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SUMMABILITY OF TRIGONOMETRIC SERIES AND CALDERON REPRODUCING FORMULAS

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Dedicated to the Life and Work of Professor S. Aljančić

Abstract. We show how to associate a Calderon reproducing formula with a number of summability methods, including Abel-Poisson, Gaussian, Riesz typical and Bochner-Riesz means. Such formula contains local/global information concerning the trigonometric series being summed whether it be a Fourier series or not. Consequently a number of classical results of Littlewood–Paley type are unified as well as related results concerning the function class Fourier character of trigonometric series. The paper also provides classical insight and heuristics concerning much contemporary work falling under the name of wavelet analysis.

1. Introduction

Given a summability kernel $\{\phi_a\}_{a>0} \subset L^1(T)$, $T = \mathbf{R}/2\pi\mathbf{Z}$ (section 2 gives explicit definition), then under a variety of function space hypothesis and/or modes of convergence one has $\phi_a * f \to f$ as $a \to 0$. A basic question underlying much of classical and modern Fourier analysis is: how are the local/global properties of functions f reflected in the family of approximations $\{\phi_a * f\}$? A second related question may be posed as follows. Suppose the Fourier coefficients of ϕ_a have the form $\widehat{\phi}_a(n) = \Phi(an)$ for some function Φ defined on $\mathbf{R}$ (necessarily $\Phi(0) = 1$, other properties are prescribed in section 2); we have a summability method for trigonometric series

$$\sum_n c(n)e^{int} \tag{TS}$$

by considering the behavior as $a \to 0$ of the functions

$$\sum_n \Phi(an)c(n)e^{int}. \tag{1.1}$$

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The second question is: how is the function class Fourier character of the trigonometric series characterized in terms of the behavior of (1.1)? Chapter 4 of [**13**] provides numerous results in this vein.

The interconnection between these questions is readily apparent within the framework of Littlewood–Paley theory [**7**], [**13**]. Another example relevant to this paper is the following result (which is essentially a paraphrase of Theorem 5.1 in [**7**, page 262]).

THEOREM A. *Let* $\{c(n)\} \subset \mathbf{C}$ *be a null sequence. Let* $u(t,a)$ *denote the series* (1.1) *where* $\Phi(\lambda) = e^{-|\lambda|}$. *Then* (TS) *is the Fourier series of some function* $f \in Lip_\alpha(T)$ *for some* $0 < \alpha \leq 1$ *if and only if* $u_t(a,t) = O(a^{\alpha-1})$ *as* $a \to 0$ *uniformly for* $t \in T$.

Underlying contemporary approaches to Littlewood-Paley theory on Euclidean space is the notion of a Calderon reproducing formula (see [**2**], [**9**]). The essential purpose of this paper is develop on T analogs of Calderon's formula using suitable summability kernels and indicate the utility of these in regard to the above questions. Section 2 gives the class of summability kernels under consideration and develops two versions of the associated Calderon formula. Section 3 gives variations and generalizations of Theorem A and local versions which are analogs of results in [**4**], [**5**], [**3**]. Section 4 gives further results and commentary. This paper has three outcomes. First, a number of classical results in regard to the above questions are unified. Secondly, two of our main results (Theorems 2.4 and 2.6) provide interpretation of the Calderon reproducing formulas (2.6) and (2.11) via summability processes; this is a classical heuristic for the "nice" behavior of wavelet expansions in regards to convergence [**6**]. Third, this paper provides the reader classical perspectives of recent ideas and methodologies falling under the name of wavelet analysis (see [**2**], [**3**], [**4**], [**5**], [**9**] and the references within).

Notation. Let $p \geq 1$; $L^p(T)$ is the usual Banach space of functions on T with norm

$$\|f\|_p = \left(\frac{1}{2\pi}\int_T |f(t)|^p dt\right)$$

dt is unnormalized Haar measure on T. Convolution is defined for appropriate functions f and g via:

$$(f * g)(t) = \frac{1}{2\pi}\int_T f(s)g(t-s)ds.$$

Fourier coefficients of an $f \in L^1(T)$ are given by

$$\widehat{f}(n) = \frac{1}{2\pi}\int_T f(t)e^{-int}dt, \;\; n \in Z.$$

With these definitions the convolution formula reads $(f * g)\widehat{\ }(n) = \widehat{f}(n)\widehat{g}(n)$ and the Parseval formula is: $\|f\|_2^2 = \sum_n |\widehat{f}(n)|^2$. We denote the Fourier transform of a function on $\mathbf{R}$ also by $\widehat{f}$, the meaning should be clear by context. For integrable functions on $\mathbf{R}$ the transform is given by

$$\widehat{f}(\lambda) = \int_{\mathbf{R}} f(x)e^{-i\lambda x}dx.$$

Absolute constants will be denoted by A, *possibly different* in each occurrence. Finally if the trigonometric series TS is the Fourier series of a function f we write $TS = S(f)$.

