptotics

The main part of the next assertion belongs to Zavialov ([**12**], see also [**6**]) His proof is original and powerful. We will present it and make some changes in relation (59) in [**12**] which have certain consequences in the rest of the proof.

THEOREM 1. *Let $f \in \mathcal{S}'$ and ρ be regularly varying at 0^+ of order $\alpha \in \mathbf{R}$ ($\rho(\varepsilon) = \varepsilon^\alpha L(\varepsilon)$). Assume that $\lim_{\varepsilon\to 0^+} (1/\rho(\varepsilon)) \langle f(\varepsilon x), \phi(x) \rangle$ exists for every $\phi \in \mathcal{D}$, and it is different from zero for some ϕ. Then, there is $(C_1, C_2) \neq (0,0)$ such that*

$$\lim_{\varepsilon\to 0^+} (1/\rho(\varepsilon)) \langle f(\varepsilon x), \phi(x) \rangle = \langle C_1 f_{\alpha+1}(x) + C_2 f_{\alpha+1}(-x),\ \phi(x) \rangle, \quad \phi \in \mathcal{S}.$$

Proof. It is enough to prove that for every $\phi \in \mathcal{S}$, such that $\phi_{|[-1,1]} = 0$, there exists the limit

$$\lim_{\varepsilon\to 0^+} \frac{1}{\rho(\varepsilon)} \langle f(\varepsilon x), \phi(x) \rangle . \tag{3}$$

The following construction is given by Zavialov. Let $\eta \in C_0^\infty$ such that $\operatorname{supp} \eta \subset [-1, -1/2] \cup [1/2, 1]$ and $\eta > 0$ in $(-1, -1/2) \cup (1/2, 1)$. Put

$$\gamma(x) = \int_0^\infty \eta(x/t)\,dt, \qquad x \in \mathbf{R}.$$

This is a smooth function with polynomially bounded derivatives as $|x| \to \infty$ or $|x| \to 0$. Moreover, there are constants $a_1, b_1, a_2, b_2 > 0$ and $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbf{R}$ such that

$$a_1 |x|^{\alpha_1} \leq \gamma(x) \leq b_1 |x|^{\beta_1}, \quad |x| \leq 1,$$
$$a_2 |x|^{\alpha_2} \leq \gamma(x) \leq b_2 |x|^{\beta_2}, \quad |x| \geq 1.$$

We have

$$\begin{aligned}\frac{1}{\rho(\varepsilon)}\langle f(\varepsilon x),\phi(x)\rangle &= \frac{1}{\rho(\varepsilon)}\left\langle f(\varepsilon x),\frac{1}{\gamma(x)}\int_0^\infty \phi(x)\eta(\frac{x}{t})dt\right\rangle\\ &= \int_0^\infty \frac{1}{\rho(\varepsilon)}\left\langle f(\varepsilon x),\frac{1}{\gamma(x)}\phi(x)\eta(\frac{x}{t})\right\rangle dt. \qquad (4)\end{aligned}$$

The limit (as $\varepsilon\to 0$) of the integrand in (4) exists for every fixed $t\in[0,\infty)$. Thus (3) will be proved if we show

$$\left|\frac{1}{\rho(\varepsilon)}\left\langle f(\varepsilon x),\frac{1}{\gamma(x)}\phi(x)\eta(\frac{x}{t})\right\rangle\right|\le s(t),\quad t\in\mathbf{R},$$

where $s\in L^1(0,\infty)$. There holds

$$\begin{aligned}F(t)&=\frac{1}{\rho(\varepsilon)}\left\langle f(\varepsilon x),\frac{1}{\gamma(x)}\phi(x)\eta(\frac{x}{t})\right\rangle=\frac{t}{\rho(\varepsilon)}\left\langle f(\varepsilon tx),\frac{\phi(tx)}{\gamma(tx)}\eta(x)\right\rangle\\ &=t\frac{\rho(\varepsilon t)}{\rho(\varepsilon)}\left\langle\frac{f(\varepsilon tx)}{\rho(\varepsilon t)},\frac{\phi(tx)}{\gamma(tx)}\eta(x)\right\rangle,\quad t\in\mathbf{R}.\end{aligned}$$

We will use the following estimate

$$\frac{\rho(\varepsilon t)}{\rho(\varepsilon)}\le\frac{1+t^{n_1}}{t^{n_2}}\quad\text{for}\quad \varepsilon<\varepsilon_0 \text{ and } t<\frac{\varepsilon_0}{\varepsilon}, \qquad (5)$$

which holds for suitable positive numbers n_1, n_2 and ϵ_0 (cf. [**2**]) and

$$\left\{\frac{f(\varepsilon tx)}{\rho(\varepsilon t)};\varepsilon t\in(0,\varepsilon_1)\right\}\text{ is bounded in }\mathcal{D}'. \qquad (6)$$