2. Calderon Formulas

The summability methods used in this paper are constructed as follows. Let $k \in L^1(\mathbf{R})$ be even and satisfy the conditions:

$$\int_{\mathbf{R}} k(x)dx = 1 \tag{2.1}$$

$$|k(x)| \leq A(1+|x|)^{-(1+\delta)},\ |\widehat{k}(\lambda)| \leq A(1+|\lambda|)^{-(1+\delta)},\ \text{for some } \delta > 0. \tag{2.2}$$

Without loss of generality we may suppose k and $\widehat{k}$ are continuous. Let $k_a(x) = a^{-1}k(x/a)$ for $a > 0$. Then k_a also satisfies the above conditions. Set $\Phi(\lambda) = \widehat{k}(\lambda)$ so that $\Phi(a\lambda) = \widehat{k}_a(\lambda)$. The Poisson summation formula [**1**] gives

$$\sum_n \Phi(an)e^{int} = 2\pi \sum_n k_a(t + 2n\pi). \tag{2.3}$$

Moreover, the growth conditions (2.2) imply both series are absolutely and uniformly convergent on T, hence in $L^1(T)$-norm. Let $\varphi_a(t)$ denote the sum function; it follows that $\varphi_a \in L^1(T) \cap C(T)$ and $\hat{\varphi}_a(n) = \Phi(an)$. The following result is well known (e.g. [**11**]).

LEMMA 2.1. *Let φ_a be as above. Let $f \in L^p(T)$ for some $1 \leq p \leq \infty$ and set*

$$u_f(t,a) = (\varphi_a * f)(t) = \sum_n \Phi(an)\widehat{f}(n)e^{-int}. \tag{2.4}$$

Then

i) *if $1 \leq p < \infty$, then $u(t,a)$ converges to $f(t)$ in $L^p(T)$-norm as $a \to 0$.*

ii) *if $f \in C(T)$, then $u(t,a)$ converges to $f(t)$ uniformly as $a \to 0$.*

iii) *if $f \in L^1(T)$, then $u(t,a)$ converges to $f(t)$ as $a \to 0$ at every Lebesgue point; in particular almost everywhere.*

We say that $\{\varphi_a\}_{a>0}$ is an *admissible summability kernel* generated by k if it is constructed in the above manner and the function k is differentiable such that $k' \in L^1(\mathbf{R})$ and satisfies estimates like (2.2). This implies that $|\widehat{k}(\lambda)| \leq A(1+|\lambda|)^{-(2+\delta)}$ and consequently $u_f(t,a)$ is differentiable and the series in (2.4) can be differentiated term by term. For a general trigonometric series TS the series in (2.4) is denoted $u(t,a)$; it may also be differentiated term by term if say the coefficients $\{c(n)\}_{|n|<\infty}$ form a null sequence. We also note that k' and its Fourier transform may be assumed to be continuous and by Poisson summation:

$$\varphi_a'(t) = 2\pi a^{-2} \sum k'\left(\frac{t+2n\pi}{a}\right).$$

The following result is the key to the Calderon type formula; the $'$-notation will always denote differentiation in the T variable.

LEMMA 2.2. *Let φ_a be admissible. If $f \in L^2(T)$, then*

$$\int_0^\infty \|u'_f(\cdot,a)\|_2^2 a da = c_\varphi \big[\|f\|_2^2 - |\widehat{f}(0)|^2\big] \text{ where } c_\varphi = \int_0^\infty |\Phi(a)|^2 a da. \qquad (2.5)$$

Proof. Note that the integral defining c_φ is convergent. From (2.4) and the above discussion $u'(\cdot,a) \in L^2(T)$ and

$$\|u'_f(\cdot,a)\|_2^2 = \sum_{n\neq 0} |\Phi(an)|^2 n^2 |\widehat{f}(n)|^2.$$

Hence applying monotone convergence,

$$\int_0^\infty \|u'_f(\cdot,a)\|_2^2 a da = \sum_{n\neq 0}\left(\int_0^\infty |\Phi(an)|^2 a da\right) n^2 |\widehat{f}(n)|^2 =$$
$$= c_\varphi \sum_{n\neq 0} |\widehat{f}(n)|^2 = c_\varphi \left[\|f\|_2^2 - |\widehat{f}(0)|^2\right].$$

This concludes the proof. ◊

Polarizing (2.5) one obtains: if $f_1, f_2 \in L^2(T)$, with $\widehat{f_1}(0) = 0$, denote by $u_1(t,a)$, $u_2(t,a)$ the respective functions (2.4), then

$$\int_0^\infty \langle u'_1(\cdot,a),\, u'_2(\cdot,a)\rangle\, a da = c_\varphi \langle f_1, f_2\rangle\,.$$

Here $\langle\ ,\ \rangle$ denotes the usual complex inner product on $L^2(T)$. Formally setting f_2 to be a delta function at t and computing the inner product one obtains the following formula (again assuming $\widehat{f}(0) = 0$):

$$f(t) = c_\varphi^{-1} \int_0^\infty \int_T u'_f(s,a)\overline{\varphi'_a(s-t)} ds\, a\, da. \qquad (2.6)$$

This is the desired analog of Calderon's formula and is made precise as follows.