Let $a\le|x|\le b$, where $0<a<b$. One can easily prove that for every $k>0$ and every $\alpha\in\mathbf{N}_0$

$$\begin{aligned}|\phi^{(\alpha)}(tx)|&=o(|t|^{-k})\quad\text{as}\quad |t|\to\infty,\\ |\phi^{(\alpha)}(tx)|&=o(|t|^{k})\quad\text{as}\quad |t|\to 0.\end{aligned}\qquad (7)$$

Let $\varepsilon_2=\min\{\varepsilon_0,\varepsilon_1\}$. We will estimate the function F defined by (4) in the following cases: (a) $\varepsilon t\le\varepsilon_2$ and (b) $\varepsilon t>\varepsilon_2$.

(a) Then (5) and (6) hold. Thus (7) implies that $|F|$ is bounded by an integrable function on $(0,\infty)$ which does not depend on ε.

(b) There exists $N\in\mathbf{N}$ such that $\{(\varepsilon t)^{-N}\eta(x)f(\varepsilon tx);\ \varepsilon t>\varepsilon_2,\ \varepsilon<\varepsilon_2\}$ is bounded in $\mathcal{D}'$. Then we have

$$\left\langle\frac{tf(\varepsilon tx)}{\rho(\varepsilon)},\frac{\phi(tx)}{\gamma(tx)}\eta(x)\right\rangle=\frac{t(\varepsilon t)^N}{\rho(\varepsilon)}\left\langle\frac{f(\varepsilon tx)}{(\varepsilon t)^N}\eta(x),\frac{\phi(tx)}{\gamma(tx)}\right\rangle.$$

Taking N large enough and $1/L(\varepsilon)\le C_s/\varepsilon^s$, $\varepsilon<\varepsilon_2$, which holds for every s and suitable $C_s>0$ (cf. [**2**]), it follows that there exist $C>0$ and $n_1,n_2>0$ such that

$$\frac{t(\varepsilon t)^N}{\rho(\varepsilon)}\le C\frac{1+t^{n_1}}{t^{n_2}},\quad \varepsilon t>\varepsilon_2,\quad \varepsilon<\varepsilon_2.$$

Note that there exist $a, b > 0$ and $k \in \mathbf{N}$ such that

$$\left|\left\langle \frac{f(\varepsilon tx)}{(\varepsilon t)^N}\eta(x), \frac{\phi(tx)}{\gamma(tx)}\right\rangle\right| \le \sup\left\{\left|\left(\frac{\phi(tx)}{\gamma(tx)}\right)^{(\alpha)}\right|; 0 < a \le x \le b,\ \alpha \le k\right\}.$$

Thus (7) implies that $|F|$ is bounded by an integrable function on $(0,\infty)$ which does not depend on ε. The proof is completed. □

THEOREM 2. *Let $f \in \mathcal{D}'$ and ρ be regularly varying at ∞ of order α. Assume* $\lim_{k\to\infty} \langle f(kx), \phi(x)\rangle / \rho(k)$ *exists for every $\phi \in \mathcal{D}$. Then:* (a) $f \in \mathcal{S}'$. (b) *there exists $(C_1, C_2) \neq (0,0)$ such that*

$$\lim_{k\to\infty} \frac{1}{\rho(k)} \langle f(kx), \phi(x)\rangle = \langle C_1 f_{\alpha+1}(x) + C_2 f_{\alpha+1}(-x), \phi(x)\rangle\,,\ \phi \in \mathcal{S}.$$

Proof. (a) This is proved in [**5**]. We present a sketch of the proof. The set $\{f(kx)/(k^\nu L(k));\ k > 0\}$ is a bounded subset of $\mathcal{D}'$. Let $\Omega = (-2,2)$ and $K = [-\varepsilon, \varepsilon]$. From Theorem XXII in [**9**, T. II, Ch. VI] it follows that there exists a non-negative integer m such that for any $\varphi \in \mathcal{D}^m_K$

$$\Omega \ni x \to ((f(kt)/(k^\nu L(k))) * \varphi(t))(x),\quad k \in (k_0, \infty),$$

is a family of functions which are continuous and uniformly bounded on Ω. Since the weakly bounded family is strongly bounded in $(\mathcal{D}^m_K)'$, it follows that for every bounded set $A \subset \mathcal{D}^m_{[-\varepsilon,\varepsilon]}$, the set of functions