Let φ_a be an admissible summability kernel corresponding to the function k as above. Define $\Omega(\lambda) = c_\varphi^{-1}\int_{|\lambda|}^\infty |\Phi(\xi)|^2 \xi d\xi$. Then $|\Omega(\lambda)| \le A(1+|\lambda|)^{-(2+2\delta)}$ and Ω is the Fourier transform of an even function $k_1 \in L^2(\mathbf{R})$. It is easily shown that Ω is differentiable and $d\Omega/d\lambda$ is in $L^2(\mathbf{R})$. It follows that $k_1 \in L^1(\mathbf{R})$ by the following inequality:

$$\int_R |k_1(x)|dx \le A\left(\int_R \big((1+x^2)|k_1(x)|\big)^2 dx\right)^{1/2} \le A\big(\|k_1\|_2 + \|d\Omega/d\lambda\|_2\big) < \infty.$$

We now have that k_1 satisfies all of the conditions to generate a summability kernel except the first inequality in (2.2). To see the latter, note that $|\Phi(\xi)|^2$ is the Fourier transform of the autocorrelation function of k defined by $F_k(x) = \int_R k(x+y)\overline{k(y)}dy$. Then $F_k \in L^1(\mathbf{R})$ is differentiable with $F'_k \in L^1(\mathbf{R})$. It follows by Fourier inversion that $k_1(x) = (c_\varphi x)^{-1} F_k'(x)$ and the desired inequality follows from the following elementary lemma whose proof is left to the reader.

LEMMA 2.3. *Let $g_1, g_2 \in C(\mathbf{R})$ satisfy the inequalities $|g_j(x)| \leq A(1+|x|)^{-(1+\delta)}$, for some $\delta > 0$. Then the function $G(x) = \int_R g_1(x \pm y) g_2(y) dy$ satisfies $|G(x)| \leq A(1+|x|)^{-\delta}$.*

We now see that k_1 generates a summability kernel on T which is denoted $\omega_a, a > 0$. The connection between this summability process and the Calderon formula (2.6) is made by truncating the outer integral in (2.6). Let $f \in L^p(T)$, for some $1 \leq p \leq \infty$ and formally define for $\varepsilon > 0$:

$$f_\varepsilon(t) = c_\varphi^{-1} \int_\varepsilon^\infty \int_T u_f'(s,a) \overline{\varphi_a(s-t)} ds\, a\, da. \tag{2.7}$$

The inner integral is given by

$$\int_T u_f'(s,a) \overline{\varphi_a(s-t)} ds = \sum_{n \neq 0} |\Phi(an)|^2 n^2 \widehat{f}(n) e^{int}. \tag{2.8}$$

and is majorized by $\sum_{n \neq 0} |\Phi(an)|^2 n^2 |\widehat{f}(n)|$. Applying dominated convergence and a change of variable we have that

$$c_\varphi^{-1} \int_\varepsilon^\infty \sum_{n \neq 0} |\Phi(an)|^2 n^2 |\widehat{f}(n)| a da = \sum_{n \neq 0} \Omega(\varepsilon n) |\widehat{f}(n)| < \infty.$$

Consequently the series in (2.8) can be integrated term by term with respect to ada, the outer integral in (2.7) is absolutely convergent, and $f_\varepsilon \in C(T)$. Furthermore we have the following formula valid if $\widehat{f}(0) = 0$:

$$f_\varepsilon(t) = \sum_{\| \neq \| 0} \Omega(\varepsilon n) \widehat{f}(n) e^{int} = (\omega_\varepsilon * f)(t).$$

The discussion is summarized in the following Theorem making precise formula (2.6).

THEOREM 2.4. *Let φ_a be an admissible summability kernel on T. Let $f \in L^p(T)$, for some $1 \leq p \leq \infty$, with $\widehat{f}(0) = 0$ and define $f_\varepsilon \in C(T)$ by (2.7). Then*

i) *if $1 \leq p < \infty$, then $f_\varepsilon \to f$ in $L^p(T)$-norm as $\varepsilon \to 0$*

ii) *if $f \in C(T)$, then $f_\varepsilon \to f$ uniformly as $\varepsilon \to 0$;*

iii) *if $f \in L^1(T)$ then $f_\varepsilon(t) \to f(t)$ as $\varepsilon \to 0$ at every Lebesgue point; in particular almost everywhere.*

Examples. All of the following examples generate admissible summability kernels.

E1) Let $k(x) = 1/(2\pi(1+x^2))$, then $\Phi(\lambda) = e^{-|\lambda|}$. This corresponds to Abel-Poisson summability.

E2) Let $k(x) = 1/2\pi^{1/2}e^{-x^2/4}$, then $\Phi(\lambda) = e^{-\lambda^2}$. We then have Gaussian summability.

E3) Let $\beta > 0$ and

$$k_\beta(x) = \frac{\Gamma(\beta+1)}{2\pi\Gamma(\beta+2)}[\mathbf{H}(1,\beta+2,ix) + \mathbf{H}(1,\beta+2,-ix)],$$

where $\mathbf{H}$ is the confluent hypergeometric function [**8**]. Then $\Phi(\lambda) = (1-|\lambda|)^\beta \times \chi_{[-1,1]}(\lambda)$ which yield the typical means [**1**] of order β. Admissibility follows from asymptotic estimates of the confluent hypergeometric function [**8**]. In this case we note that $u(t,a) = \sigma_n^\beta(t)$, for $(n+1)^{-1} \le a < n^{-1}$ and $u(t,a) = 0$ for $a > 1$ where

$$\sigma_n^\beta(t) = \sum_{|k|\le n}\left(1 - \frac{|k|}{n+1}\right)^\beta c(k)e^{ikt}.$$

When $\beta = 1$, this is the classical $(C,1)$ means of the trigonometric series TS. Denote the $(C,1)$ means of the Fourier series of f by $\sigma_n(f,t)$. Then the Calderon formula takes the following curious form:

$$f(t) = \sum_{k=1}^\infty \frac{2k+1}{k^2(k+1)^2}\int_T \sigma_k'(f,t)\overline{F_k'(s-t)}ds,$$

where F_k is the Fejer kernel.