$$\{\Omega \ni x \to ((f(kt)/(k^\nu L(k))) * \varphi(t))(x);\ \ k > 0,\ \varphi \in A\}$$

is a bounded family of continuous functions on Ω. Let $\psi \in \mathcal{D}^m_{[-\varepsilon,\varepsilon]}$ and $\varphi_k(x) = \psi(kx)/k^m$, $x \in \mathbf{R}$, $k \ge 1$. Then $A = \{\varphi_k(x);\ k \ge 1\}$ is a bounded family in $\mathcal{D}^m_{[-\varepsilon,\varepsilon]}$ and

$$\{((f(kt)/(k^\nu L(k))) * \varphi_r(t))(x);\ \ k > 0,\ r \ge 1\}$$

is a bounded family of continuous functions on Ω. Taking $r = k$, we obtain that for some $M > 0$

$$|((f(kt)/(k^\nu L(k))) * (\psi(kt)/k^m))(x)| \le M,\ \ x \in (-2,2),\ \ k \ge 1.$$

Since $(f(kt) * \psi(kt))(x) = k^{-1}(f * \psi)(kx)$, it follows that

$$|(f * \psi)(kx)/(k^{\nu+m+1}L(k))| < M \quad \text{for} \quad x \in (-2,2),\ \ k \ge 1.$$

By taking $x = 1$ and $x = -1$ it follows that for any $\psi \in \mathcal{D}^m_{[-\varepsilon,\varepsilon]}$ there exists $M_\psi > 0$ such that

$$|(f * \psi)(x)| \le M_\psi(1 + |x|^{\nu+m+1}L(|x|)),\ \ x \in \mathbf{R}.$$

By (6; 22) in [**9**, T. II Ch. VI], we obtain

$$f = \frac{d^{2s}}{dx^{2s}}(\varphi_1 * f) - \varphi_2 * f,$$

where $\varphi_1 \in \mathcal{D}^m_{[-\varepsilon,\varepsilon]}$ and $\varphi_2 \in \mathcal{D}_{[-\varepsilon,\varepsilon]}$. This implies that $f \in \mathcal{S}'$.

(b) The proof is the same as the proof of the second part of Theorem 1 except that we use

$$\frac{\rho(kt)}{\rho(k)} \le C\frac{1+t^{n_1}}{t^{n_2}} \quad \text{for} k > k_0, \quad t > \frac{k_0}{k},$$

instead of (5), and at the end, consider separately the cases: (a) $kt \ge T$, and (b) $kt < T$.

4. Relations between the S-asymptotics

Definition 2. A distribution $f \in \mathcal{D}'$ has the $S-$asymptotics in $\mathcal{D}'$ with respect to a positive measurable function $c(h)$, $h > h_0$ if for every $\phi \in \mathcal{D}$ there exists the limit

$$\lim_{h\to\infty} \left\langle \frac{f(x+h)}{c(h)},\ \phi(x) \right\rangle. \tag{8}$$

If $f \in \mathcal{S}'$ (resp. $\mathcal{K}'{}_1(\mathbf{R})$) and (8) exists for every $\phi \in \mathcal{S}$ (resp $\mathcal{K}_1$), then f has the S-asymptotics in $\mathcal{S}'$(resp. $\mathcal{K}'{}_1$) with respect to $c(h)$.

It is well known that in the case of S-asymptotics in $\mathcal{D}'$ there exist $C \ne 0$ and $\alpha \in \mathbf{R}$ such that

$$\lim_{h\to\infty} \frac{f(x+h)}{c(h)} \to Ce^{\alpha\, x} \quad \text{in } \mathcal{D}' \tag{9}$$

$$c(h) = e^{\alpha h}L(e^h), \quad h > h_0, \tag{10}$$

where L is a slowly varying function. If f has the S-asymptotics in $\mathcal{S}'$, then $\alpha = 0$ in (8) and (9).

The following theorem is a part of the results given in [**7**].

Theorem 3. *Let* $\alpha, \beta \in \mathbf{R}$, $c_1(h) = h^\alpha L(h)$ *and* $c_2(h) = h^\alpha L(h)e^{\beta h}$, $h > h_0$.