E4) Let $\beta > 0$ and $k_\beta(x) = (2\pi)^{-1}2^{-(\beta+1)}\Gamma(\beta+1)x^{-(\beta+1/2)}\mathbf{J}_{\beta+1/2}(x)$ where $\mathbf{J}_\mu$ is Bessel function of order μ. Then $\Phi(\lambda) = (1-\lambda^2)^\beta\chi_{[-1,1]}(\lambda)$ corresponding to the Bochner-Riesz means [1] of order β.

Let $\alpha > 0$. An admissible summability kernel φ_a generated by k is said to be *α-admissible* if $\int_R |k'(x)||x|^\alpha dx < \infty$ and is *α'-admissible* if it is *α-admissible* and k is twice continuously differentiable with $|k''(x)| \le A(1+|x|)^{-(1+\delta)}$, for some $\delta > 0$. Examples E1 and E2 are α'-admissible for all $\alpha > 0$. Examples E3 and E4 are α'-admissible for $\alpha < \beta$ (one uses the asymptotics of the special functions [**8**] involved to prove this).

In some applications a more local version of the Calderon formula is needed. Let $g, r \in L^1(\mathbf{R})$ satisfy (2.2). Let $\Psi = \widehat{g}$ and $\Theta = \widehat{r}$ and suppose that

$$\int_0^\infty \Psi(\xi)\overline{\Theta(\xi)}\frac{d\xi}{\xi} = 1 = \int_0^\infty \Psi(-\xi)\overline{\Theta(-\xi)}\frac{d\xi}{\xi}. \tag{2.9}$$

As before we define the families of continuous functions on T, $\{\psi\}_a$ and $\{\theta\}_a$ for $a > 0$ corresponding to g and r respectively. Given a function g as above, note that there are many choices for the function r, in particular one may choose r to be smooth and compactly supported. In a similar fashion to Lemma 2.2 we have:

LEMMA 2.5. *Let ψ_a and θ_a be as above. Then for $f \in L^2(T)$,*

$$\int_0^\infty \int_T (\psi_a * f)(s)\overline{(\theta_a * f)(s)}ds\frac{da}{a} = \|f\|_2^2 - |\widehat{f}(0)|^2. \tag{2.10}$$

Our primary interest in this formula is in the case when $g = k'$ where k generates an admissible summability kernel φ_a. In which case $\psi_a = a\varphi_a'$ and we call $\{\varphi_a, \theta_a\}$ an *admissible summability pair.* Formula (2.10) is now

$$\int_0^\infty \int_T u_f'(s,a)\overline{(\theta_a * f)(s)}ds\,da = \|f\|_2^2 - |\widehat{f}(0)|^2. \tag{2.11}$$

where $u(s,t)$ is given by (2.4), and the corresponding Calderon formula becomes:

$$f(t) = \int_0^\infty \int_T u_f'(s,a)\overline{\theta_a(s-t)}ds\,da \tag{2.12}$$

Formula (2.12) is made precise in a fashion similar to (2.6) by considering the family of truncations given by

$$f_\varepsilon^\theta = \int_\varepsilon^\infty \int_T u_f'(s,a)\overline{\theta_a(s-t)}ds\,da \tag{2.13}$$

It is left to the reader to verify the series representation

$$f_\varepsilon^\theta = \sum_{n\neq 0} \Omega(an)\widehat{f}(n)e^{int}, \quad \text{where} \quad \Omega(\lambda) = \int_{|\lambda|}^\infty i\xi\Phi(\xi)\overline{\Theta(\xi)}\frac{d\xi}{\xi}.$$

Furthermore the outer integral in (2.13) is absolutely convergent, and $\Omega(\lambda)$ is the Fourier transform of an $L^1(\mathbf{R})$ function k which generates a summability kernel satisfying the hypothesis of Lemma 2.1. Summarizing these facts yields the following generalization of Theorem 2.4.

THEOREM 2.6. *Let $\{\varphi_a, \theta_a\}$ be an admissible summability pair. Let $f \in L^p(T)$, for some $1 \le p \le \infty$, with $\widehat{f}(0)$ and define $f_\varepsilon^\theta \in C(T)$ by* (2.13). *Then*

i) *if $1 \le p < \infty$, then $f_\varepsilon^\theta \to f$ in $L^p(T)$-norm as $\varepsilon \to 0$;*

ii) *if $f \in C(T)$, then $f_\varepsilon^\theta \to f$ uniformly as $\varepsilon \to 0$;*

iii) *if $f \in L^1(T)$ then $f_\varepsilon^\theta(t) \to f(t)$ as $\varepsilon \to 0$ at every Lebesgue point; in particular almost everywhere.*

Remark. In the recent paper [**6**] norm and a.e. convergence properties of wavelet expansions were established under very mild assumptions on the wavelet. Although our assumptions are stronger than those in [**6**] (φ_a' plays the role of the wavelet), Theorem 2.4 and Theorem 2.6 provide a classical heuristic for the results in [**6**], namely wavelet expansions on the Fourier transform side behave somewhat like summability processes.

3. Local and Global Properties of Functions

Let $0 < \alpha \le 1$, $\Lambda_\alpha = \Lambda_\alpha(T)$ denotes the class of Lipschitz functions of order α i.e. the subclass of continuous functions f such that

$$\sup_{h\neq 0, t\in T} \frac{|f(t+h) - f(t)|}{|h|^\alpha} < \infty.$$

It is well known that a necessary condition for $f \in \Lambda_\alpha$ is that $\widehat{f}(n) = O(n^{-\alpha})$. The following is a characterization of this class.

THEOREM 3.1. *Let* $0 < \alpha < 1$.

(i) *Let* φ_a *be an* α*-admissible summability kernel. If* $f \in \Lambda_\alpha$ *then* $u_f(t,a) = O(a^{\alpha-1})$ *uniformly in* t.

(ii) *Let* φ_a *be an* α'*-admissible summability kernel and let* $\{c(n)\} \subset \mathbf{C}$ *be a null sequence. If* $u(t,a) = O(a^{\alpha-1})$ *uniformly in* t, *then* $\{f_\varepsilon\}$ *converges uniformly to* $f \in \Lambda_\alpha$ *and* $TS = S(f)$.

Proof. (i) This is an application of the Poisson summation formula as follows:

$$\begin{aligned} u_f'(t,a) &= \int_T f(s)\varphi_a'(t-s)ds = \int_T [f(t-s) - f(t)]\varphi_a'(s)ds = \\ &= 2\pi a^{-2} \int_T [f(t-s) - f(t)] \sum_n k'\left(\frac{s+2n\pi}{a}\right) ds = \\ &= 2\pi a^{-2} \sum_n \int_0^{2\pi} [f(t-s) - f(t)] k'\left(\frac{s+2n\pi}{a}\right) ds = \\ &= 2\pi a^{-2} \sum_n \int_{2\pi n}^{2\pi(n+1)} [f(t-s) - f(t)] k'\left(\frac{s}{a}\right) ds = \\ &= 2\pi a^{-2} \int_{\mathbf{R}} [f(t-s) - f(t)] k'\left(\frac{s}{a}\right) ds. \end{aligned}$$

In the first line above we used the fact that $\int_T \varphi_a'(s)da = 0$. Applying the hypothesis we get $|u_f(t,a)| \le Aa^{-2}\int |s|^\alpha |k'\left(\frac{s}{a}\right)| da \le Aa^{\alpha-1}$ as desired.

(ii) Let $0 < \eta < \varepsilon$. Then from (2.6), the hypothesis, and applying the Poisson summation formula as above we get:

$$|f_\varepsilon(t) - f_\eta(t)| \le A \int_\eta^\varepsilon a^\alpha \int_T |\varphi_a'(s)| ds\, da = A\int_\eta^\varepsilon a^{\alpha-1} \int_R |k'(s)| ds\, da \le A(\varepsilon^\alpha - \eta^\alpha).$$

It follows that $\{f_\varepsilon\}$ is uniformly Cauchy on T and hence converges uniformly to $f_\varepsilon \in C(T)$ (note that $f_\varepsilon \in C(T)$ for each $\varepsilon > 0$). It is immediate that $TS = S(f)$ using the series form of f_ε. The conclusion that $f \in \Lambda_\alpha$ will be arrived at by showing that $\{f_\varepsilon\}$ is uniformly Lipschitz i.e. $|f_\varepsilon(t+h) - f_\varepsilon(t)| \le A|h|^\alpha$ where A is independent of t and ε. Let $0 < \varepsilon < 1$ and suppose without loss of generality that $0 < h < 1$. Then

$$f_\varepsilon(t) = \int_\varepsilon^\infty \int_T u'(s,a)\overline{\varphi_a'(s-t)} ds\, a\, da$$

(recall the outer integral is absolutely convergent). Split the outer integral into two integrals over $[\varepsilon, 1]$ and $[1, \infty)$. Using the series form it is easily shown that the second piece defines a $C^1(T)$ function, hence in Λ_α. Denote the first piece by $g_\varepsilon(t)$; by the hypothesis and Poisson summation formula, this double integral is absolutely convergent with ε replaced by 0. Hence,

$$\begin{aligned}|g_\varepsilon(t+h)-g_\varepsilon(t)| &\le \int_0^1\int_T |u'(s,a)||\varphi_a'(s-(t+h))-\varphi'(s-t)|ds\,a\,da \le \\ &\le \int_0^h\int_T |u'(s,a)||\varphi_a'(s-(t+h))|ds\,a\,da+ \\ &\quad+\int_0^h\int_T |u'(s,a)||\varphi_a'(s-t)|ds\,a\,da+ \qquad (3.1)\\ &\quad+\int_h^1\int_T |u'(s,a)||\varphi_a(s-(t+h))-\varphi_a'(t-s)|ds\,a\,da.\end{aligned}$$