(a) *Let* $T \in \mathcal{S}'$ *(resp.* $\mathcal{K}_1'$*). If*

$$\{T(x+h)/c_1(h);\ h > h_0\} \quad (\textit{resp.}\ \{T(x+h)/c_1(h); h > h_0\}) \tag{11}$$

is bounded in $\mathcal{D}'$*, then this set is bounded in* $\mathcal{S}'$ *(resp.* $\mathcal{K}_1'$*).*

(b) *Let* $T \in \mathcal{D}'$ *such that* $\operatorname{supp} T \subset [0,\infty)$*. If the set in* (10) *is bounded in* $\mathcal{D}'$*, then this set is bounded in* $\mathcal{S}'$ *(resp.* $\mathcal{K}_1'$*).*

(c) *Let* $T \in \mathcal{D}'$ *such that* $\operatorname{supp} T \subset [0,\infty)$*. If* T *has the* S*-asymptotics in* $\mathcal{D}'$ *with respect to* $c_1(h)$ *(resp. with respect to* $c_2(h)$*), then* T *has the* S*-asymptotics in* $\mathcal{S}'$ *(resp.* $\mathcal{K}'{}_1$*) with respect to* $c_1(h)$ *(resp.* $c_2(h)$*).*

Proof. We will prove only the assertion (b) for a family $\{f(x+h)/c_2(h),$ $h > h_0\}$, where $c_2(h)$ is of the given form.

As in the proof of Theorem 2, the use of a suitable parametrix implies that there exists $m \in \mathbf{N}$ such that for every $\psi \in \mathcal{D}^m_{[-1,1]}$

$$h \mapsto \frac{(T*\psi)(h)}{c_2(h)}, \quad h > h_0$$

is a bounded function and that for large enough $N \in \mathbf{N}$,

$$T(x) = \Big(\frac{d}{dx}\Big)^{2N} T_1(x) + T_2(x),\ x \in \mathbf{R},$$

where T_1 and T_2 are contionuous functions such that

$$\max\{|T_1(x)|, |T_2(x)|\} \le c_2(x), \quad x > h_0.$$

The structural theorem for $\mathcal{K}'_1$ given in (1) implies that there is $k \in \mathbf{N}$ such that

$$\max\{|T_1(x)|, |T_2(x)|\} \le e^{k|x|},\ \ x \in \mathbf{R}. \tag{12}$$

Let $\phi \in \mathcal{K}_1$, We have to estimate

$$\Big\langle \frac{T(x+h)}{c_2(h)}, \phi(x) \Big\rangle = \Big\langle \frac{T_1(x+h)}{c_2(h)}, \phi^{(2N)}(x) \Big\rangle + \Big\langle \frac{T_2(x+h)}{c_2(h)}, \phi(x) \Big\rangle .$$

We will estimate only the first member on the right-hand side because the estimate for the second one is the same. By (11) we have

$$\begin{aligned}\int_{-\infty}^{\infty} \Big|\frac{T(x+h)}{c_2(h)}\Big| |\phi^{(2N)}(x)| dx &= \Big(\int_{-\infty}^{h_0-h} + \int_{h_0-h}^{\infty}\Big)\Big(\frac{|T(x+h)|}{c_2(h)}\Big)|\phi^{(2N)}(x)|dx \\ \le \int_{-\infty}^{h_0-h} \frac{e^{k|x+h|}}{c_2(h)} |\phi^{(2N)}(x)|dx &+ \int_{h_0-h}^{\infty} \Big|\frac{T_1(x+h)}{c_2(x+h)}\Big| \frac{c_2(x+h)}{c_2(h)} |\phi^{(2N)}(x)|dx.\end{aligned}$$

Note that $|x+h| \le |x| - |h-h_0| + |h_0|$, for $x < h_0 - h$, and for every $\varepsilon > 0$ there is $h_1(> h_0)$ such that $L(e^h) \ge (e^h)^{-\epsilon}$ for $h > h_1$, which implies that there exist $C_1 > 0$ and $k_1 > 0$ such that

$$1/c_2(h) \le C_1 e^{-k_1 h}, \quad h > h_1. \tag{13}$$

Also, there exist $C_2 > 0$ and $k_2 > 0$ such that

$$\frac{c_2(x+h)}{c_2(h)} \le C_2 e^{k_2|x|} \quad \text{for } x > h_0 - h > 0.$$

Since $\Big|\frac{T_1(x+h)}{c_2(x+h)}\Big|$ is bounded for $x + h > h_0$, (11), (12) and (13) imply the boundedness of integrals given above because $\phi^{(2N)}(x)$ decreases faster than any negative power of $e^{|x|}$ as $|x| \to \infty$.

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Institut za matematiku
Trg Dositeja Obradovića
21000 Novi Sad
Yugoslavia

(Received 16 01 1995)