The first two integrals are handled in the same fashion using Poisson summation e.g.

$$\int_0^h\int_T |u'(s,a)||\varphi_a'(s-t)|ds\,a\,da \le A\int_0^h a^{\alpha-1}\int_R |k'(s)|ds\,da = A\|k'\|_1|h|^\alpha.$$

For the third integral we have:

$$\begin{aligned}&\int_h^1\int_T |u'(s,a)||\varphi_a'(s-(t+h))-\varphi_a'(s-t)|ds\,a\,da \le \\ &\le A\int_h^1 a^\alpha\int_T |\varphi_a'(s-h)-\varphi_a'(s)|ds\,da.\end{aligned}$$

Again applying Poisson summation and the mean value theorem the inner integral equals

$$\begin{aligned}&a^{-2}\int_R\left|k'\left(\frac{s-h}{a}\right)-k'\left(\frac{s}{a}\right)\right|ds = \\ &= ha^{-3}\int_R\left|k''\left(\frac{s-\tau}{a}\right)\right|ds = ha^{-2}\int_R\left|k''\left(s-\frac{\tau}{a}\right)\right|ds.\end{aligned}$$

Here we have that τ satisfies $0<\tau<h$ and of course depends on s. From the estimate $|k''(x)|\le A(1+|x|)^{-(1+\delta)}$ and the fact that $\tau/a<1$ the integral $\int_R\left|k''\left(s-\frac{\tau}{a}\right)\right|ds$ is uniformly bounded for $0<a<1$. Consequently,

$$\int_h^1\int_T |u'(s,a)||\varphi_a'(s-(t+h))-\varphi_a'(s-t)|ds\,a\,da \le Ah\int_h^1 a^{\alpha-2}da \le Ah^\alpha.$$

Putting the estimates together we have $|g_\varepsilon(t+h)-g_\varepsilon(t)|\le A|h|^\alpha$ uniformly on T where A is independent of ε as desired. ◊

Using similar methods we have the following result concerning the class $\Lambda_\alpha^p = \Lambda_\alpha^p(T)$ of $L^p(T)$ functions which satisfy

$$\sup_{h\neq 0}\frac{\|f(\cdot+h)-f(\cdot)\|_p}{|h|^\alpha}<\infty.$$

THEOREM 3.2. *Let* $0 < \alpha < 1$.

(i) *Let* φ_a *be an* α*-admissible summability kernel. If* $f \in \Lambda_\alpha^p$ *then* $\|u_f'(\cdot, a)\|_p = O(a^{\alpha-1})$.

(ii) *Let* φ_a *be an* α'*-admissible summability kernel and let* $\{c(n)\} \subset \mathbf{C}$ *be a null sequence. If* $\|u(\cdot, a)\|_p = O(a^{\alpha-1})$, *then* $\{f_\varepsilon\}$ *converges in* L^p *norm to* $f \in \Lambda_\alpha^p$ *and* $TS = S(f)$.

Notice that part (i) of both Theorems 3.1 and 3.2 are valid for $\alpha = 1$. These theorems have as corollaries known results concerning the Abel means and the $(C, 1)$ means of trigonometric series ([**13**], Chapter 7). We now turn our attention to local results.

By an open or closed subinterval of T we mean a subset of T given by the map $t \to e^{it}$ applied to an open or closed subinterval of any subinterval of $\mathbf{R}$ of length 2π. Let J be a subinterval of T, we denote by $\Lambda_\alpha^{loc}(J)$ the class of Lipschitz functions of order α on J i.e. the class of $L^1(T)$ functions f such that $|f(t+h) - f(t)| = O(|h|^\alpha)$ uniformly for $t \in J$. The following result can be viewed as a local version of Theorem 3.1.

THEOREM 3.3. *Let* $\{\varphi_a\}$ *be an* α*-admissible summability kernel for some* $0 < \alpha < 1$.

(i) *Let* f *be a bounded measurable function on* T *with* $f \in \Lambda_\alpha^{loc}(J)$ *for some subinterval* J *of* T. *Then* $u_f'(t, a) = O(a^{\alpha-1})$ *uniformly for* $t \in J$.

(ii) *Let* f *be a bounded measurable function on* T *with* $\widehat{f}(0) = 0$ *such that* $u_f'(t, a) = O(a^{\alpha-1})$ *uniformly on an open subinterval* J *of* T. *Let* $\{\varphi_a, \theta_a\}$ *be an admissible summability pair with* θ *generated by a* C^2 *compactly supported function* r. *Then* $f_\varepsilon^\theta(t) \to f(t)$ *uniformly on any closed subinterval* J' *of* J *and* $f \in \Lambda_\alpha^{loc}(J')$.

Proof. Part (i) is clear by previous techniques. For part (ii), let J' be a closed subinterval of J, without loss of generality suppose J' is centered at e^{i0} and write $J' = [-c, c]$ and $J = [-p, q]$. Let $d = \min\{p - c, q - c\}$. We may suppose $\operatorname{supp}(r) \subset [-1/2, 1/2]$; it follows that $\theta_a(s) = 0$ for $a/2 \leq |s| \leq \pi$. Let $0 < \eta < \varepsilon < d$, then for $t \in J'$

$$\begin{aligned}
|f_\varepsilon^\theta(t) - f_\eta^\theta(t)| &= \left| \int_\eta^\varepsilon \int_T u'(s, a)\overline{\theta_a(s-t)}\,ds\,da \right| \leq \\
&\leq \int_\eta^\varepsilon \int_T |u'(s-t, a)||\theta_a(s)|\,ds\,da \leq \\
&\leq A \int_0^\infty a^{\alpha-1} \int_T |\theta_a(s)|\,ds\,da \leq A\|r\|_{L(R)}(\varepsilon^\alpha - \eta^\alpha).
\end{aligned}$$

Hence $\{f_\varepsilon^\theta\}$ is uniformly Cauchy for $t \in J'$ and converges uniformly to f as $\varepsilon \to 0$. To show that $\{f_\varepsilon^\theta\} \subset \Lambda_\alpha^{loc}(J')$ we use the same ideas as in Theorem 3.1 (ii). In particular the outer integral in (2.13) is split into two pieces over intervals $[\varepsilon, d]$ and $[d, \infty)$. The first piece defines a $C^1(T)$ function, the second is handled as in the proof of Theorem 3.1. Details are left to the reader. ◊

Further localization in Theorem 3.3 is achievable under stronger hypothesis. The following result is analogous to that of Jaffard [4], [5] for wavelet expansions and to that of Holschneider and Tchamitchian [3] for the continuous wavelet transform on $\mathbf{R}$. Our proof is an adaptation to T of that in [3].

THEOREM 3.4. *Let $\{\varphi_a\}$ be an α-admissible summability kernel for some $0 < \alpha < 1$.*

(i) *Let f be a bounded measurable function on T such that $f(t_0+h)-f(t_0) = O(|h|^\alpha)$ for some $t_0 \in T$. Then $u'(t_0+b,a) = O(a^{\alpha-1} + a^{-1}|b|^\alpha)$ $(a, |b| \to 0)$.*

(ii) *Let $0 < \gamma < \alpha$ and suppose $f \in \Lambda_\gamma$. If for some $t_0 \in T, u'(t_0+b,a) = O\left(a^{\alpha-1} + \dfrac{|b|^\alpha}{a|\log b|}\right)$ $(a, |b| \to 0)$, then $f(t_0+h) - f(t_0) = O(|h|^\alpha)$.*

Proof. Part (i) is now easy and left to the reader. For part (ii), let $\{\varphi_a, \theta_a\}$ be an admissible summability pair with θ_a generated by C^1 function r with $\mathrm{supp}(r) \subset [-1/2, 1/2]$. Without loss of generality let $t_0 = 0$. As in the proofs of Theorem 3.1 and 3.3 it suffices to consider the function

$$g(t) = \int_0^{1/2} \int_T u'(s,a)\overline{\theta_a(s-t)}ds\,da.$$

Let $h > 0$ and write

$$\begin{aligned} g(h) - g(0) = &\int_0^{h^{\alpha/\gamma}} \int_{-\pi}^{\pi} u'(s,a)\overline{\theta_a(s-h)}ds\,da + \int_{h^{\alpha/\gamma}}^{h} \int_{-\pi}^{\pi} u'(s,a)\overline{\theta_a(s-h)}ds\,da \\ &- \int_0^h \int_{-\pi}^{\pi} u'(s,a)\overline{\theta_a(s)}ds\,da + \int_h^{1/2} \int_{-\pi}^{\pi} u'(s,a)[\overline{\theta_a(s-h)} - \overline{\theta_a(s)}]ds\,da. \end{aligned}$$

We denote the four integrals by $J_i,\ I = 1,2,3,4$. For J_1 we have $u'(s,a) = O(a^{\gamma-1})$ uniformly in s by Theorem 3.1. Hence,

$$|J_1| \le A \int_0^{h^{\alpha/\gamma}} a^{\gamma-1} \int_R |r(x)|dxda = O(h^{\alpha-1}).$$

The inner integral for J_2 is over an interval contained in $[-2h, 2h]$ by the support condition on r. It follows that

$$|J_2| \le A \int_{h^{\alpha/\gamma}}^{h} \left(a^{\alpha-1} + a^{-1}\frac{h^\alpha}{|\log h|}\right) \int_T |\theta_a(s)|ds = O(h^\alpha).$$

For the third integral, the inner integral is over the interval $[-a/2, a/2]$ and by the hypothesis we have $u'(s,a) = O(a^{\alpha-1})$ for $|s| \le a/2$. Immediately we get $J_3 = O(h^{\alpha-1})$. Finally, for the fourth integral we apply the mean value theorem to get $J_4 = \int_h^{1/2} \int_{-\pi}^{\pi} u'(s,a)h\overline{\theta'_a(s-\tau)}ds\,da$ where $-h < \tau < 0$ and of course τ depends on s. Again by the support condition on r, the inner integral is over an interval contained in $[-(h+a/2), a/2]$. Moreover the growth condition imposed on r, (2.2) and Poisson summation show that $|\theta'_a(s)| = O(a^{-1})$ uniformly in s. Consequently,

$$\begin{aligned}
J_4 &= O\left(h\int_h^{1/2}\int_{-(h+a/2)}^{a/2}\left(a^{\alpha-1}+a^{-1}\frac{|s|^\alpha}{|\log|s||}\right)a^{-1}ds\,da\right)\\
&= O(h^\alpha)+O\left(h\int_h^{1/2}\frac{(h+a/2)^\alpha}{|\log(h+a/2)|}a^{-2}da\right)\\
&= O(h^\alpha)+O\left(\frac{h}{|\log h|}\int_h^{1/2}a^{\alpha-2}\right)=O(h^\alpha).
\end{aligned}$$

Combining the estimates concludes the proof. ♢

4. Further Results and Commentary

Two additional topics are mentioned in this section.

A) Littlewood-Paley Theory. We have the following result generalizing the well known theorem of Littlewood and Paley [**7**], [**13**] for Poisson integrals.

THEOREM 4.1. *Let $\{\varphi_a\}$ be an admissible summability kernel. If $f \in L^p(T)$ for some $1 < p < \infty$ with $\widehat{f}(0) = 0$, then*

$$A_p\|f\|_p \le \left[\int_T\left(\int_0^\infty |u'_f(s,a)|^2 a da\right)^{p/2}\right]^{1/p} \le B_p\|f\|_p. \tag{4.1}$$

where A_p and B_p are constants dependant only on p. The right hand inequality holds without the hypothesis $\widehat{f}(0) = 0$.

The proof is omitted; it is based on standard techniques centered on the Marcinkiewicz interpolation theorem [**2**], [**10**], [**12**]. However the following corollary is of some interest relative to the perspective of this paper.

COROLLARY 4.2. *Let $\{\varphi_a\}$ be an admissible summability kernel and let $\{c(n)\} \subset \mathbf{C}$ be a null sequence. If for some $1 < p < \infty$*

$$\left[\int_T\left(\int_0^\infty |u'(s,a)|^2 a da\right)^{p/2}\right]^{1/p} < \infty, \tag{4.2}$$

then (i) *$\{f_\varepsilon\}$ as defined by* (2.7) *is Cauchy in $L^p(T)$ norm, hence converges to say f; and* (ii) *$TS = S(f)$.*

Proof. Let $0 < \eta < \varepsilon$, then

$$f_\varepsilon(t) - f_\eta(t) = \int_\eta^\varepsilon \int_T u'(s,a)\overline{\varphi_a(s-t)}ds\,a\,da,$$

where the double integral is absolutely convergent. Let $g \in L^q(T)$ where q is the conjugate index to p. Then $\langle f_\varepsilon - f_\eta, g\rangle = \int_\eta^\varepsilon \int_T u'(s,a)\overline{u_g(s,a)}ds\,a\,da$. Applying Hölder's inequality and Theorem 4.1 we get:

$$|\langle f_\varepsilon - f_\eta, g\rangle| \le$$

$$\le \left[\int_T \left(\int_\eta^\varepsilon |u'(s,a)|^2 a da\right)^{p/2} ds\right]^{1/p} \left[\int_T \left(\int_0^\infty |u_g'(s,a)|^2 a da\right)^{q/2} ds\right]^{1/q} \le$$

$$\le B_q \|g\|_q \left[\int_T \left(\int_\eta^\varepsilon |u'(s,a)|^2 a da\right)^{p/2} ds\right]^{1/p}.$$

Consequently $\langle f_\varepsilon - f_\eta, g\rangle \to 0$ for every $g \in L^q(T)$. The conclusions are immediate from this and the series representation of f_ε. $\diamondsuit$

It is interesting to observe that (4.2) becomes the following for the case of $(C,1)$ summability:

$$\int_T \left(\sum_1^\infty |\sigma_n'(s)|^2/n^3\right)^{p/2} ds < \infty.$$

B) Generalization to the Torus T^N. The results of this paper have extension to the torus in N dimensions. Here we are content to state the natural generalization of the Calderon formula (2.6). An admissible summability kernel is defined in much the same way as in one dimension. Let $k \in L^1(\mathbf{R}^N)$ be radial and satisfy the conditions:

$$\int_{\mathbf{R}^N} k(x)dx = 1$$

$$|k(x)| \le A(1+|x|)^{-(N+\delta)},\ |\widehat{k}(\lambda)| \le A(1+|\lambda|)^{-(N+\delta)}, \text{ for some } \delta > 0. \quad (4.3)$$

Let $k_a(x) = a^{-N}k(x/a)$ and let $\Phi(\xi) = \widehat{k}(\xi)$, note that Φ is also radial. We define the family of continuous functions $\{\varphi_a\}$ on T^N using the N-dimensional version of the Poisson summation formula [11] (just like (2.3) in one dimension). We say the family $\{\varphi_a\}$ is an admissible summability kernel on T^N if it is generated in the above manner from a function k which is differentiable on $\mathbf{R}^N$ and whose partial derivatives satisfy (4.3). The N-dimensional version of Calderon's identity on the torus now takes the form:

$$f(t) = c_\varphi^{-1} \int_0^\infty \int_{T^N} \nabla u(s,a) \cdot \overline{\nabla \varphi_a(s-t)} ds\, a\, da.$$

Here ∇ denotes the gradient operator and $c_\varphi = \int_{\mathbf{R}^N} |\Phi(\xi)|^2 |\xi| d\xi$. Interpretation of this formula is carried out like Theorem 2.2. Further development of results on the torus will be done in later work.

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